

Klein Quartics as Eigenschemes of the Fano Adjacency Tensor

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Abstract

For the normalized order-three adjacency tensor of the Fano plane, we identify the projective eigenscheme at each root of $\lambda^2 + \lambda + 2$ with the bicanonical Klein quartic. An explicit Fourier change of coordinates converts the seven eigen-equations into an adjugate equation for a symmetric 3×3 matrix and one additional quadric. A displayed integer polynomial identity excludes every component outside the zero-sum hyperplane. On that hyperplane the equations become the Veronese surface cut by the Klein quartic, and the adjugate identity removes possible infinitesimal thickening. This proves the identification for the whole projective eigenscheme, rather than only constructing a family of eigenvectors. A previous computational preprint reported smooth degree-eight, genus-three eigencurves at these eigenvalues; the contribution here is their explicit classical identification and a direct certificate-based proof. We do not recompute the full characteristic polynomial or its resultant multiplicities.

1 The two exceptional eigenschemes

Label the vertices of the Fano plane by $\mathbb{Z}/7\mathbb{Z}$, with lines

$$\mathcal{E} = \{ \{j, j+1, j+3\} : j \in \mathbb{Z}/7\mathbb{Z} \}.$$

We use the adjacency normalization of Cooper and Dutle [1]: the symmetric tensor entry is $1/2$ on each permutation of a line, and zero otherwise. Thus the homogeneous eigen-equations are

$$F_j(\lambda, x) = \lambda x_j^2 - \sum_{\substack{e \in \mathcal{E} \\ j \in e}} \prod_{k \in e \setminus \{j\}} x_k = 0, \quad 0 \leq j \leq 6. \quad (1)$$

For a fixed $\lambda \in \mathbb{C}$, let

$$X_\lambda = \text{Proj } \mathbb{C}[x_0, \dots, x_6] / (F_0(\lambda, x), \dots, F_6(\lambda, x)).$$

This definition retains the projective scheme structure of the eigen-equations.

Clark's question about the homogeneous adjacency spectrum appears in Cooper's October 2020 problem list [2], and as AMR-030-0030 in UnsolvedMath 1.6.0 [5]. An anonymous computational preprint [3] reports the full factorization and, at the two roots of $\lambda^2 + \lambda + 2$, smooth irreducible eigencurves of degree eight and genus three. Its proof uses saturation, Jacobian calculations and a free resolution. It explicitly leaves identification with the Klein quartic open. We credit those earlier curve assertions. Our result is the explicit identification below, proved directly from (1); no resultant factorization from [3] is a premise.

Choose a primitive seventh root of unity ζ , and put

$$q = \zeta + \zeta^2 + \zeta^4, \quad q^2 + q + 2 = 0. \quad (2)$$

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Let

$$\mathcal{K} = \{[a : b : c] \in \mathbb{P}_{\mathbb{C}}^2 : K(a, b, c) = a^3c + ab^3 + bc^3 = 0\}. \quad (3)$$

After interchanging coordinates this is the standard Klein quartic; see, for example, [4].

Theorem 1. *The morphism $\Phi_q : \mathcal{K} \rightarrow \mathbb{P}_{\mathbb{C}}^6$ with coordinates*

$$(\Phi_q)_j = a^2\zeta^j + b^2\zeta^{2j} + qbc\zeta^{3j} + c^2\zeta^{4j} + qab\zeta^{5j} + qac\zeta^{6j}, \quad 0 \leq j \leq 6, \quad (4)$$

is an isomorphism onto X_q , including its scheme structure. It is the bicanonical embedding of $\mathcal{K}$, followed by a linear embedding of $\mathbb{P}^5$ as $\sum_j x_j = 0$ in $\mathbb{P}^6$. In particular X_q is smooth and geometrically irreducible, with degree eight and genus three. Replacing ζ by ζ^3 gives the same assertions for the other root $-1 - q$.

The identification is over $\mathbb{Q}(\zeta)$, hence over $\mathbb{C}$. We make no descent assertion over the smaller field $\mathbb{Q}(q)$. The theorem concerns these two fixed eigenschemes; it does not establish the complete spectrum or explain an exponent in a characteristic resultant.

2 Fourier coordinates and a matrix equation

Set $d = 2q + 1$. Besides (2), we shall use

$$d^2 = -7, \quad q(q - 3) = -2d, \quad d(q + 2) = 3q - 2. \quad (5)$$

All the scalars $q, d, 3q - 2$ are nonzero. Make the invertible linear change

$$x_j = qh + A\zeta^j + B\zeta^{2j} + qF\zeta^{3j} + C\zeta^{4j} + qE\zeta^{5j} + qD\zeta^{6j}. \quad (6)$$

Define the following seven quadrics in h, A, B, C, D, E, F :

$$g_1 = AB - E^2 - hF, \quad g_2 = AC - D^2 - hE, \quad g_3 = BC - F^2 - hD, \quad (7)$$

$$g_4 = 2BD - 2EF + hA, \quad g_5 = 2CE - 2FD + hB, \quad g_6 = 2AF - 2ED + hC, \quad (8)$$

$$g_7 = AD + BE + CF - 2h^2. \quad (9)$$

Let $I = (g_1, \dots, g_7)$.

Lemma 2. *Under (6), the ideal generated by $F_j(q, x)$ is exactly I . Moreover $\sum_{j=0}^6 x_j = 7qh$.*

Proof. Write $\hat{F}_k = 7^{-1} \sum_j F_j(q, x) \zeta^{-jk}$. Expanding the three neighbor products at vertex zero, whose pairs are $(1, 3), (6, 2), (4, 5)$, and translating indices gives

$$(\hat{F}_0, \dots, \hat{F}_6) = (dqg_7, dqg_4, dqg_5, (3q - 2)g_1, dqg_6, (3q - 2)g_2, (3q - 2)g_3). \quad (10)$$

For example, in unrestricted Fourier coordinates $x_j = t + \sum_{k=1}^6 u_k \zeta^{jk}$, the first two coefficients are

$$(q - 3)t^2 + d(u_1u_6 + u_2u_5 + u_3u_4), \quad dtu_1 + 2du_2u_6 + (q - 3)u_3u_5.$$

Substitution of $t = qh$ and $(u_1, \dots, u_6) = (A, B, qF, C, qE, qD)$ gives the first two entries of (10), using (5); the remaining five follow by the same expansion. The discrete Fourier transform is invertible and the displayed scalar factors are nonzero. Summing (6) proves the last assertion. $\square$

Introduce symmetric matrices

$$M = \begin{pmatrix} A & E & D \\ E & B & F \\ D & F & C \end{pmatrix}, \quad L(M) = \begin{pmatrix} D & B/2 & A/2 \\ B/2 & E & C/2 \\ A/2 & C/2 & F \end{pmatrix}. \quad (11)$$

A direct cofactor expansion gives

$$\operatorname{adj}(M) - hL(M) = R := \begin{pmatrix} g_3 & -g_5/2 & -g_4/2 \\ -g_5/2 & g_2 & -g_6/2 \\ -g_4/2 & -g_6/2 & g_1 \end{pmatrix}. \quad (12)$$

Thus the first six equations are an adjugate equation.

3 Eliminating the extra coordinate

Lemma 3. *In $\mathbb{Q}[h, A, B, C, D, E, F]$, one has $h^6 \in I$. Consequently every geometric point of $\operatorname{Proj}(\mathbb{C}[h, A, B, C, D, E, F]/I)$ has $h = 0$.*

Proof. Put $G_i = g_i(1, A, B, C, D, E, F)$. Appendix A lists seven integer polynomials C_i , of degree at most four, with the identity

$$\sum_{i=1}^7 C_i G_i = 224.$$

For each C_i , form its degree-four homogenization $\tilde{C}_i = h^4 C_i(A/h, B/h, C/h, D/h, E/h, F/h)$. Homogenizing the identity to degree six gives

$$\sum_{i=1}^7 \tilde{C}_i g_i = 224h^6.$$

Since the ground field has characteristic zero, the assertion follows. The identity is an explicit coefficient certificate; obtaining it requires no assumption about a numerical root or a computed Gröbner basis. $\square$

Set $h = 0$. By (12), the first six equations are the six distinct 2×2 minors of the symmetric matrix M . Their projective zero scheme is the Veronese surface, parametrized by

$$M = vv^\top, \quad v = (a, b, c)^\top, \quad (A, B, C, D, E, F) = (a^2, b^2, c^2, ac, ab, bc). \quad (13)$$

This is also a scheme-theoretic statement: the kernel of the map sending the six matrix entries to the six quadratic monomials is generated by these minors. Alternatively, on the chart $A \neq 0$ the minors give $B = E^2/A$, $C = D^2/A$, $F = ED/A$; the charts $B \neq 0$ and $C \neq 0$ complete the same verification. These charts cover the rank-one projective locus.

The remaining equation pulls back to

$$(AD + BE + CF)|_{M=vv^\top} = a^3c + ab^3 + bc^3 = K(a, b, c).$$

It follows already that the support of X_q is the image in (4). To prove the theorem for schemes, it remains to exclude an infinitesimal thickening in the h -direction.

Lemma 4. *The plane quartic $\mathcal{K}$ is smooth over $\mathbb{C}$. At a nonzero vector v on its affine cone,*

$$ML(M)|_{M=vv^\top} = \frac{1}{2}v \nabla K(v)^\top$$

is a nonzero rank-one matrix.

Proof. The partial derivatives are

$$K_a = 3a^2c + b^3, \quad K_b = 3ab^2 + c^3, \quad K_c = a^3 + 3bc^2.$$

If one coordinate vanishes and all three derivatives vanish, the other coordinates vanish as well. If all coordinates are nonzero, multiplication of the three derivative equations would give $27a^3b^3c^3 = -a^3b^3c^3$, impossible in characteristic zero. This proves smoothness. Substitution in (11) gives $v^\top L(vv^\top) = \frac{1}{2}\nabla K(v)^\top$, which proves the matrix formula and its asserted rank. $\square$

Proposition 5. *The ideal sheaves defined by I and $I + (h)$ in $\mathbb{P}_{\mathbb{C}}^6$ are equal.*

Proof. Multiplying (12) by M , and using $M \operatorname{adj}(M) = \det(M) \operatorname{Id}_3$, gives

$$hML(M) = \det(M) \operatorname{Id}_3 - MR. \quad (14)$$

Every entry of R belongs to I . Hence, modulo I , h annihilates each off-diagonal entry of $ML(M)$ and every difference of two diagonal entries.

By Lemma 3, each closed projective point of the scheme lies at $h = 0$, where it is represented as $M = vv^\top$ for a point of $\mathcal{K}$. Lemma 4 says that $ML(M)$ there is a nonzero rank-one matrix. Such a 3×3 matrix cannot be scalar. Therefore at least one of the preceding off-diagonal entries or diagonal differences has nonzero value at the point. On any affine projective chart containing it, that polynomial, after dehomogenization, is a unit in the local ring. Equation (14) then implies $h = 0$ in that local ring. These neighborhoods cover the projective scheme, so the ideal sheaves coincide. $\square$

Proof of Theorem 1. The two preceding sections identify X_q , as a scheme, with the Veronese image of the smooth quartic $\mathcal{K}$. The coordinate formula is exactly (6) with $h = 0$ and (13). The Veronese map has no base points and is an embedding; the Fourier transformation is invertible. Thus this formula defines the stated isomorphism.

A smooth plane quartic is irreducible: distinct positive-degree components would intersect by Bézout's theorem and yield a singular point. The genus formula gives genus three, and adjunction gives $\omega_{\mathcal{K}} \simeq \mathcal{O}_{\mathcal{K}}(1)$. Consequently the six quadratic coordinates give the complete bicanonical linear system $\omega_{\mathcal{K}}^{\otimes 2} \simeq \mathcal{O}_{\mathcal{K}}(2)$, of degree eight. Its image spans $\mathbb{P}^5$, since a smooth quartic has no nonzero quadratic equation. The Fourier hyperplane is $\sum_j x_j = 0$. Finally the field automorphism $\zeta \mapsto \zeta^3$ sends q to $\zeta^3 + \zeta^5 + \zeta^6 = -1 - q$, giving the other case. $\square$

4 Scope and reproducibility

The argument proves the complete projective scheme assertion at the two specified eigenvalues directly from Fano incidence. It neither validates nor replaces the full characteristic-polynomial calculation in [3]; in particular a resultant multiplicity does not follow merely from the degree and genus of an eigencurve. There can be additional structure at the vertex of an unsaturated affine cone, and no assertion about that vertex is needed here.

The accompanying standard-library Python checker expands the original incidence equations over $\mathbb{Q}[\zeta]/(\Phi_7)$, checks the Fourier identity, verifies the integer certificate and its homogenization, and checks the matrix and Veronese substitutions. It uses exact rational arithmetic and requires no external computer algebra package. The written local-ring argument supplies the passage from those identities to the scheme-theoretic conclusion. The certificate was found using symbolic linear algebra; its validity is determined by the displayed polynomial identity.

AI assistance was used for the derivation, literature search, manuscript preparation and exact symbolic checks. This is a self-audited preprint, without a claim of independent review,

proof-assistant verification or exhaustive priority clearance. The prior computational report is credited, and the frozen source’s whole-spectrum question is not counted here as a newly solved problem.

A An integer certificate for the chart $h = 1$

For $G_i = g_i(1, A, B, C, D, E, F)$, define

$$\begin{aligned}
C_1 &= 732BCE + 1968C^2E^2 - 132C^2F - 2112CDEF \\
&\quad + 103C + 366DE, \\
C_2 &= -732B^2E - 1968BCE^2 + 132BCF + 1824BDEF \\
&\quad + 493B + 918CE + 768DE^2 - 930DF, \\
C_3 &= 288ADEF - 596A + 1836CDE - 432D^2E^2 \\
&\quad - 1992D^2F - 756E^3 + 996EF, \\
C_4 &= -366BDE - 918C^2E - 768CDE^2 + 1062CDF \\
&\quad + 912D^2EF - 484D + 378E^2F - 498F^2, \\
C_5 &= -732BE^2 + 984CE^3 + 1461D^2 - 840DE^2F + 996DF^2 + 605E, \\
C_6 &= -168D^2E + 144DEF^2 - 189E^2 - 49F, \\
C_7 &= -132CF - 288DEF - 112.
\end{aligned}$$

Straight expansion, collecting monomials in A, B, C, D, E, F , gives

$$\boxed{\sum_{i=1}^7 C_i G_i = 224}.$$

This also specifies a certificate over the integers, so its check is independent of the number-field Fourier calculation. The accompanying JSON stores $C_i/224$; the checker verifies all coefficients using rational arithmetic and rejects a changed constant term.

References

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