

Polynomial–Exponential Counterexamples to a Derivative-Zero Half-Plane Classification

Alper Ferudun*

11 October 2026

Abstract

Let $\mathcal{P}$ be the compact-open closure of the polynomials whose zeros lie in the closed lower half-plane. The entire function $e^{iz} + z + 2i$ has every zero of every derivative in that half-plane, but lies neither in $\mathcal{P}$ nor in the exceptional families Ce^{az} and $C(e^{ibz} - a)$, where C, a are complex and b is real. This gives a negative answer to the precise alternative stated in Eremenko’s version of a problem of B. Ya. Levin. More generally, for every positive integer d we construct an affine holomorphic d -parameter family of such exceptions, all of whose derivatives are zero-free even in the closed upper half-plane. Hence these exceptions cannot be covered by countably many holomorphically parameterized families of uniformly bounded finite dimension. The arguments use polynomial dominance and a logarithmic-derivative condition for $\mathcal{P}$. No classification of all admissible entire functions, or absolute priority claim, is made. This manuscript is unrefereed.

1 The precise question and its scope

Set $H_+ = \{z : \operatorname{Im} z > 0\}$ and $H_- = \{z : \operatorname{Im} z \leq 0\}$. Let $\mathcal{P} \subset \mathcal{H}(\mathbb{C})$ be the closure, for locally uniform convergence on $\mathbb{C}$, of the nonzero polynomials with all zeros in H_- . The zero function is then included as a limit. Let $\mathcal{D}$ consist of the nonzero entire functions whose every nonzero derivative, including derivative order zero, is zero-free in H_+ . None of the derivatives in the examples below is identically zero, so the convention for zero derivatives is irrelevant to the counterexamples.

Eremenko’s note dated September 10, 2026 [1] asks whether $f \in \mathcal{D}$ implies that $f \in \mathcal{P}$ or that f has one of the forms

$$Ce^{az}, \quad C(e^{ibz} - a), \quad C, a \in \mathbb{C}, \quad b \in \mathbb{R}. \quad (1)$$

The parameters in the two alternatives are independent. The related formulation LO in the survey of Eremenko and Sodin [3, pp. 11–12 of the author version] refers to the exceptional families in their real-line formulation P2.

The frozen record AMR-036-0017 in `ulamai/UnsolvedMath` version 1.6.0 uses e^{id} , $d \in \mathbb{R}$, instead of the arbitrary constant a in the second family. Our example is outside the larger family in (1), so its force does not depend on that wording difference.

There is an important antecedent. Buhovski’s companion note [2] shows that $e^{iz} - 1 \in \mathcal{D} \setminus \mathcal{P}$ by using closure of $\mathcal{P}$ under products and differentiation. That function belongs to (1). The point here is to escape the specified exceptional families as well. The attribution to Buhovski follows the link text retained in the HTML source of Eremenko’s index [4]; the companion PDF itself has no author line.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

The note [1] also suggests replacing the precise alternatives by an explicitly described finite-dimensional exceptional family. We address a specified holomorphic-parametrization interpretation in Section 4. The informal phrase alone does not define a mathematical class of parametrizations, and we do not claim a classification of $\mathcal{D}$.

2 A closure obstruction and one counterexample

Lemma 2.1. *Every nonzero $g \in \mathcal{P}$ is zero-free in H_+ and satisfies*

$$\operatorname{Im} \frac{g'(z)}{g(z)} \leq 0 \quad (z \in H_+). \quad (2)$$

Every nonzero derivative of a function in $\mathcal{P}$ also belongs to $\mathcal{P}$.

Proof. For an approximating polynomial $p(z) = c \prod_j (z - \rho_j)$, with $\operatorname{Im} \rho_j \leq 0$, its logarithmic derivative gives

$$\operatorname{Im} \frac{p'(z)}{p(z)} = - \sum_j \frac{\operatorname{Im} z - \operatorname{Im} \rho_j}{|z - \rho_j|^2} \leq 0 \quad (z \in H_+).$$

The sum is empty for a nonzero constant. Hurwitz's theorem shows that a nonzero locally uniform limit g is zero-free in H_+ . Cauchy's integral formula gives $p' \rightarrow g'$ locally uniformly, and hence $p'(z)/p(z) \rightarrow g'(z)/g(z)$ at each point of H_+ . This proves (2). Gauss–Lucas puts all zeros of each nonzero p' in H_- . If $g' \not\equiv 0$, an approximating sequence for g' has an eventually nonzero subsequence, proving $g' \in \mathcal{P}$. Iteration proves the last assertion. $\square$

Lemma 2.2. *If $A, B \in \mathbb{C} \setminus \{0\}$, then $A + Be^{iz} \notin \mathcal{P}$.*

Proof. Choose $x \in \mathbb{R}$ so that Be^{ix}/A is positive real. For any $y > 0$, write $t = Be^{ix-y}/A > 0$. At $z = x + iy$,

$$\frac{(A + Be^{iz})'}{A + Be^{iz}} = \frac{it}{1 + t}.$$

Its imaginary part is positive, contrary to Lemma 2.1. $\square$

Theorem 2.3. *The function*

$$f(z) = e^{iz} + z + 2i \quad (3)$$

belongs to $\mathcal{D} \setminus \mathcal{P}$ and has neither form in (1).

Proof. If $y = \operatorname{Im} z \geq 0$, then $|e^{iz}| = e^{-y} \leq 1$, whereas $|z + 2i| \geq y + 2 \geq 2$. Thus f has no zeros in the closed upper half-plane. Its first derivative is $1 + ie^{iz}$, whose zeros are exactly

$$z = \frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}.$$

For $n \geq 2$, $f^{(n)}(z) = i^n e^{iz}$ is zero-free. Therefore $f \in \mathcal{D}$.

By Lemma 2.2, $f' \notin \mathcal{P}$, so Lemma 2.1 implies $f \notin \mathcal{P}$. Finally, f' is nonzero and has zeros. Every nonconstant function of either form in (1) has a zero-free first derivative. The constant cases have identically zero first derivative. Both possibilities are excluded. $\square$

3 Uniform polynomial perturbations

For integers $k \geq n \geq 0$, write $(k)_n = k!/(k-n)!$, including $(k)_0 = 1$.

Theorem 3.1. *Fix any integer $d \geq 1$. For $a = (a_0, \dots, a_{d-1}) \in \mathbb{C}^d$ with $|a_k| < 1$, define*

$$F_a(z) = e^{iz} + 2(z+2i)^d + \sum_{k=0}^{d-1} a_k(z+2i)^k. \quad (4)$$

Every derivative of F_a , including order zero, is zero-free in $\{\operatorname{Im} z \geq 0\}$. Nevertheless $F_a \notin \mathcal{P}$, and F_a has neither form in (1).

Proof. Put $w = z + 2i$ and $Q_a(w) = 2w^d + \sum_{k=0}^{d-1} a_k w^k$. For $0 \leq n \leq d$ and $\operatorname{Im} z \geq 0$, we have $|w| \geq 2$. Writing $m = d - n$, the triangle inequality and $(k)_n \leq (d)_n$ give

$$\begin{aligned} |Q_a^{(n)}(w)| &\geq (d)_n |w|^m \left(2 - \sum_{k=n}^{d-1} |w|^{k-d} \right) \\ &\geq (d)_n |w|^m \left(2 - \sum_{j=1}^m 2^{-j} \right) \\ &= (d)_n |w|^m (1 + 2^{-m}) > 1. \end{aligned}$$

The sum is empty when $n = d$; then the last quantity equals $2d!$. Since $|i^n e^{iz}| \leq 1$, the derivative $F_a^{(n)}(z) = i^n e^{iz} + Q_a^{(n)}(z+2i)$ cannot vanish there. For $n > d$ it is simply $i^n e^{iz}$, also zero-free.

On the other hand,

$$F_a^{(d)}(z) = 2d! + i^d e^{iz}$$

is not in $\mathcal{P}$, by Lemma 2.2. Differentiation closure excludes $F_a \in \mathcal{P}$. This same nonzero derivative has zeros, because $e^{iz} = -2d!i^{-d}$ has solutions. Every derivative of positive order of a function in (1) is either identically zero or zero-free. This excludes the two exceptional forms. $\square$

Remark 3.2. The examples have exponential type 1. Indeed, the polynomial part does not change the upper bound for the type, and along $z = -ir$ the term e^r dominates the polynomial as $r \rightarrow \infty$. Thus a restriction to finite order or finite exponential type does not remove them.

4 What finite-dimensional parametrizations cannot cover

We say that a map $\Phi : U \rightarrow \mathcal{H}(\mathbb{C})$, for $U \subset \mathbb{C}^N$ open, is holomorphic in parameters if $u \mapsto \Phi(u)(z)$ is holomorphic for each fixed z . Only this evaluation property is needed below.

Corollary 4.1. *There is no finite integer N and countable collection of maps $\Phi_j : U_j \rightarrow \mathcal{H}(\mathbb{C})$, holomorphic in parameters on open sets $U_j \subset \mathbb{C}^{n_j}$ with $n_j \leq N$, whose images cover $\mathcal{D} \setminus \mathcal{P}$.*

Proof. Take $d > N$ and evaluate entire functions at any d distinct complex points $z_1, \dots, z_d$. The map

$$a \mapsto (F_a(z_1), \dots, F_a(z_d))$$

has an invertible affine linear part: its matrix has entries $(z_j + 2i)^k$, $1 \leq j \leq d$, $0 \leq k < d$, and is a Vandermonde matrix. Its image on the coefficient polydisc is therefore a nonempty open subset of $\mathbb{C}^d$.

For each j , evaluation composed with Φ_j is a holomorphic map $U_j \rightarrow \mathbb{C}^d$ from complex dimension less than d . Its image has $2d$ -dimensional Lebesgue measure zero. For completeness, exhaust U_j by countably many compact cubes contained in the domain. On each cube the map is Lipschitz, since its first derivatives are bounded on a slightly larger compact neighborhood. Subdivision into cubes of side ε covers the image by $O(\varepsilon^{-2n_j})$ balls of radius $O(\varepsilon)$; their total $2d$ -dimensional volume tends to zero. The case $n_j = 0$ is immediate. A countable union of these images cannot contain a nonempty open set in $\mathbb{C}^d$. Theorem 3.1 now gives a contradiction. $\square$

This excludes parametrizations of uniformly bounded holomorphic dimension, including those on second-countable complex manifolds by passing to countable atlases. It does not exclude a countable union of families of unbounded dimensions, a nonregular set-theoretic parametrization, or some other meaning of the informal suggestion in [1]. In particular, our result is not a classification of $\mathcal{D}$.

5 Provenance and verification

The proof is analytic and holds for all derivative orders and all parameters stated. The accompanying exact-arithmetic regression script checks the falling-factorial bounds for a finite range; those checks are not the proof of the unrestricted theorem. The primary sources [1, 2, 3] and the current index [4] were checked on October 11, 2026. The two Levin links are commented out in that index's HTML; we do not infer an explanation or a literature status from this. A bounded search did not identify a prior statement of the polynomial-perturbation family, but does not certify novelty or priority.

AI assistance was used in developing the examples, checking the arguments, searching sources, and preparing this text. The manuscript has been self-audited in this workflow; no independent expert review or proof-assistant certification is claimed.

References

- [1] A. Eremenko, *A problem of B. Ya. Levin*, note dated September 10, 2026, <https://www.math.purdue.edu/~eremenko/dvi/levin.pdf>.
- [2] L. Buhovski, *A counterexample*, undated companion note linked with this attribution from [4], <https://www.math.purdue.edu/~eremenko/dvi/levin-b.pdf>.
- [3] A. Eremenko and M. Sodin, *Iossif Ostrovskii's work on entire functions*, J. Math. Phys. Anal. Geom. **20** (2024), doi:10.15407/mag20.04.02. Author version dated January 28, 2024: <https://www.math.purdue.edu/~eremenko/dvi/ostr6.pdf>.
- [4] A. Eremenko, *My favorite unsolved problems*, author-maintained index, <https://www.math.purdue.edu/~eremenko/uns1.html>, accessed October 11, 2026.