

# Radial Barriers for Discrete Newtonian Potentials: Counterexamples and Sufficient Conditions

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## Abstract

For a locally finite configuration of positive masses in  $\mathbb{R}^n$ ,  $n \geq 3$ , we study the existence of spheres on which the Newtonian potential is uniformly small. Unit masses at the positive integers contradict the dimensional extension of an auxiliary lemma of Clunie, Eremenko and Rossi for every  $n \geq 4$ ; the same example nevertheless has infinitely many nondegenerate equilibrium points. Under potential summability, we give valid radial criteria involving the accumulated  $(n-2)$ th roots of the masses when  $n \geq 4$ , and a mass-entropy sum when  $n = 3$ . Each criterion implies infinitely many equilibria escaping to infinity. Finite-total-mass examples show that the corresponding sublinear growth requirements cannot be replaced by linear bounds in these radial assertions. These results neither refute the equilibrium-existence theorem of the cited paper nor settle the general positive-mass question under force summability alone. This is an unrefereed manuscript; no absolute priority claim is made.

## 1 The equilibrium question and the radial estimate

Let  $x_k \in \mathbb{R}^n \setminus \{0\}$  be distinct, with  $|x_k| \rightarrow \infty$ , and let  $a_k > 0$ . Eremenko's problem [2] asks whether, under

$$\sum_k \frac{a_k}{|x_k|^{n-1}} < \infty, \quad (1)$$

the field

$$F(x) = \sum_k a_k \frac{x_k - x}{|x - x_k|^n} \quad (2)$$

must vanish away from the masses. The problem includes  $n = 2$ ; our results concern only  $n \geq 3$  and use the stronger potential-summability condition below. They do not resolve the general question.

Put  $p = n - 2$  and  $r_k = |x_k|$ . Unless explicitly stated otherwise, assume

$$\sum_k \frac{a_k}{r_k^p} < \infty, \quad U(x) = \sum_k \frac{a_k}{|x - x_k|^p}. \quad (3)$$

On compact sets avoiding the poles, this series and all its derivatives converge uniformly. Indeed, for all sufficiently large  $k$ ,  $|x - x_k| \geq r_k/2$ , and an additional derivative contributes an additional factor bounded by a constant times  $r_k^{-1}$ . Consequently  $U$  is smooth and harmonic there, and

$$\nabla U = pF. \quad (4)$$

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We extend  $U$  by  $+\infty$  at its poles. Define

$$B(t) = \max_{|x|=t} U(x), \quad \Psi(t) = \sum_k \frac{a_k}{|t - r_k|^p}. \quad (5)$$

Values of  $+\infty$  are allowed. The reverse triangle inequality gives  $B(t) \leq \Psi(t)$ . A *radial barrier sequence* means radii  $t_j \rightarrow \infty$  such that  $B(t_j) \rightarrow 0$ .

Clunie, Eremenko and Rossi [1] state in their Theorem 2.11 that (3), together with  $a_k \geq 1$ , implies infinitely many critical points. Their proof uses a radial barrier sequence in equation (8.1), obtained from Lemma 8.1. The latter asserts that, for strictly increasing positive radii and  $p = n - 2 \geq 1$ , the assumptions  $a_k \geq 1$  and (3) imply  $\liminf_{t \rightarrow \infty} \Psi(t) = 0$ . Its displayed proof treats  $p = 1$  and asserts that higher dimensions are similar. Section 2 shows that the assertion is false for every  $p > 1$ . This observation concerns that auxiliary lemma and that radial step, not the truth or falsity of the existence conclusion of Theorem 2.11.

Our replacement estimates appear in Section 3. The subsequent deformation argument is the classical bounded-superlevel mechanism used in [1, Sections 7–8]. The new input here is the explicit radial analysis and its limitations, not a new mountain-pass principle. The planar positive-residue conjecture discussed in [1, Conjecture 2.7] and [3, Conjecture 1] is a separate part of the general problem.

## 2 The integer lattice obstruction

**Proposition 2.1.** *For every real  $p > 1$ , the radii  $r_k = k$  and masses  $a_k = 1$ ,  $k \geq 1$ , satisfy  $\sum_k a_k/r_k^p < \infty$ , but*

$$\liminf_{t \rightarrow \infty} \sum_{k=1}^{\infty} |t - k|^{-p} = 2(2^p - 1)\zeta(p) > 0. \quad (6)$$

*In particular, Lemma 8.1 of [1], as printed, is false in every dimension  $n \geq 4$ .*

*Proof.* The summability assertion is  $\zeta(p) < \infty$ . Every  $t \geq 1$  is within  $1/2$  of a positive integer, so the sum is at least  $2^p$ . For the exact value, consider

$$G_p(s) = \sum_{j \in \mathbb{Z}} |s - j|^{-p}, \quad 0 < s < 1.$$

Local uniform convergence of the differentiated series shows that  $G_p$  is strictly convex. It is symmetric about  $1/2$ , and hence its minimum is

$$G_p(1/2) = 2 \sum_{j=0}^{\infty} (j + 1/2)^{-p} = 2(2^p - 1)\zeta(p).$$

For  $t = m + s$ ,  $m \geq 1$  an integer and  $0 < s < 1$ , the difference between  $G_p(s)$  and the one-sided sum is a nonnegative tail bounded above by  $\sum_{j \geq m} j^{-p}$ . This bound tends to zero uniformly in  $s$ . Comparison with the minimum of  $G_p$ , followed by evaluation at  $s = 1/2$ , proves (6).  $\square$

**Proposition 2.2.** *For  $n \geq 4$ , the configuration  $x_k = ke_1$ ,  $a_k = 1$  in  $\mathbb{R}^n$  has no radial barrier sequence, but has exactly one equilibrium in each interval  $(j, j+1)e_1$ ,  $j \geq 1$ , and no other equilibria. All these equilibria are nondegenerate.*

*Proof.* At  $te_1$  the potential equals the sum in Proposition 2.1, so  $B(t) \geq U(te_1) \geq 2^{n-2}$  for  $t \geq 1$ . Write  $x = (t, y)$  with  $y \perp e_1$ . The transverse component is

$$F_\perp(x) = -y \sum_{k \geq 1} |x - ke_1|^{-n}.$$

It vanishes only when  $y = 0$ . On each pole gap the axial field has derivative

$$\frac{d}{dt} F_1(te_1) = (n-1) \sum_{k \geq 1} |t - k|^{-n} > 0,$$

and its limits at the left and right endpoints are  $-\infty$  and  $+\infty$ . Thus there is exactly one zero in each gap. For  $t < 1$  every axial term is positive. At an axial zero, put  $S = \sum_k |t - k|^{-n} > 0$ . The Jacobian eigenvalues are  $(n-1)S$  in the axial direction and  $-S$  in each of the  $n-1$  transverse directions, proving nondegeneracy.  $\square$

The example itself therefore satisfies the conclusion of [1, Theorem 2.11]. It is not admissible in dimension three under (3), because  $\sum_k 1/k$  diverges.

### 3 Valid small-value criteria

For  $R \geq e$ , let

$$\begin{aligned} M(R) &= \sum_{r_k \leq R} a_k, & H(R) &= \sum_{r_k \leq R} a_k \log^+(1/a_k), \\ A_p(R) &= \sum_{r_k \leq R} a_k^{1/p}, & \log^+ u &= \max\{0, \log u\}. \end{aligned} \tag{7}$$

All logarithms are natural. Distinct points may have equal radii.

**Theorem 3.1.** *Under (3), either of the following conditions implies a radial barrier sequence:*

$$p > 1 : \quad \liminf_{R \rightarrow \infty} \frac{A_p(R)}{R} = 0; \tag{8}$$

$$p = 1 : \quad \liminf_{R \rightarrow \infty} \frac{M(R) \log R + H(R)}{R} = 0. \tag{9}$$

*In fact, the radii can be chosen so that  $\Psi(t_j) \rightarrow 0$ .*

*Proof.* Fix  $0 < \delta \leq 1$  and consider  $I_R = [R, 2R]$ . For  $r_k > 4R$  and  $t \in I_R$ ,  $|t - r_k| \geq r_k/2$ , so

$$\sum_{r_k > 4R} \frac{a_k}{|t - r_k|^p} \leq 2^p \sum_{r_k > 4R} \frac{a_k}{r_k^p} \longrightarrow 0 \tag{10}$$

uniformly on  $I_R$ . For the finite near sum define

$$C_\delta(t) = \sum_{r_k \leq 4R} \min \left\{ \delta, \frac{a_k}{|t - r_k|^p} \right\}.$$

If  $C_\delta(t) < \delta$ , every unclipped term is strictly less than  $\delta$ . Thus the near sum equals  $C_\delta(t)$  and is less than  $\delta$ . This strict implication is essential.

Suppose  $p > 1$ . Splitting the integral at  $|s| = (a/\delta)^{1/p}$  gives

$$\int_{\mathbb{R}} \min\{\delta, a|s|^{-p}\} ds = \frac{2p}{p-1} \delta^{1-1/p} a^{1/p}. \quad (11)$$

Consequently

$$\int_{I_R} C_\delta(t) dt \leq \frac{2p}{p-1} \delta^{1-1/p} A_p(4R).$$

Condition (8), with a change of scale from  $4R$  to  $R$ , makes this less than  $\delta R$  along an unbounded sequence of radii. There is a  $t \in I_R$  with  $C_\delta(t) < \delta$ . Taking  $R$  sufficiently large also makes (10) less than  $\delta$ , so  $\Psi(t) < 2\delta$ .

For  $p = 1$ , the translated interval  $I_R - r_k$  is contained in  $[-4R, 4R]$ . Direct integration, including the case  $a \geq 4R\delta$ , gives

$$\begin{aligned} \int_{-4R}^{4R} \min\{\delta, a/|s|\} ds &\leq 2a[1 + \log^+(4R\delta/a)] \\ &\leq 2a[1 + \log(4R) + \log^+(1/a)]. \end{aligned}$$

It follows that

$$\int_{I_R} C_\delta(t) dt \leq 2[(1 + \log(4R))M(4R) + H(4R)].$$

By (9), this is less than  $\delta R$  along an unbounded sequence. The same strict clipping argument and tail estimate give  $\Psi(t) < 2\delta$ . In both cases, let  $\delta$  tend to zero and choose successively larger radii. No chosen radius is a pole of  $\Psi$ .  $\square$

**Corollary 3.2.** *For  $p > 1$ , the condition*

$$\sum_k \frac{a_k^{1/p}}{r_k} < \infty \quad (12)$$

*implies potential summability and a radial barrier sequence. For  $p = 1$ , the same conclusion follows from*

$$\sum_k \frac{a_k[1 + \log^+(1/a_k)]}{r_k} < \infty. \quad (13)$$

*Proof.* If  $b_k > 0$  and  $\sum_k b_k/r_k < \infty$ , then  $\sum_{r_k \leq R} b_k = o(R)$ . Indeed, split at a fixed finite index: the finite initial sum divided by  $R$  tends to zero, while the remaining sum divided by  $R$  is bounded by the corresponding tail of  $\sum_k b_k/r_k$ . Apply this with  $b_k = a_k^{1/p}$  to obtain (8). Also an  $\ell^1$  sequence belongs to  $\ell^p$  for  $p > 1$ , so (12) implies (3).

Under (13), the same argument gives  $H(R) = o(R)$ . Tonelli's theorem gives

$$\int_e^\infty \frac{M(t)}{t^2} dt < \infty.$$

If  $\liminf M(R) \log R/R$  were positive, this integral would dominate a positive multiple of  $\int_e^\infty dt/(t \log t)$ , which diverges. Choose the resulting subsequence and use  $H(R) = o(R)$  to obtain (9).  $\square$

*Remark 3.3.* In dimension three, (3) and  $H(R) = o(R)$  already suffice, by the same Tonelli argument. In particular, when  $a_k \geq 1$ ,  $H = 0$  and one recovers the valid three-dimensional radial step in [1]. In higher dimensions our conditions are sufficient replacements, not consequences of the original hypotheses of Lemma 8.1.

## 4 Infinitely many equilibrium points

**Theorem 4.1.** *Under (3), a radial barrier sequence implies that  $U$  has critical points  $z_j$  with  $U(z_j) \rightarrow 0$  and  $|z_j| \rightarrow \infty$ . In particular, every condition in Theorem 3.1 or Corollary 3.2 implies infinitely many distinct equilibria of (2).*

*Proof.* For  $c > 0$ , let  $E_c = \{x : U(x) > c\}$ , including the poles. This is an open subset of  $\mathbb{R}^n$ . Every component is bounded: any one of its points can be enclosed by a sufficiently large sphere on which  $U < c$ , and a path in the component cannot cross that sphere. Here we use that components of an open subset of  $\mathbb{R}^n$  are path connected. Each such bounded component contains only finitely many poles. Since there are infinitely many poles, infinitely many components contain poles.

We claim that there is a critical point with  $0 < U \leq c$ . Suppose otherwise. Take poles in two different components of  $E_c$  and join them by a compact path avoiding all other poles. Such a path exists because the pole set is locally finite and  $n \geq 3$ . The potential has a positive minimum  $b$  on this path, with its infinite endpoint values understood. Necessarily  $b \leq c$ . The path is contained in a bounded component  $D$  of  $E_{b/2}$ .

The strip

$$K = \{x \in \overline{D} : b \leq U(x) \leq c\}$$

is compact, disjoint from the poles and from  $\partial D$ . Our assumption implies that  $|\nabla U|$  has a positive lower bound there. Moreover, for some  $\varepsilon > 0$  the enlarged strip  $b \leq U \leq c + \varepsilon$  in  $\overline{D}$  is also compact and critical-point-free. Otherwise critical points with values tending down to  $c$  would have a limit in  $K$ , a contradiction.

On this strip the normalized gradient flow

$$x' = \frac{\nabla U(x)}{|\nabla U(x)|^2}$$

raises  $U$  at unit speed. Flow every point of the path for time  $\max\{0, c + \varepsilon - U(x)\}$ ; leave points with higher potential, including neighborhoods of the two endpoint poles, fixed. All nontrivial flow segments stay in the compact enlarged strip, so the construction exists and depends continuously on the initial point. The resulting path lies in  $E_c$  and joins the original poles. This contradicts their belonging to different components of  $E_c$ .

Taking  $c \downarrow 0$  gives critical values approaching zero. On any fixed compact set, even one term  $a_k |x - x_k|^{-p}$  gives a strictly positive lower bound for  $U$ . Thus the critical points escape every compact set. Equation (4) identifies them with equilibria. No nondegeneracy assumption is needed.  $\square$

## 5 Finite mass and sharp radial obstructions

**Proposition 5.1.** *There is a locally finite positive-mass configuration in  $\mathbb{R}^3$  with total mass one for which  $B(t) > 8$  for every  $t \geq 1$ . It satisfies (3), but has no radial barrier sequence. For this configuration*

$$M(R) \leq 1, \quad \frac{H(R)}{R} \rightarrow 16 \log 2.$$

*For every  $0 < c < 8$ , the pole-free superlevel set  $\{U > c\}$  has an unbounded component.*

*Proof.* For each integer  $j \geq 1$ , put

$$N_j = 2^{2^{j+4}}, \quad x_{j,k} = \left(j + \frac{k + 1/2}{N_j}\right) e_1, \quad a_{j,k} = \frac{2^{-j}}{N_j} \quad (0 \leq k < N_j). \quad (14)$$

Each block is finite, so the pole set is locally finite. Its total mass is  $\sum_{j \geq 1} 2^{-j} = 1$ , and its radii exceed one; both potential and force summability follow.

Fix  $t \in [j, j+1]$ , not a pole. At least  $N_j/2$  grid points lie on one side of  $t$ . The  $\ell$ th point on that side is at distance at most  $\ell/N_j$ . Writing  $h_N = \sum_{\ell=1}^N 1/\ell$ ,

$$U(te_1) \geq 2^{-j} h_{N_j/2}.$$

Grouping harmonic terms into dyadic blocks gives  $h_{2^L} \geq 1 + L/2$ . Since  $N_j/2 = 2^L$  for  $L = 2^{j+4} - 1$ , the last lower bound is at least  $8 + 2^{-j-1} > 8$ . The extended potential is infinite at poles. Thus  $B(t) > 8$  on the whole ray.

For  $c < 8$ , each compact segment of the ray can bypass its finitely many poles through small punctured balls on which  $U > c$ . Such balls exist because of the singular positive mass term, and their punctured versions are path connected. These paths show that the pole-free superlevel set has one unbounded component.

The entropy contribution of block  $j$  is exactly

$$\sum_{k < N_j} a_{j,k} \log(1/a_{j,k}) = (j2^{-j} + 16) \log 2.$$

The completed blocks therefore contribute  $16R \log 2 + O(1)$ , up to the bounded effect of the block intersecting the sphere of radius  $R$ . This proves the entropy limit.  $\square$

The same finite-mass configuration obstructs radial barriers in every  $n \geq 3$ : within one block the distances are at most one, and  $d^{-(n-2)} \geq d^{-1}$  there. The three-dimensional entropy limit shows that, under potential summability, the sufficient hypothesis  $H(R) = o(R)$  cannot be replaced by  $H(R) = O(R)$  in a geometry-free radial assertion.

**Proposition 5.2.** *For every integer  $p \geq 2$ , there is a locally finite configuration in  $\mathbb{R}^{p+2}$  with finite total mass,  $B(t) \geq 2^p$  for all  $t \geq 1$ , and  $A_p(R)/R \rightarrow 1$ .*

*Proof.* Use the midpoint grids in (14), now with  $N_j = 2^j$  and mass  $a_{j,k} = 2^{-pj}$  at every point. Their total mass is

$$\sum_{j \geq 1} 2^{-(p-1)j} = \frac{1}{2^{p-1} - 1}.$$

Every  $t \in [j, j+1]$  is within  $1/(2N_j)$  of a grid point, whose contribution to the potential is at least  $2^{-pj}(2N_j)^p = 2^p$ . Every complete block contributes  $N_j(2^{-pj})^{1/p} = 1$  to  $A_p$ , and a partial block contributes between zero and one. Hence  $A_p(R)/R \rightarrow 1$ .  $\square$

Thus the sublinear condition in (8) cannot be replaced by a linear upper bound, even with finite total mass. These are sharp *order obstructions to the radial method*, not necessary conditions for equilibrium existence. Indeed, in both grid constructions the transverse-field and strictly increasing axial-field arguments of Proposition 2.2 give exactly one equilibrium in each consecutive pole gap. Neither example is a zero-free field.

## 6 Scope and reproducibility

The source problem is recorded as AMR-036-0023 in `ulamai/UnsolvedMath` version 1.6.0. The unrestricted question (1)–(2) remains open in the scope of this work. In particular, we do not infer an answer from the failure of a radial argument, nor assert that the stated equilibrium conclusion in

[1] is false. The elementary lattice counterexample, corrected sufficient conditions and finite-mass radial obstructions are the specific results proved here.

The accompanying exact rational and integer checker tests lattice lower bounds, certified axial sign brackets, clipped-kernel identities, the strict clipping implication and midpoint-grid estimates. Its 56,395 finite conditions pass identically in ordinary and optimized Python. These computations are regression checks, not substitutes for the infinite arguments above or proof-assistant certification.

AI assistance was used for source retrieval, mathematical exploration, drafting and computational checks. The author is responsible for the manuscript. This preprint has not undergone independent peer review.

## References

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