

Integer Decompositions for Independent Local Quotas on Forests

Alper Ferudun*

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Abstract

Let F be a finite forest. At each vertex v , choose a subset S_v of its neighbors and an integer interval $[\ell_v, u_v]$. We consider independent sets T for which $\ell_v \leq |T \cap S_v| \leq u_v$ whenever $v \notin T$. Their convex hull is described by vertex bounds, edge inequalities and the perspective inequalities $\ell_v(1 - x_v) \leq x(S_v) \leq u_v(1 - x_v)$. We give a constructive integer decomposition for every integer point in every positive integral dilation. The decomposition uses at most $3|V(F)| - 1$ distinct admissible sets, with integer multiplicities, and can be computed in time polynomial in the binary input size. The ordinary forest-kernel description is a known specialization. The result does not characterize kernel polytopes of arbitrary digraphs.

1 Local quotas and the kernel problem

Throughout, $F = (V, E)$ is a finite simple forest with $n = |V| \geq 1$. Its open neighborhood at v is denoted by $N_F(v)$. For each v , fix one subset $S_v \subseteq N_F(v)$ and integers $0 \leq \ell_v \leq u_v$. An independent set $T \subseteq V$ is *admissible* if

$$\ell_v \leq |T \cap S_v| \leq u_v \quad (v \in V \setminus T). \quad (1)$$

There is no quota condition at a selected vertex. The intervals may vary with the vertex, and S_v need not be the whole neighborhood. Write $x(S) = \sum_{w \in S} x_w$.

Define the bounded rational polyhedron

$$P = \{x \in \mathbb{R}^V : 0 \leq x_v \leq 1 \quad (v \in V), \quad (2)$$

$$x_v + x_w \leq 1 \quad (vw \in E), \quad (3)$$

$$\ell_v(1 - x_v) \leq x(S_v) \leq u_v(1 - x_v) \quad (v \in V)\}. \quad (4)$$

Every admissible incidence vector belongs to P . The converse need not hold on graphs containing cycles, even for ordinary independent domination.

A kernel of a loopless digraph is an independent absorbing set: each unselected vertex has an outgoing arc to a selected vertex. The EGRES problem [1] asks for classes admitting explicit linear descriptions of the convex hull of kernels. Chen, Chen and Zang [2] characterize digraphs for which the nonnegativity, edge and absorption system is integral on every induced subdigraph. Their Theorem 1.5 also establishes total dual integrality for that hereditary class. In particular, the ordinary

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint, version 1.0. Licensed under CC BY 4.0.

forest-kernel description is not a new consequence to be claimed here. Their characterization concerns the edge system; it is not a characterization of every possible description of arbitrary kernel polytopes. For undirected cycles, Mahjoub and Mailfert [3] give a description of the independent dominating set polytope that requires additional inequalities.

This note gives a direct construction for the vertex-dependent interval system (2)–(4), including an integer decomposition and a support bound independent of the dilation factor. The proof uses classical tree-marginal gluing; see [4, Section 2.5.2] for the general junction-tree principle. All steps needed here, including preservation of integer multiplicities, are proved below. No absolute priority claim is made.

2 The decomposition theorem

A lattice polytope Q has the *integer decomposition property* if every $a \in qQ \cap \mathbb{Z}^V$, for every integer $q \geq 1$, is a sum of q lattice points of Q .

Theorem 1. *For every forest and every choice of neighbor subsets and interval quotas above,*

$$P = \text{conv}\{\mathbf{1}_T : T \text{ is admissible}\}.$$

For each integer $q \geq 1$ and $a \in qP \cap \mathbb{Z}^V$, there exist admissible sets $T_1, \dots, T_s$ and positive integers $b_1, \dots, b_s$ such that

$$\sum_{i=1}^s b_i = q, \quad a = \sum_{i=1}^s b_i \mathbf{1}_{T_i}, \quad s \leq 3n - 1. \quad (5)$$

The weighted list in (5) is computable using polynomially many bit operations in the binary input size. In particular, P , when nonempty, is an integer-decomposition polytope.

The data q and a are written in binary. The algorithm outputs a compressed list of sets and multiplicities, not an expanded list of q sets.

2.1 Local schedules on a star

A weighted schedule on a vertex set B is a list of binary assignments $z \in \{0, 1\}^B$ with positive integer weights. Its total mass is the sum of the weights. Its marginal at $w \in B$ is the weighted number of assignments having $z_w = 1$.

Lemma 2. *Suppose $a \in qP \cap \mathbb{Z}^V$. For each v , put $B_v = \{v\} \cup N_F(v)$. There is a schedule on B_v of mass q , with marginal a_w at each $w \in B_v$, such that each assignment is independent on the star and satisfies v 's quota whenever v is absent. This schedule has at most $\deg_F(v) + 2$ entries and is computable without iterating through q positions.*

Proof. Set $m = q - a_v$. The scaled edge inequalities give $0 \leq a_w \leq m$ for every neighbor w . Allocate a_v positions to the assignment in which v is selected and every neighbor is absent. If $m = 0$, all neighboring marginals vanish and the construction is finished.

Suppose $m > 0$, and enumerate S_v as $w_1, \dots, w_d$. In the remaining m positions, indexed by residues modulo m , assign w_i to the residues of

$$\sum_{j < i} a_{w_j}, \sum_{j < i} a_{w_j} + 1, \dots, \sum_{j \leq i} a_{w_j} - 1. \quad (6)$$

Empty blocks are allowed. Since $a_{w_i} \leq m$, the residues within one block are distinct. Each w_i therefore has its prescribed marginal.

Put $A = \sum_{w \in S_v} a_w$. The concatenated blocks in (6) contain precisely the integers from 0 to $A - 1$. Hence every residue occurs either $\lfloor A/m \rfloor$ or $\lceil A/m \rceil$ times. The scaled quota inequalities say $\ell_v m \leq A \leq u_v m$. Since ℓ_v, u_v are integers, each residual position satisfies v 's quota. For each neighbor outside S_v , choose the first a_w residual positions. This does not affect that quota. Distinct neighbors are nonadjacent because F is a forest.

To compress the schedule, cut the interval $[0, m)$ at the cumulative block endpoints modulo m , and at each a_w for $w \notin S_v$. The circular blocks contribute at most d internal cut points, and the other neighbors at most $\deg(v) - d$. There are at most $\deg(v) + 1$ intervals on which the assignment is constant. Their lengths give integer weights. The central selected assignment adds at most one further entry. All endpoints are obtained by integer arithmetic and sorting. $\square$

2.2 Gluing with integer multiplicities

Lemma 3. *Let two schedules, on B and C , have the same total mass and identical masses for each binary assignment on $B \cap C$. They can be joined to a schedule on $B \cup C$ preserving both marginal schedules and integer weights. If their support sizes are r, t , and h shared patterns have positive mass, the joined schedule has at most $r + t - h$ entries.*

Proof. Group each schedule by its restriction to $B \cap C$. In one group, choose one remaining assignment from each list and join them with weight equal to the smaller of their remaining weights. Their values agree on the intersection. Subtract that weight and remove exhausted entries. The equal total masses of the groups ensure that both lists are exhausted together.

All operations preserve integer weights and both marginal lists. A group with r_j, t_j entries requires at most $r_j + t_j - 1$ joins: each join exhausts at least one entry, and the final join exhausts one from each side. Summing over the h groups proves the support bound. When the intersection is empty, there is one shared pattern, so the same argument applies. $\square$

2.3 Proof of the theorem

Proof of Theorem 1. Construct the local schedules from Lemma 2. For an edge vw , independence implies that the masses of the four patterns on its endpoints, in the order $(1, 0), (0, 1), (0, 0), (1, 1)$, are

$$a_v, \quad a_w, \quad q - a_v - a_w, \quad 0. \tag{7}$$

Thus all schedules containing that edge have identical pair marginals. Zero masses present no difficulty; no division is used.

Root each component and process its vertices in breadth-first order. When the star B_v is added after its parent's star, its intersection with all earlier bags in that component is exactly $\{p, v\}$, where p is its parent. Indeed, v has already been introduced by B_p , but every other neighbor of v is a child not adjacent to a previously processed vertex. Such an adjacency would create a cycle. The first star of a new component has empty intersection with earlier components. Consequently (7) supplies precisely the agreement needed in Lemma 3.

Successive gluing gives a schedule on V with marginals a . Each of its assignments restricts to a valid local assignment on every bag. It is therefore independent and satisfies every quota. This proves the first two identities in (5).

Let c be the number of components. The sum of the local support bounds is

$$\sum_{v \in V} (\deg(v) + 2) = 2(n - c) + 2n = 4n - 2c.$$

The n local schedules require $n - 1$ joins, including joins between components. Each join saves at least one entry relative to the sum of the two support sizes. The final support is at most $3n - 2c + 1 \leq 3n - 1$. Duplicate assignments may be merged.

All intermediate support sizes are $O(n)$. Every weight lies between 1 and q , and every assignment has at most n bits. The local arithmetic, grouping by shared patterns, and greedy mass matching consequently take polynomially many bit operations. This proves the algorithmic assertion.

Finally, each rational $x \in P$ has the form a/q and hence is a convex combination of admissible incidence vectors. Every vertex of a nonempty bounded rational polyhedron is rational, so this proves the convex-hull identity. The empty case is immediate. Repeating T_i exactly b_i times gives the usual q -summand integer decomposition. $\square$

3 Kernel specialization and boundary examples

Corollary 4. *Let D be a finite loopless digraph whose underlying simple graph is a forest. Antiparallel arcs are allowed. Its kernel polytope is described by vertex bounds, edge inequalities and*

$$x_v + x(N_D^+(v)) \geq 1 \quad (v \in V).$$

It has the integer decomposition of Theorem 1, with at most $3n - 1$ distinct kernels.

Proof. Choose $S_v = N_D^+(v)$, $\ell_v = 1$, and $u_v = \max(1, |S_v|)$. The upper quota is redundant. The admissible sets are exactly the kernels. In particular, a vertex with no outgoing arc is forced into every kernel. $\square$

As noted above, the ordinary polyhedral description in this corollary is a known special case of [2]. Theorem 1 also allows upper quotas, exact quotas $\ell_v = u_v$, zero lower quotas, and asymmetric neighbor subsets. It does not assert total dual integrality for that generalized system.

Cycles. Replace F by the bidirected cycle C_8 with ordinary kernel quotas. The constant vector $x_v = 1/3$ satisfies the edge and absorption inequalities. But every independent dominating set contains at least three vertices, since one selected vertex covers at most three vertices. Thus $\sum_v x_v = 8/3 < 3$ separates this fractional point from the kernel polytope. Equivalently, the all-one integer vector lies in the third dilation of the relaxation but cannot be the sum of three kernels. Extra cycle inequalities are classical; see [3].

Noninterval quotas. On the star with center v and leaves a, b , set $x_v = 0$ and permit either zero or two selected leaves at v . There are no restrictions at the leaves except independence. The interval-hull relaxation admits $(x_v, x_a, x_b) = (0, 1/2, 0)$. Every admissible set with center absent has equal leaf coordinates, so this point is outside the convex hull. Sets containing the center cannot contribute to a convex combination with center marginal zero.

Weighted quotas. On the same star, require the exact weighted value $2x_a + 3x_b = 3$ when the center is absent. The fractional vector $(0, 1, 1/3)$ satisfies this equality and all edge bounds. The only binary leaf solution is $(x_a, x_b) = (0, 1)$, so the vector is not a convex combination of valid assignments. Uniform cardinalities, not arbitrary weights, are used in the circular-block proof.

4 Exact verification and scope

The accompanying Python implementation uses only integer arithmetic. It computes compressed local schedules, joins equal separator patterns, and returns integer multiplicities. A separate checker tests the returned sets against independence and the original conditional quotas, then checks all marginals, total mass and the support bound.

The tests exhaust all labelled forests on at most four vertices, all nonempty edge orientations including antiparallel arcs, and all integer vectors in dilation boxes with $q \leq 4$ for ordinary kernels. They also exhaust all neighbor selectors, all intervals within $[0, |S_v| + 1]$, and dilation boxes with $q \leq 3$ on at most three vertices. There are 6,012 feasible ordinary instances and 86,142 feasible quota instances; every one is decomposed. Eighty further forest instances with very large integer weights test the compressed representation. Five corrupt witnesses and two invalid input graphs are rejected. Normal and optimized Python runs produce identical results. These tests are reproducibility checks, not a substitute for the unrestricted-in- n proof.

The theorem settles the stated forest interval-quota problem. It does not settle the open-ended EGRES question for arbitrary digraph classes, nor does it provide a characterization with unrestricted local constraints. The manuscript was developed with AI assistance for derivation, implementation, literature search and exposition; the author is responsible for its claims. It is an unrefereed preprint, not a formally verified proof or a certification of novelty.

References

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