

Independent Arborescences in Acyclic Digraphs: The Path Condition Suffices for All Roots and All Convex Sets

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October 11, 2026

Abstract

Let D be a finite acyclic digraph with root-nodes $r_1, \dots, r_k$, not necessarily distinct, and convex node sets $U_1, \dots, U_k$ with $r_i \in U_i$. The EGRES Open collection lists as open the question, proposed by S. Fujishige, N. Kamiyama and N. Katoh, whether there are independent r_i -arborescences F_i with $V(F_i) = U_i$ if and only if for every vertex v there are pairwise openly node-disjoint paths from r_i to v , one for each i with $v \in U_i$. This is Question 2 of Frank, Fujishige, Kamiyama and Katoh (Discrete Math. 313 (2013)) restricted to acyclic digraphs; their paper proves three cases and does not treat the general acyclic case. We prove that the answer is yes, for all finite acyclic digraphs (parallel arcs allowed), all roots and all convex sets, in the reading of the 2013 paper: the open node-disjointness of the EGRES problem page is used for the paths and for the root paths of the arborescences. With different notions on the two sides the equivalence fails for trivial reasons. Both conditions are equivalent to a local one, a bipartite matching problem at every vertex. The proof assigns to every vertex a real vector, chosen along a topological order by a minimum-cost assignment of the entering arcs and a strictly feasible dual solution; for two sets this is the classical ordering argument. We make no claim that the method is new: it was not found in the texts we read, among them Huck's paper of 1999 on spanning arborescences, whose proof removes one arborescence at a time; the dissertation of A. Hoyer (2019) was not accessible to us beyond its abstract. The equivalence is also proved for the weaker disjointness of Kamiyama's survey, which gives the same conditions on simple digraphs. The statement is false without convexity, and for digraphs with directed cycles (Huck 1995, known to us through the 2013 paper). No answer to the question was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. Computations, among them one that covers all instances with at most seven vertices and three sets, are tests and no part of the proof. This is an unrefereed note.

1 Introduction

The question. Let $D = (V, A)$ be an acyclic digraph with designated root-nodes $r_1, \dots, r_k \in V$, and let $U_1, \dots, U_k \subseteq V$ be convex node sets with $r_i \in U_i$. The page “Independent arborescences in acyclic digraphs” of the EGRES Open collection [EGR15] asks: is it true that there exist independent r_i -arborescences F_i with $V(F_i) = U_i$ if and only if for each $v \in V$ there exist openly node-disjoint paths P_i from r_i to v , one for each i with $v \in U_i$? The page states that the problem was proposed by S. Fujishige, N. Kamiyama and N. Katoh. It is Question 2 of Frank, Fujishige, Kamiyama and Katoh [FFKK13, p. 454], which is posed for arbitrary digraphs, restricted to acyclic digraphs. The question is listed as open on the EGRES page; the paper [FFKK13] proves three cases (its Theorems 2, 4 and 6) and does not treat the general acyclic case. The problem is record AMR-029-0044 of the UnsolvedMath collection [ula26], whose statement is that of the EGRES page.

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Let (ARB) denote the existence of the arborescences and (PC) the path condition (Section 2.1), and let (LOC) be the local condition of Definition 3.1: at every vertex the sets through it can be given pairwise different entering arcs, subject to three rules on their tails.

Theorem 1.1. *Let $D = (V, A)$ be a finite acyclic digraph, in which parallel arcs are allowed, let $r_1, \dots, r_k \in V$ be not necessarily distinct, and let $U_1, \dots, U_k \subseteq V$ be convex sets with $r_i \in U_i$. Then (ARB), (PC) and (LOC) are equivalent.*

Corollary 1.2. *The answer to the question of [EGR15] is yes: in an acyclic digraph, independent r_i -arborescences F_i with $V(F_i) = U_i$ exist if and only if for every vertex v there are pairwise openly node-disjoint paths from r_i to v , one for each i with $v \in U_i$. Here open node-disjointness is that of the EGRES problem page, both for the paths and for the root paths of the arborescences, and a path without arcs ($v = r_i$) imposes no condition.*

Reading, credits and limits. The corollary is stated in the reading of [FFKK13], where one notion of disjointness is used on both sides. The EGRES page “Arborescence”, to which the problem page refers for “independent”, says “internally node-disjoint” without defining it; with different notions on the two sides the equivalence fails for trivial reasons (Section 2.2). Known special cases are the two acyclic cases of [FFKK13] and the two theorems of Huck [Huc99]: one root and spanning arborescences, and pairwise distinct roots with arborescences that contain all vertices except the other roots (Sections 2.3 and 5). The dissertation [Hoy19] was not accessible to us beyond its abstract. No answer to the question was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and no priority is claimed (see “Scope and priority” at the end).

The method. The proofs we read [Huc99, FFKK13, Kam14, BF09, BK11] order the vertices so that one arborescence runs upwards and everything else downwards, and remove one arborescence at a time. Here every vertex v receives a vector $f(v)$ with one coordinate for each set containing v , such that $f_i - f_j$ increases along F_i and decreases along F_j inside $U_i \cap U_j$, for all pairs i, j ; the vectors are chosen along a topological order, each as a strictly feasible dual solution of a minimum-cost assignment of the entering arcs. We make no claim that the method is new: it was not found in the texts we read, for two sets it is the classical ordering argument, and the dissertation [Hoy19], of which we know only the abstract, may contain related ideas.

2 The statement

2.1 Objects and definitions

Throughout, $D = (V, A)$ is a finite digraph without loops, parallel arcs being allowed as in [FFKK13, p. 453], and D is *acyclic*. Then every walk is a path and every closed walk is *trivial* (has no arc), so the walks of [FFKK13] and the paths of [EGR15] are the same objects. $\Gamma^-(v)$ is the set of arcs entering v , and $x \rightsquigarrow y$ means that there is a path from x to y ($x \rightsquigarrow x$ holds). As D is acyclic, $x \rightsquigarrow y \rightsquigarrow x$ implies $x = y$, and V has an enumeration $v_1, \dots, v_n$ in which the tail of every arc comes before its head; we fix one, and “earlier”, “later” refer to it.

The *root-nodes* $r_1, \dots, r_k \in V$ need not be distinct [FFKK13, Question 2]. A set $U \subseteq V$ is *convex* if it satisfies (C): $x, z \in U$ and $x \rightsquigarrow y \rightsquigarrow z$ imply $y \in U$. This is the definition of [FFKK13, p. 453] (the intermediate vertices of every walk between two vertices of U lie in U) and of the EGRES page “Convex node set” (no node outside U is reachable from U and reaches U). The sets $U_1, \dots, U_k$ are convex and $r_i \in U_i$; nothing else is assumed: they may coincide, be nested or consist of the root alone, and a root may lie in other sets. For $v \in V$ let $I(v) := \{i : v \in U_i\}$ and $J(v) := \{i \in I(v) : v \neq r_i\}$.

An r -arborescence is a subgraph F of D containing r such that no arc of F enters r and exactly one arc of F enters every other vertex of F [FFKK13, p. 454]; the EGRES page “Arborescence” describes the same subgraphs. Going backwards from a vertex v of F along arcs of F never repeats a vertex, as D is acyclic, and can stop only at r ; so F contains exactly one path from r to v , the *root path* rFv .

Definition 2.1. Let $i \neq j$, let P_i be a path from r_i to v and P_j a path from r_j to v . They are *openly node-disjoint* if they have no arc in common and $V(P_i) \cap V(P_j) = \{v\} \cup (\{r_i\} \cap \{r_j\})$. Arborescences $F_1, \dots, F_k$, where F_i is an r_i -arborescence, are *independent* if for all $i \neq j$ and all $v \in V(F_i) \cap V(F_j)$ the root paths $r_i F_i v$ and $r_j F_j v$ are openly node-disjoint.

The first part is the definition of the EGRES problem page: the paths meet only in v , and in their first vertex if $r_i = r_j$. The second part is our reading of “independent” (Section 2.2). The two conditions of the question are:

(ARB) there are independent r_i -arborescences F_i ($i = 1, \dots, k$) with $V(F_i) = U_i$;

(PC) for every $v \in V$ there are paths P_i ($i \in I(v)$), where P_i goes from r_i to v , which are pairwise openly node-disjoint.

If $v = r_i$, then P_i is the trivial path, which is openly node-disjoint from every other path to v ; so at v only the indices of $J(v)$ matter. Reachability is not assumed: if some vertex of U_i is not reachable from r_i , then (ARB) and (PC) both fail. A vertex in no set imposes nothing.

2.2 The reading

For the word “independent” the problem page links to the EGRES page “Arborescence”, by which two arborescences are independent if the root paths to every common node are “internally node-disjoint”; this term is not defined there. In [FFKK13] one notion, internal disjointness of walks, is used on both sides of Question 2, and the “only if” part is called immediate. We read the question in the same way, with the open node-disjointness of Definition 2.1 on both sides.

Edge-disjoint or distinct. In [FFKK13, p. 453], distinct walks P_i from u_i to v_i with vertex sets W_i are internally disjoint if $W_i \cap W_j = (\{u_i\} \cap \{u_j\}) \cup (\{v_i\} \cap \{v_j\})$ for $i \neq j$, without a condition on arcs.

Lemma 2.2. *Let P be a path from r_i to v and Q a path from r_j to v in the acyclic digraph D with $V(P) \cap V(Q) = \{v\} \cup (\{r_i\} \cap \{r_j\})$. Then P and Q have a common arc if and only if $P = Q$ and this path is not trivial.*

Proof. Let e be a common arc, from x to y . Then $x \neq y$ are common vertices, so $\{x, y\} = \{v, r\}$ with $r := r_i = r_j \neq v$. As $r \rightsquigarrow v$ and D is acyclic, $x = r$ and $y = v$. The part of P before e is a closed walk at r , and the part after e a closed walk at v ; both are trivial, so P consists of e alone, and so does Q . The converse is clear. $\square$

Hence, given the condition on vertices, “no common arc” [EGR15] and “distinct” [FFKK13] mean the same unless $P = Q$ is trivial, that is, $v = r_i = r_j$: Definition 2.1 accepts two trivial paths at a common root, the literal wording of [FFKK13] does not, and read literally it would make (ARB) and (PC) both false as soon as two roots coincide. The paper [FFKK13] does not use it so (its condition (1) concerns $v \neq r$, and the proof of its Theorem 6 works with $J(v)$); we follow Definition 2.1.

Passing through another root. Kamiyama’s survey [Kam14, p. 64] asks only that the interiors of two arc-disjoint paths are disjoint, so that a path may pass through the root of another one. This *weak* open disjointness is defined in Section 7, where the equivalence is proved for the weak notion on both sides as well; on digraphs without parallel arcs nothing changes.

Mixed readings of the two sides fail for trivial reasons:

Remark 2.3 (two examples). (a) Let D consist of one arc from r to v , and let $r_1 = r_2 = r$, $U_1 = U_2 = \{r, v\}$. If independence is read as a condition on vertices only, then $F_1 = F_2 = D$ are independent; but (PC) fails, as it asks for two paths from r to v without a common arc.

(b) Let D have the vertices r_1, r_2, v , an arc $r_1 r_2$ and two parallel arcs e, e' from r_2 to v , and let $U_1 = \{r_1, r_2, v\}$, $U_2 = \{r_2, v\}$; both sets are convex. The arborescence F_1 with the arcs $r_1 r_2, e$ and the arborescence F_2 with the arc e' have no common arc, and the interiors of their root paths are disjoint; so they are independent in the weak sense, and the weak path condition holds too. But every path from r_1 to v passes through $r_2 \neq v$, so (PC) fails at v , and with it (ARB). Thus “only if” fails for weak independence with (PC), and “if” for (ARB) with the weak path condition.

Under every reading the following holds: Theorem 1.1 derives from (PC) arborescences that are independent in the strongest of these senses, hence in every weaker one, and the converse holds whenever both sides use the same notion. If the condition on arcs is dropped on both sides, the equivalence remains true: replace every arc of D by k parallel copies, let the path P_i use the i -th copy of each of its arcs, apply Theorem 1.1, and replace the copies in the arborescences by the original arcs; the convex sets and the vertex sets of all paths do not change.

2.3 What is known

By [FFKK13, Theorems 2, 4 and 6], (ARB) and (PC) are equivalent (1) for an arbitrary digraph, if all roots are equal and every vertex other than the root lies in at most two of the sets; (2) if D is acyclic and all roots are equal; (3) if D is acyclic and every vertex lies in at most two of the sets. For $U_1 = \dots = U_k = V$, case (2) is Theorem 1 of Huck [Huc99]; his Theorem 2 treats pairwise distinct roots in a simple acyclic digraph and arborescences that contain all vertices except the other roots (Section 5). For one root and spanning arborescences in an arbitrary digraph the equivalence holds for $k \leq 2$ by a theorem of Whitty [Whi87] and fails for every $k \geq 3$ by counterexamples of Huck [Huc95], which have directed cycles; we take both facts from [Huc99, p. 246], [FFKK13, p. 454] and [BF09, p. 24]. Kamiyama’s survey [Kam14, Theorem 3.9] lists the cases (1)–(3) and no other (its remark on p. 65 that the equivalence fails when a vertex lies in three sets refers to Huck’s counterexamples). So what these sources leave open is the case of acyclic digraphs with at least two different roots and a vertex in at least three of the sets, apart from the situation of [Huc99, Theorem 2], which is that of [BF09, Theorem 5.1] (Section 5). We call an instance a *three-set instance* if for some vertex v the set $J(v)$ has at least three elements whose roots are not all equal; such instances fall under neither (2) nor (3).

3 The local condition and the proof

3.1 Admissible assignments

Definition 3.1. An *admissible assignment* at a vertex v is a family of arcs $a_i \in \Gamma^-(v)$ ($i \in J(v)$), with tails t_i , such that

(A1) $t_i \in U_i$ for every $i \in J(v)$;

(A2) $a_i \neq a_j$ for $i \neq j$, and if $t_i = t_j$ for some $i \neq j$, then $t_i = r_i = r_j$;

(A3) $t_i \neq r_j$ for all $i, j \in J(v)$ with $r_j \neq r_i$.

(LOC) says that every vertex has an admissible assignment (the empty family if $J(v) = \emptyset$).

Remark 3.2 ((LOC) is a matching condition). Let $R(v) := \{r_j : j \in J(v)\}$. By (A3) a tail $u \in R(v)$ can be used only by indices i with $r_i = u$, each with its own arc from u to v by (A2); a tail $u \notin R(v)$

can be used by at most one index, by (A2), and only by an index i with $u \in U_i$, by (A1); and every choice of arcs obeying these two rules satisfies (A1)–(A3). So the admissible assignments at v correspond to the matchings covering $J(v)$ in a bipartite graph in which $i \in J(v)$ is joined to every arc from r_i to v and to one node for each in-neighbour $u \notin R(v)$ of v with $u \in U_i$ (an assignment uses at most one arc from such a u , and it does not matter which). So (LOC), hence by Theorem 1.1 also (ARB) and (PC), is decided by $|V|$ bipartite matching problems.

3.2 The two easy implications

Lemma 3.3. (ARB) implies (PC), and (PC) implies (LOC).

Proof. Let $F_1, \dots, F_k$ be as in (ARB). For $v \in V$ and $i \in I(v)$ put $P_i := r_i F_i v$, which is defined as $v \in U_i = V(F_i)$. For $i \neq j$ in $I(v)$ we have $v \in V(F_i) \cap V(F_j)$, so P_i and P_j are openly node-disjoint by independence. This is (PC).

Assume (PC), let $v \in V$, and let P_i ($i \in I(v)$) be paths as in (PC). For $i \in J(v)$ the path P_i is not trivial; let $a_i \in \Gamma^-(v)$ be its last arc and $t_i \neq v$ the tail of a_i . (A1): $r_i \rightsquigarrow t_i \rightsquigarrow v$ and $r_i, v \in U_i$, so $t_i \in U_i$ by (C). (A2): P_i and P_j have no common arc, so $a_i \neq a_j$; and if $t_i = t_j$, this vertex lies in $V(P_i) \cap V(P_j) = \{v\} \cup (\{r_i\} \cap \{r_j\})$ and is not v , so it equals $r_i = r_j$. (A3): let $i, j \in J(v)$ with $r_j \neq r_i$ and suppose that $t_i = r_j$. Then r_j lies on P_i and on P_j , so $r_j \in \{v\} \cup (\{r_i\} \cap \{r_j\}) = \{v\}$, contradicting $j \in J(v)$. $\square$

3.3 Potentials

It remains to derive (ARB) from (LOC). A *potential* f assigns to every vertex v a vector $f(v) = (f_i(v))_{i \in I(v)} \in \mathbb{R}^{I(v)}$; a *partial potential* is defined only on $v_1, \dots, v_p$ for some p .

A *cyclic configuration* is a tuple $\kappa = (w; i_1, \dots, i_m; u_1, \dots, u_m)$ with $m \geq 2$, where $w \in V$; $i_1, \dots, i_m$ are pairwise distinct elements of $J(w)$; $u_1, \dots, u_m$ are pairwise distinct vertices, each of them the tail of some arc entering w ; and $u_s \in U_{i_s} \cap U_{i_{s+1}}$ for $s = 1, \dots, m$, where $i_{m+1} := i_1$. Its *value* under f is

$$\text{val}_f(\kappa) := \sum_{s=1}^m (f_{i_{s+1}}(u_s) - f_{i_s}(u_s)).$$

There are finitely many cyclic configurations, and they are determined by D and the sets U_i . A potential or partial potential f is *generic* if $\text{val}_f(\kappa) \neq 0$ for every cyclic configuration κ all of whose vertices $u_1, \dots, u_m$ lie in the domain of f . The vertex w of κ need not lie in the domain: the condition concerns every w , in particular all later ones.

Lemma 3.4 (one vertex). *Let $v = v_p$, let f be a generic partial potential on $v_1, \dots, v_{p-1}$, and let $(a_i)_{i \in J(v)}$, with tails t_i , be an admissible assignment at v whose cost $\sum_{i \in J(v)} f_i(t_i)$ is minimum among all admissible assignments at v . Then the set X of all $x \in \mathbb{R}^{I(v)}$ with*

$$x_i - x_j > f_i(t_i) - f_j(t_i) \quad \text{for all } i \neq j \text{ in } J(v) \text{ with } t_i \neq r_i \text{ and } t_i \in U_j \quad (1)$$

is open and non-empty.

The cost and the right-hand sides are defined: t_i is earlier than v , $t_i \in U_i$ by (A1), and $t_i \in U_j$ in (1). An assignment of minimum cost need not be unique (assignments with the same tails, which exist when there are parallel arcs, have the same cost); the lemma holds for each of them, and X depends on the assignment only through its tails.

Proof. X is open, being defined by finitely many strict linear inequalities. Let Δ be the digraph with vertex set $J(v)$ and an arc (i, j) for all $i \neq j$ with $t_i \neq r_i$ and $t_i \in U_j$; give this arc the length $\ell(i, j) := f_j(t_i) - f_i(t_i)$. Then (1) reads $x_j - x_i < \ell(i, j)$ for every arc (i, j) of Δ .

(i) If $i \in J(v)$ and $t_i \neq r_i$, then $t_i \neq r_l$ for every $l \in J(v)$, and $t_l \neq t_i$ for every $l \in J(v) \setminus \{i\}$. Indeed, $t_i \neq r_l$ holds by (A3) if $r_l \neq r_i$ and by assumption if $r_l = r_i$; and $t_l = t_i$ with $l \neq i$ would give $t_i = r_i$ by (A2).

(ii) Every directed cycle of Δ has positive length. Let $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_m \rightarrow i_1$ be a cycle of Δ with pairwise distinct $i_1, \dots, i_m$; then $m \geq 2$, as Δ has no loops, and indices are read modulo m . An arc of Δ leaves every i_s , so $t_{i_s} \neq r_{i_s}$, and (i) applies to every i_s . Define a new family by $a'_{i_{s+1}} := a_{i_s}$ for $s = 1, \dots, m$ and $a'_l := a_l$ for the other $l \in J(v)$. It is admissible:

(A1): the tail of $a'_{i_{s+1}}$ is t_{i_s} , which lies in $U_{i_{s+1}}$ because (i_s, i_{s+1}) is an arc of Δ .

(A2): the arcs a'_l are the arcs a_l in another order, hence pairwise different. Suppose that $l \neq l'$ have the same tail under a' . If one of them, say $l = i_{s+1}$, lies on the cycle, this tail is t_{i_s} , which by (i) is the tail of no arc a_q with $q \neq i_s$; but $a'_{l'}$ is such an arc. So neither l nor l' is on the cycle; there $a' = a$, and (A2) holds for a .

(A3): for l outside the cycle neither its tail nor any root has changed; and the tail t_{i_s} of $a'_{i_{s+1}}$ is not the root of any index of $J(v)$, by (i).

The cost of a' minus the cost of a is

$$\sum_{s=1}^m (f_{i_{s+1}}(t_{i_s}) - f_{i_{s+1}}(t_{i_{s+1}})) = \sum_{s=1}^m (f_{i_{s+1}}(t_{i_s}) - f_{i_s}(t_{i_s})) = \sum_{s=1}^m \ell(i_s, i_{s+1}),$$

because $\sum_s f_{i_{s+1}}(t_{i_{s+1}})$ and $\sum_s f_{i_s}(t_{i_s})$ are the same cyclic sum. This is the length of the cycle, and it is ≥ 0 since a has minimum cost. Moreover $\kappa := (v; i_1, \dots, i_m; t_{i_1}, \dots, t_{i_m})$ is a cyclic configuration: the t_{i_s} are pairwise distinct by (i), they are tails of arcs entering v , and $t_{i_s} \in U_{i_s} \cap U_{i_{s+1}}$ by (A1) and the definition of Δ . Its vertices are earlier than v , and $\text{val}_f(\kappa)$ is the length of the cycle. As f is generic, this length is not 0; hence it is positive.

(iii) A point of X . If Δ has no directed cycle, put $\varepsilon := 1$; otherwise let $\mu > 0$ be the minimum length of a directed cycle of Δ and $\varepsilon := \mu/|J(v)|$. For the lengths $\ell' := \ell - \varepsilon$ every directed cycle, having at most $|J(v)|$ arcs, has length at least $\mu - |J(v)|\varepsilon = 0$; hence every closed walk has non-negative ℓ' -length, and for $j \in J(v)$ the minimum x_j of the ℓ' -lengths of all directed walks in Δ ending in j is well defined (the walk without arcs has length 0, and the minimum is attained by a path). For an arc (i, j) of Δ , appending it to a shortest walk ending in i gives $x_j \leq x_i + \ell(i, j) - \varepsilon < x_i + \ell(i, j)$. The coordinates x_i with $i \in I(v) \setminus J(v)$ are arbitrary. $\square$

Lemma 3.5 (keeping the potential generic). *Let $v = v_p$, let f be a generic partial potential on $v_1, \dots, v_{p-1}$, and let $X \subseteq \mathbb{R}^{I(v)}$ be open and non-empty. Then there is $x \in X$ such that the extension of f by $f(v) := x$ is generic.*

Proof. Let $\kappa = (w; i_1, \dots, i_m; u_1, \dots, u_m)$ be a cyclic configuration with all u_s among $v_1, \dots, v_p$. If v is not among them, then $\text{val}(\kappa) \neq 0$ by assumption. Otherwise $v = u_s$ for exactly one s , the u_s being pairwise distinct, and $\text{val}(\kappa) = x_{i_{s+1}} - x_{i_s} + c_\kappa$, where c_κ depends only on the values of f at earlier vertices. Here $i_s \neq i_{s+1}$, as $m \geq 2$, and both indices lie in $I(v)$, as $u_s \in U_{i_s} \cap U_{i_{s+1}}$. So $\text{val}(\kappa) = 0$ is the equation of an affine hyperplane of $\mathbb{R}^{I(v)}$. A non-empty open set is not covered by finitely many hyperplanes; choose $x \in X$ outside all of them. $\square$

The construction. Assume (LOC) and treat $v = v_1, \dots, v_n$ in this order. Suppose that a generic partial potential f on the earlier vertices has been constructed; for v_1 it is empty. By (LOC) there are admissible assignments at v , finitely many; let $a^v = (a_i^v)_{i \in J(v)}$, with tails $t_i(v)$, be one of minimum cost. Lemma 3.4 provides the open non-empty set X (which is $\mathbb{R}^{I(v)}$ if $J(v) = \emptyset$), and Lemma 3.5

a point $f(v) \in X$ for which the extended potential is generic. The genericity secured at this step concerns all cyclic configurations in which v is the latest of the vertices u_s , whatever their vertex w is; and Lemma 3.4, applied later at a vertex w , uses exactly the configurations $(w; \dots)$, whose vertices u_s are tails of arcs entering w and have all been treated by then. At the end we have admissible assignments a^v and a potential f such that

$$(P) \quad f_i(v) - f_j(v) > f_i(t_i(v)) - f_j(t_i(v)) \quad \text{for } v \in V \text{ and } i \neq j \text{ in } J(v) \text{ with } t_i(v) \neq r_i, t_i(v) \in U_j.$$

For $i = 1, \dots, k$ let F_i be the subgraph of D with vertex set U_i and arc set $\{a_i^v : v \in U_i, v \neq r_i\}$.

Lemma 3.6. *F_i is an r_i -arborescence with $V(F_i) = U_i$. In particular $r_i \rightsquigarrow v$ for every $v \in U_i$.*

Proof. For $v \in U_i \setminus \{r_i\}$ we have $i \in J(v)$, and a_i^v enters v and has its tail in U_i by (A1). So F_i is a subgraph of D with vertex set U_i . Arcs chosen at different vertices have different heads; hence exactly one arc of F_i enters each $v \in U_i \setminus \{r_i\}$, and none enters r_i . $\square$

Lemma 3.7. *The arborescences $F_1, \dots, F_k$ are independent.*

Proof. Let $i \neq j$ and $v \in U_i \cap U_j$; put $P := r_i F_i v$, $Q := r_j F_j v$, and let $g := f_i - f_j$, a function on $U_i \cap U_j$. We first record two consequences of (P):

$$\begin{aligned} &\text{if } (u, y) \text{ is an arc of } F_i \text{ with } u, y \in U_j, u \neq r_i \text{ and } y \neq r_j, \text{ then } g(y) > g(u); \\ &\text{if } (u, y) \text{ is an arc of } F_j \text{ with } u, y \in U_i, u \neq r_j \text{ and } y \neq r_i, \text{ then } g(y) < g(u). \end{aligned} \tag{2}$$

For the first line: $y \in U_i$ and $y \neq r_i$, because an arc of F_i enters y ; so $i, j \in J(y)$, and $u = t_i(y)$ satisfies $u \neq r_i$ and $u \in U_j$; now (P) at y for the pair (i, j) is $g(y) > g(u)$. The second line is (P) at y for the pair (j, i) .

If $v = r_i$ or $v = r_j$, one of the two paths is trivial and there is nothing to prove. Let $w \neq v$ be a common vertex of P and Q ; then $w \in U_i \cap U_j$. We show that $w = r_i = r_j$.

Case a: $w \neq r_i$ and $w \neq r_j$. Let $w = x_0, x_1, \dots, x_p = v$ be the part of P from w to v and $w = y_0, y_1, \dots, y_q = v$ the part of Q from w to v ; here $p, q \geq 1$. Every x_s lies in U_i , being a vertex of F_i , and in U_j by (C), as $w \rightsquigarrow x_s \rightsquigarrow v$ and $w, v \in U_j$. For $s \geq 1$ we have $x_{s-1} \neq r_i$, since $x_0 = w \neq r_i$ and, for $s \geq 2$, an arc of F_i enters x_{s-1} ; and $x_s \neq r_j$, since $r_j \rightsquigarrow w$ by Lemma 3.6 for F_j and $w \rightsquigarrow x_s$ by a path with at least one arc, so that $x_s = r_j$ would give a directed cycle. By (2), $g(x_s) > g(x_{s-1})$ for $s = 1, \dots, p$, hence $g(v) > g(w)$. In the same way every y_s lies in U_j and, by (C) for U_i , in U_i ; $y_{s-1} \neq r_j$; and $y_s \neq r_i$, since $r_i \rightsquigarrow w$ by Lemma 3.6 for F_i . By (2), $g(y_s) < g(y_{s-1})$ for $s = 1, \dots, q$, hence $g(v) < g(w)$, a contradiction.

Case b: $w = r_i \neq r_j$. As $w \neq v$, the vertex r_i has a successor b on Q , and (r_i, b) is the arc of F_j entering b . Hence $b \in U_j$ and $b \neq r_j$, so $j \in J(b)$, and $t_j(b) = r_i$. Moreover $r_i \rightsquigarrow b \rightsquigarrow v$ and $r_i, v \in U_i$, so $b \in U_i$ by (C), and $b \neq r_i$; thus $i \in J(b)$. But (A3) at b for the pair (j, i) says $t_j(b) \neq r_i$, because $r_i \neq r_j$. Contradiction.

Case c: $w = r_j \neq r_i$. This is case b with i and j exchanged.

The cases are exhaustive, so $V(P) \cap V(Q) \subseteq \{v\} \cup (\{r_i\} \cap \{r_j\})$, and the other inclusion is clear. Finally, a common arc of P and Q has its two different ends in this set; so $r_i = r_j =: r \neq v$ and the arc joins r and v . It goes from r to v , as $r \rightsquigarrow v$ and D is acyclic, and is therefore the last arc of P and of Q : $a_i^v = a_j^v$ with $i \neq j$ in $J(v)$, contradicting (A2). So P and Q are openly node-disjoint. $\square$

Proof of Theorem 1.1 and Corollary 1.2. Lemma 3.3 gives $(ARB) \Rightarrow (PC) \Rightarrow (LOC)$, and the construction with Lemmas 3.6 and 3.7 gives $(LOC) \Rightarrow (ARB)$. The corollary is the equivalence of (ARB) and (PC) . $\square$

Remark 3.8. (a) Lemmas 3.6 and 3.7 use only that the assignments a^v are admissible and that (P) holds. So a family of admissible assignments with a potential satisfying (P) is a certificate for (ARB) that is verified by finitely many comparisons; minimality of the cost and genericity serve only to find one (Section 6).

(b) If some directed cycle of Δ has length ≤ 0 , then $X = \emptyset$ (add the inequalities (1) along the cycle). So the conclusion of Lemma 3.4 fails for an admissible assignment that has a rotation, as in step (ii), of smaller or of equal cost; genericity excludes equal cost.

4 The hypotheses

Convexity is used for (A1) in Lemma 3.3 and in the proof of Lemma 3.7 (in case a for U_j and for U_i , and in cases b and c). Without it the statement is false already for $k = 1$: let D be the path $r \rightarrow x \rightarrow v$ and $U_1 = \{r, v\}$. Then (PC) holds, but no r -arborescence has the vertex set $\{r, v\}$, as there is no arc from r to v . This instance is minimal, since on two vertices every set is convex. Convexity is also needed beyond (A1): for the arcs 02, 12, 13, 23, 24, 34 on the vertices $0, \dots, 4$ and $r_1 = 1$, $U_1 = \{1, 2, 4\}$, $r_2 = 0$, $U_2 = \{0, 2, 3, 4\}$, the set U_1 is not convex, (PC) and (LOC) hold, and (ARB) fails (both root paths of 4 pass through 2).

Acyclicity is used for the enumeration along which the potential is built, in Lemma 3.6, and in Lemma 3.7 (case a and the direction of a common arc); with directed cycles the equivalence of (ARB) and (PC) fails (Section 2.3). *Finiteness* is used for the induction and in Lemma 3.5. Nothing is assumed about the number of sets through a vertex, the roots, simplicity of D , or reachability, which follows from (LOC) by Lemma 3.6.

The path condition at every vertex. In the Steiner version of [BK11, §2] the arborescences need only reach given terminals, and disjoint paths are required only to the terminals; [BK11, Theorem 2.8] gives an acyclic digraph with three internally node-disjoint paths from the root to each of two terminals and without three edge-independent Steiner arborescences (by a figure, which we did not examine). This does not contradict Theorem 1.1: a convex set containing the root and a terminal contains every vertex on a path between them, and (PC) is required at each of these vertices.

5 The known cases as special cases

One root [FFKK13, Theorem 4]. Let $r_1 = \dots = r_k =: r$; then (A3) is void. The proof in [FFKK13] assumes, without loss of generality, that every vertex lies in some set and that all parallel arcs leave r (its Assumptions 1 and 2), and reduces the path condition to its condition (1): $|\Gamma^-(v)| \geq |I(v)|$ for all $v \neq r$. Under these assumptions (1) is equivalent to (LOC): (LOC) gives $|I(v)|$ different arcs entering v ; conversely, (1) makes every vertex reachable from r , so every tail of an arc entering v lies in every U_i containing v , by (C), and any $|I(v)|$ arcs entering v form an admissible assignment (parallel arcs leave r).

Huck's theorems [Huc99]. In Huck's paper digraphs are finite and may have multiple edges, branchings are directed towards their root, and two edge-disjoint paths are *openly disjoint* if every common vertex is an end of both paths, that is, if $V(P_1) \cap V(P_2) \subseteq \{x_1, y_1\} \cap \{x_2, y_2\}$ for paths P_i from x_i to y_i [Huc99, p. 245]. For two paths from a vertex x to roots t_i and t_j this is Definition 2.1 in the reversed digraph, as $\{x, t_i\} \cap \{x, t_j\} = \{x\} \cup (\{t_i\} \cap \{t_j\})$. Theorem 1 of [Huc99] says: if D is acyclic and every vertex other than t has n pairwise openly disjoint paths to t , then D has n pairwise independent spanning t -branchings. For the reversed digraph this is the implication from (PC) to (ARB) of Theorem 1.1 for one root and $U_1 = \dots = U_k = V$, parallel arcs included. It is restated for simple digraphs in [BF09, Theorem 5.3] (a simple acyclic digraph has k independent spanning arborescences rooted at r if and only if it is rooted k -node-connected) and for acyclic digraphs D with $D - r$ simple

in [BK11, Theorem 1.2(ii)]. So for one root and spanning arborescences the admission of parallel arcs is not new; it is also made in [FFKK13, Theorem 4].

Several roots [Huc99, Theorem 2]. Huck derives his Theorem 1, by splitting the root, from his Theorem 2: if D is a simple acyclic digraph, $t_1, \dots, t_n$ are pairwise distinct vertices and every other vertex has out-degree at least n , then there are pairwise independent t_i -branchings B_i with $V(B_i) = V(D) \setminus \{t_j : j \neq i\}$. No edge leaving one of the vertices t_j lies in a branching, so these edges may be deleted; in the reversed digraph the roots are then the sources, and the statement is [BF09, Theorem 5.1], attributed to Huck there. Let D be simple and acyclic with the set $S = \{s_1, \dots, s_k\}$ of sources, and let every vertex of $U := V \setminus S$ have in-degree at least k ; then there are independent s_i -arborescences F_i with $V(F_i) = U \cup \{s_i\}$. This is Theorem 1.1 for $r_i = s_i$ and $U_i = U \cup \{s_i\}$: these sets are convex, as no arc enters a source, and (LOC) holds, since by (A1) and (A3) a source s_j entering $u \in U$ can serve the index j only, while every other in-neighbour of u can serve every index, so that k in-neighbours suffice. Huck proves Theorem 2 by induction on n : his Lemma 1 gives a t_n -branching B_n on the vertices other than $t_1, \dots, t_{n-1}$ and a numbering of the vertices other than $t_1, \dots, t_n$ that is topological for $B_n - t_n$ and, in reversed order, for the digraph on these vertices that remains when the edges of B_n are deleted. Convex sets, roots that lie in other sets, and coinciding roots together with different ones do not occur in [Huc99].

At most two sets per vertex [FFKK13, Theorem 6]. (LOC) implies the conditions (4) and (5) of the proof in [FFKK13]: $|\Gamma^-(v)| \geq |J(v)|$, and for every $i \in J(v)$ some arc entering v has its tail in U_i . That proof derives (ARB) from (4) and (5) under its Assumptions 3 and 4. So in this case, as for one root, a local condition equivalent to the path condition is in [FFKK13]. In our proof, for $J(v) = \{i, j\}$ and $g = f_i - f_j$, condition (1) reads $g(t_i) < g(v) < g(t_j)$ (each inequality only if the tail in question lies in $U_i \cap U_j$ and is not the root of its index): g orders $U_i \cap U_j$ so that F_i runs upwards and F_j downwards, as in the proofs for two arborescences in [FFKK13, BF09, BK11]. For two sets the method is not new.

6 A worked example

Let D have the vertices $0, \dots, 9$ and the arcs given by the in-neighbours in Table 1; the vertices $0, 1, 2$ are sources. Let $r_1 = 0$, $r_2 = 1$, $r_3 = 2$, and let $U_1 = \{0, 3, 4, 5, 6, 7, 9\}$, $U_2 = \{1, 3, 4, 5, 6, 7, 8, 9\}$, $U_3 = \{2, 3, 5, 6, 7, 9\}$. The sets are convex, the roots are different and the vertices $3, 5, 6, 7, 9$ lie in all three sets; so this is a three-set instance, and it is not of the type of [BF09, Theorem 5.1], as U_1 and U_3 omit vertices that are not sources.

v	in-neighbours	$I(v)$	tails of the arcs of F_1, F_2, F_3 entering v	$(f_1(v), f_2(v), f_3(v))$
3	0, 1, 2	1, 2, 3	0, 1, 2	(1, 0, -2)
4	0, 1	1, 2	0, 1, -	(2, 0, -)
5	0, 2, 4	1, 2, 3	0, 4, 2	(-2, 0, 0)
6	3, 4, 5	1, 2, 3	5, 4, 3	(0, 0, 1)
7	3, 4, 6	1, 2, 3	6, 4, 3	(1, 0, -1)
8	6	2	-, 6, -	(-, 0, -)
9	5, 6, 7	1, 2, 3	5, 6, 7	(-1, 0, 0)

Table 1: The worked example with a certificate in the sense of Remark 3.8(a); $f_i(r_i) = 0$, and a dash means that v is not in the set.

The last two columns of Table 1 are a certificate: the assignment in each row satisfies (A1)–(A3), and (P) holds. For instance, at $v = 9$ the tails 5, 6, 7 lie in all three sets, and the six inequalities of (P),

written as “value at 9 > value at the tail”, are $-1 > -2$ for $f_1 - f_2$ and for $f_1 - f_3$ (tail 5); $1 > 0$ for $f_2 - f_1$ and $0 > -1$ for $f_2 - f_3$ (tail 6); $1 > -2$ for $f_3 - f_1$ and $0 > -1$ for $f_3 - f_2$ (tail 7). The root paths to 9 are 0, 5, 9; 1, 4, 6, 9; 2, 3, 7, 9. By Remark 3.8(a) the three arborescences are independent. By enumeration, the instance has exactly 6 families of independent arborescences, and of the 5 families on the vertices 0, ..., 8 two cannot be extended to vertex 9: choosing the entering arcs vertex by vertex without the costs can get stuck. For the potential of the table the listed assignment is the only admissible assignment of minimum cost at every vertex (at vertex 9 the six admissible assignments cost $-3, -1, -1, 0, 1, 2$). These statements were recomputed by the program `check_note.py` of the package and by the three verification runs.

7 The weaker notion of disjointness

This section is not needed for Corollary 1.2. Kamiyama’s survey [Kam14, p. 64] defines internal disjointness for arc-disjoint paths P_i from u_i to v_i with vertex sets X_i by $(X_i \setminus \{u_i, v_i\}) \cap (X_j \setminus \{u_j, v_j\}) = \emptyset$. Accordingly, we call P_i and P_j as in Definition 2.1 *weakly openly disjoint* if they have no common arc and $V(P_i) \cap V(P_j) \subseteq \{v, r_i, r_j\}$; this is the same, as a common vertex outside $\{v, r_i, r_j\}$ is an interior vertex of both paths, and conversely. Let (ARB_w) and (PC_w) be (ARB) and (PC) with this notion, and let (LOC_w) be (LOC) with (A3) dropped and (A2) replaced by

$$(\text{A2}_w) \quad a_i \neq a_j \text{ for } i \neq j, \text{ and if } t_i = t_j \text{ for some } i \neq j, \text{ then } t_i \in \{r_i, r_j\}.$$

Such assignments are called *weakly admissible*.

Proposition 7.1. *Under the hypotheses of Theorem 1.1, (ARB_w) , (PC_w) and (LOC_w) are equivalent.*

Proof. We follow Section 3 and give the changes.

From (ARB_w) to (PC_w) to (LOC_w) . As in Lemma 3.3; now a common tail of the last arcs a_i, a_j of P_i, P_j is a common vertex of these paths other than v , hence r_i or r_j .

Lemma 3.4 is stated for a weakly admissible assignment of minimum cost among the weakly admissible ones, with “ $t_i \neq r_i$ and $t_i \in U_j$ ” replaced by “ $t_i \notin \{r_i, r_j\}$ and $t_i \in U_j$ ” in (1) and in the definition of Δ ; cyclic configurations and genericity are as before. In the proof, (i) is replaced by two observations about a cycle $i_1 \rightarrow \dots \rightarrow i_m \rightarrow i_1$ of Δ . First, $t_{i_s} \neq r_{i_s}$ for all s , as an arc leaves i_s ; hence the tails t_{i_s} are pairwise distinct, because a common tail of a_{i_s} and $a_{i_{s'}}$ would be r_{i_s} or $r_{i_{s'}}$ by (A2_w) . Second, the rotated family a' is weakly admissible: (A1) holds as before; the arcs are permuted; two rotated arcs have different tails; and if a rotated arc $a'_{i_{s+1}} = a_{i_s}$ has the same tail t as an arc $a'_l = a_l$ with l outside the cycle, then $t \in \{r_{i_s}, r_l\}$ by (A2_w) for a , and $t \neq r_{i_s}$, so $t = r_l$, as (A2_w) requires for the pair i_{s+1}, l . The cost identity, the cyclic configuration $(v; i_1, \dots, i_m; t_{i_1}, \dots, t_{i_m})$ and step (iii) are unchanged.

Construction. Lemma 3.5 is unchanged. The construction yields weakly admissible assignments a^v and a potential f such that, at every vertex v ,

$$(\text{P}_w) \quad f_i(v) - f_j(v) > f_i(t_i(v)) - f_j(t_i(v)) \quad \text{for } i \neq j \text{ in } J(v) \text{ with } t_i(v) \notin \{r_i, r_j\}, t_i(v) \in U_j,$$

and Lemma 3.6, whose proof uses only (A1) , shows that the F_i are r_i -arborescences with $V(F_i) = U_i$.

Weak independence. Let $i \neq j$, $v \in U_i \cap U_j$, $P := r_i F_i v$, $Q := r_j F_j v$ and $g := f_i - f_j$. As in (2), (P_w) gives: if (u, y) is an arc of F_i with $u, y \in U_j$, $u \notin \{r_i, r_j\}$ and $y \neq r_j$, then $g(y) > g(u)$; and if (u, y) is an arc of F_j with $u, y \in U_i$, $u \notin \{r_i, r_j\}$ and $y \neq r_i$, then $g(y) < g(u)$. Suppose that $w \notin \{v, r_i, r_j\}$ is a common vertex of P and Q , and let $w = x_0, \dots, x_p = v$ and $w = y_0, \dots, y_q = v$ be the parts of P and Q from w to v . As in case a of Lemma 3.7, all x_s and y_s lie in $U_i \cap U_j$, and for $s \geq 1$ we have $x_s \neq r_i$ (an arc of F_i enters x_s) and $x_s \neq r_j$ (as $r_j \rightsquigarrow w \rightsquigarrow x_s$ with at least one

arc), and likewise $y_s \neq r_j$ and $y_s \neq r_i$. With $w \notin \{r_i, r_j\}$ this gives $x_{s-1}, y_{s-1} \notin \{r_i, r_j\}$ for all $s \geq 1$. Hence g increases strictly along $x_0, \dots, x_p$ and decreases strictly along $y_0, \dots, y_q$, which is impossible. So $V(P) \cap V(Q) \subseteq \{v, r_i, r_j\}$. Finally, the head of a common arc of P and Q lies in $\{v, r_i, r_j\}$; it is not r_i , as no arc of P enters r_i , and not r_j ; so it is v , and $a_i^v = a_j^v$, contradicting (A2_w). $\square$

Lemma 7.2. *If D has no parallel arcs, then (LOC_w) implies (LOC). Consequently, on such digraphs (PC) and (PC_w) are equivalent, and so are (ARB) and (ARB_w).*

Proof. Assume (LOC_w). As in Lemma 3.6, every vertex of U_j is reachable from r_j . Fix v , and call an index $i \in J(v)$ *bad* for a weakly admissible assignment at v if $t_i = r_j \neq r_i$ for some $j \in J(v)$. Take a weakly admissible assignment with the least number of bad indices. As D has no parallel arcs, its tails are pairwise distinct; so (A2) holds, and it remains to show that no index is bad. Suppose that i is bad, with $t_i = r_j \neq r_i$. Then $t_j \neq t_i = r_j$, and $r_j \rightsquigarrow t_j \rightsquigarrow v$, as $t_j \in U_j$, with $r_j = t_i \in U_i$ and $v \in U_i$; hence $t_j \in U_i$ by (C). Exchange the two arcs: $a'_i := a_j$ and $a'_j := a_i$. The new family is weakly admissible, since $t_j \in U_i$, $r_j \in U_j$, and the tails are still distinct. Now j has its own root as tail and is not bad; i has the tail t_j , and if i is bad now, that is, $t_j = r_l \neq r_i$ for some $l \in J(v)$, then $r_l \neq r_j$ and j was bad before. The other indices are not affected. So the number of bad indices has decreased, a contradiction. The second statement follows from Theorem 1.1, Proposition 7.1 and the trivial implication from (LOC) to (LOC_w). $\square$

By Remark 2.3(b) the two notions differ when parallel arcs are present.

8 Computations

No statement of this note depends on a computation. The computations are tests: programs that do not use the proof decide (ARB), (PC) and (LOC) on many instances and compare them, and the construction of Section 3.3 is run with random points in place of the generic choice. A search that finds no counterexample is not a proof. The package contains the programs and recorded outputs of the first tests (folder `original/`) and of three independent verification runs, called A, B and 2, each of which wrote its own programs (Python 3; the construction uses exact rational arithmetic). In the exhaustive runs the vertices are $0, \dots, n-1$, arcs go from smaller to larger numbers, each pair carries 0 to m parallel arcs, and an instance is a multiset of k pairs (r, U) with U convex and $r \in U$; “ $|U| \geq 2$ ” means that sets $\{r\}$, which change none of the conditions, are omitted. An instance is *tight* if exactly $|J(v)|$ arcs enter every vertex v and they form an admissible assignment at v , and *arc-minimal* if it satisfies (PC) and loses it when any arc is deleted.

First tests. Sets with a vertex not reachable from the root, for which all three conditions fail, are omitted. Exhaustive: $n \in \{3, 4\}$, $m = 3$, $k \leq 4$ (3,713,044 instances); $n = 5$, $m = 1$, $k \leq 4$ (2,813,016); $n = 5$, $m = 2$, $k = 3$, $|U| \geq 2$ (12,442,412); $n = 6$, $m = 1$, $2 \leq k \leq 4$, $|U| \geq 2$ (121,272,546). Random: 1,900,000 instances with up to 10 vertices and 7 sets; 162,821 arc-minimal three-set instances with up to 11 vertices; 360,000 tight instances with 14 to 28 vertices. Of these 142,663,839 instances, 15,675,286 satisfy (PC), among them 2,596,738 three-set instances. (PC) and (LOC) agree on every instance, and with (ARB) wherever it was searched (not on the tight instances with more than 16 vertices); on every instance with (LOC) the construction, run with 64 random bits per coordinate, passed separate tests of independence, of (A1)–(A3) and of (P). For 37,517 random instances all 1,305,650 families of admissible assignments were enumerated: the 1,102,205 families for which a linear programme finds a potential with (P) are independent (Remark 3.8(a)), and for 7,277 independent families it finds none. (PC_w) and (ARB_w) agree on 1,471,647 instances with $n \leq 5$, of which 348,034 lie in ranges with parallel arcs; only these go beyond Theorem 1.1 (Lemma 7.2).

An incident in the first run. At first the random points had numerators between -10^6 and 10^6 only. The exhaustive runs made then (all ranges above except $n = 6, k = 4$), the arc-minimal runs and five of the six general runs showed no failure (the general run with seed 5 was stopped because the program was slow then), but on tight instances the construction stopped twice with a cycle of Δ that was not positive. Both cycles have length exactly 0: two admissible assignments that differ by a rotation have the same cost (in one case the two arcs of the cycle have the lengths $\mp 178572/3000001$), as the random potential lies on a hyperplane excluded in Lemma 3.5. So the random stand-in for the generic choice failed, not the theorem: both instances satisfy (ARB), no cycle of negative length occurred, and with other random numbers the same program succeeded on both. The program was changed to 64 random bits and to start again after an exact tie (which has not happened since), and all runs were repeated; the numbers above are from the repeated runs. Run B recomputed both ties, and met ties again with 5 or 8 random bits (23 and 9 on 1,500 instances), none from 12 bits on.

Run A. (PC) is decided by enumeration of all paths and (ARB) by two procedures, for both notions of disjointness. Exhaustive, with all convex sets, also those with unreachable vertices: $n \leq 4, m = 2, k \leq 3$ (2,628,111 instances); $n = 4, m = 2, k = 4$ (17,719,680); $n = 5, m = 1, k = 3$ (42,775,104). On these 63,122,895 instances (ARB), (PC) and (LOC) agree, and so do their weak forms. With the conventions of the first tests, run A reproduced 15 of their 16 exhaustive rows (all but $n = 6, k = 4$; 42,457,616 instances) with the same numbers of instances, of instances with (PC) and of three-set instances with (PC). Random instances with a vertex in three sets with three different roots: 2,000,000 with 5 to 7 vertices, 614,746 arc-minimal ones with 5 to 11 vertices and 12,000 with 8 or 9 vertices; no disagreement, and its implementation of the construction returned independent families whenever (LOC) holds. Further tests concern Lemma 2.2 (141,722 pairs of paths), step (ii) of Lemma 3.4 (2,531,200 abstract local configurations), Remark 3.8(a) (654,881 families), the conditions of Section 5 (3,523,097 instances), non-convex sets (Section 4) and Lemma 7.2 (4,772,828 instances on simple digraphs).

Run B. Exhaustive, by enumeration of paths with maximum flow as a cross-check, for sets whose vertices are all reachable from the root: $n = 3, m = 3, k \leq 4$; $n = 4, m = 3, k \leq 5$, and $k = 6$ with $|U| \geq 2$; $n = 5, m = 1, k \leq 5$; $n = 5, m = 2, 2 \leq k \leq 5, |U| \geq 2$; $n = 6, m = 1, 2 \leq k \leq 5, |U| \geq 2$. Of these 739,798,070 instances, 30,951,388 satisfy (PC), and all of them (ARB); no family was found on the 10,358,436 instances without (PC) that were searched, and in the 16 ranges shared with the first tests the counts agree. (LOC) was compared with (PC) and (ARB) on 16,776,860 instances of this kind and on 39,338,592 instances with arbitrary convex sets (38,742,646 with an unreachable vertex), and for the weak notion on 36,720,116 and 2,352,144 instances; no disagreement. Its implementation of the construction was verified on 1,995,229 instances of the enumeration, in 646,452 runs on random instances built to be hard and on 64,858 arc-minimal instances with 14 to 32 vertices; with points that were non-generic on purpose it stopped with an exact tie in 31,720 of 161,616 runs and in no other way. With arbitrary random potentials, the set X of Lemma 3.4 was non-empty for all 2,418,919 minimum-cost assignments tested and empty for 1,814,005 of 4,135,458 others (Remark 3.8(b)). Among 2,291,311 random instances (tight, arc-minimal, general) none satisfies (PC) without (ARB), and all 505,251 arc-minimal ones are tight; a local search over 1,618,157 tight instances met none without a family of independent arborescences. Run B also repeated the first tests on 18,967,988 of their instances; all 28 outputs are identical to the recorded ones.

Run 2. This run was made on the text as it was before the passages on [Huc99] were rewritten (see “Verification” below), with programs written from the sources and from the statements of the note. Exhaustive, with all convex sets: the 2,628,111 instances with $n \leq 4, m = 2, k \leq 3$ and the 42,775,104 instances with $n = 5, m = 1, k = 3$, as in run A. On every instance (ARB), (PC) and (LOC) were decided from the definitions, in the strong and in the weak form and with the condition on arcs dropped on both sides, and Remark 3.2 was compared with Definition 3.1 at every vertex;

on the 389,276 instances with (LOC) the construction was run with a potential that was verified to be generic (all cyclic configurations were enumerated), and its output was tested for (P) and, with explicit paths, for independence. (ARB), (PC) and (LOC) were also compared, and the construction was run, on 100,000 arc-minimal random instances with 3 to 6 sets, pairwise different roots and 6 to 13 vertices, all of them tight, and, together with the weak forms, on 450,000 further random instances, 300,000 of them built from the degenerate situations (coinciding roots, roots inside other sets, parallel arcs from roots, sets $\{r\}$); on 592 of these without (PC) (745 without (PC_w)) the search for a family was stopped at a bound. No disagreement occurred. Lemma 2.2 (141,722 pairs of paths), the comparisons of Section 5, the reduction of Lemma 8.1 (519,379 reductions of 130,870 instances), the worked example and the enumeration of tight instances (below) were tested separately.

Tight instances. The tight instances suffice for a test of the implication from (LOC) to (ARB):

Lemma 8.1. *If every tight instance with at most n vertices and k sets, in which every vertex lies in some set, satisfies (ARB), then (LOC) implies (ARB) for every instance with at most n vertices and at most k sets, with any number of arcs.*

Proof. Let an instance satisfy (LOC). Choose an admissible assignment at every vertex and delete all other arcs. The sets remain convex, since deleting arcs only destroys paths, and the chosen assignments are admissible in the new digraph, of which they are all the arcs: the new instance is tight. Its arborescences are arborescences of the given instance with the same root paths. A vertex in no set is isolated in the tight instance, because by (A1) both ends of every chosen arc lie in sets; it can be deleted. Finally, adding sets $\{r_1\}$ with the root r_1 brings the number of sets to k and changes none of the conditions. $\square$

Run B enumerated all tight instances of Lemma 8.1 vertex by vertex, carrying with each of them the list of all its families of independent arborescences, for $k = 3$ and $n \leq 7$, $k = 4$ and $n \leq 6$, $k = 5$ and $n \leq 5$. Its program tells the indices apart: an instance, a multiset of k pairs (r, U) , is counted once for every arrangement of the pairs in a sequence with non-decreasing roots. Counted in this way, there are 31,502, 718,840 and 22,305,130 tight instances with $k = 3$ on exactly 5, 6 and 7 vertices, 15,675, 482,021 and 19,358,910 with $k = 4$ on 4, 5 and 6 vertices, and 6,298,238 with $k = 5$ on 5 vertices; instances on fewer vertices occur as intermediate steps and were tested as well. None of these 49,210,316 instances is without (ARB). Run 2 repeated the enumeration with a program of its own and found the same seven totals; counted as multisets they are 14,270,699 instances. By Lemmas 3.3 and 8.1 this confirms Theorem 1.1 for every instance with at most 3 sets and at most 7 vertices, at most 4 sets and at most 6 vertices, or at most 5 sets and at most 5 vertices, with any number of parallel arcs. The program of run B reproduced a filter of the full enumeration for $(n, k) = (3, 3)$, $(4, 2)$, $(4, 3)$, $(5, 2)$ and was cross-checked against the other programs of run B on 139,382 sampled instances. For $n = 8$, $k = 3$ the enumeration of run B is *incomplete*: 20 of its 24 parts were run (770,506,630 tight instances on 8 vertices in its way of counting, none without (ARB)), and the other four were stopped without result; this is a sample and not an exhaustive result. Run 2 completed this range with its own program: 421,315,865 tight instances as multisets (953,971,137 in the count of run B), none without (ARB). With Lemmas 3.3 and 8.1 this extends the test to all instances with at most 3 sets and at most 8 vertices; for 8 vertices it rests on one program only.

Verification. The results were first obtained with a written proof and the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each worked from the sources with its own programs, and neither had the report of the other. Run A compared the statement with the texts of [EGR15, FFKK13, Kam14, BF09, BK11], re-derived Lemmas 3.3 to 3.7 and Proposition 7.1 line by line, examined the hypotheses and the special cases, and proved the implication from (LOC) to (PC) directly, by Menger’s theorem, for one root and for two sets. Run B wrote its own implementations

of the conditions and of the construction, read and partly repeated the first tests, confirmed the account of the incident, and repeated the search of the literature. Neither run found an error or a counterexample. Their corrections concern wording and completeness: the reading of “independent” with the mixed readings, the status of the question, steps that are now explicit in the proofs of Lemmas 3.4, 3.5 and 3.7, Remark 3.2, a complete proof of Proposition 7.1, and the statements on the literature. All were made; the paper [Huc99], which both runs wanted to be read before publication, was read after run 2 (see below). Lemma 2.2, Remark 2.3(b) and Lemma 7.2 were found by both runs independently (the proof of Lemma 7.2 given here is that of run B); Lemma 8.1 and the enumeration of tight instances are due to run B.

The present text was written after these two runs. A further AI-assisted verification run, called run 2 in the package, then examined it as written. It compared the statement and the citations with the sources again; checked every proof line by line, among them the parts that the first two runs had not seen in this form (the wording of the proofs in Sections 3 and 7, Remark 3.2, the one-directional Lemma 8.1, the comparisons in Section 5 and the argument at the end of Section 2.2); went through the degenerate situations (coinciding roots, a root inside another set, $v = r_i$, sets $\{r\}$, parallel arcs from a common root); and wrote its own programs (Section 8). It found no error in a proof and no counterexample. Its corrections: the abstract of [Huc99], which it could read, speaks of multigraphs, so the sentences on Huck’s formulation and on parallel arcs were rewritten; Theorem 1.1 was said to contain Theorem 2 of [FFKK13], a statement about arbitrary digraphs; the totals of tight instances are now given with their way of counting; Remark 3.2, step (ii) of Lemma 3.4 and one sentence of Section 2.3 were made exact. After run 2 the paper [Huc99] was read completely, and the passages on it (in the abstract, the introduction, Sections 2.3 and 5 and the paragraphs below) were rewritten from the paper; no definition, statement or proof of the note was changed, and these passages were not seen by the three runs. The program `check_note.py` of the package tests Remark 3.2 on 20,000 random local configurations and checks the instances of Remark 2.3 and Section 4.

Scope and priority. The note answers the question of [EGR15], that is, Question 2 of [FFKK13] for acyclic digraphs, in the affirmative, for all roots and all convex sets, in the reading of Section 2.2; it says nothing new about digraphs with directed cycles. The question is due to Frank, Fujishige, Kamiyama and Katoh; the EGRES page names S. Fujishige, N. Kamiyama and N. Katoh as proposers. The problem is listed as open on the EGRES page, last modified in 2015; the paper [FFKK13] proves three cases and does not treat the general acyclic case, and it does not itself call that case an open problem. Known cases of the question are Theorems 2, 4 and 6 of [FFKK13], of which Theorem 2 is a statement about arbitrary digraphs, and the two theorems of Huck for acyclic digraphs [Huc99, Theorems 1 and 2], which are restated for simple digraphs in [BF09, Theorems 5.3 and 5.1], the first also for $D - r$ simple in [BK11, Theorem 1.2]. Theorem 1.1 contains Theorems 4 and 6 of [FFKK13], the two theorems of Huck, and Theorem 2 of [FFKK13] only for acyclic digraphs. Parallel arcs are already admitted in [FFKK13] and in [Huc99, Theorem 1]. A local condition equivalent to the path condition is known in the two acyclic cases of [FFKK13] (conditions (1) and (4)–(5) there). Related, for one root in arbitrary digraphs, is Lovász’s theorem on flames (some spanning subgraph preserves the local edge-connectivities from the root and has them as in-degrees), which we know only from the abstract of the preprint [Sze25]; nothing else of that preprint was read.

Read: [FFKK13] completely, in the copy on the first author’s web page; [Huc99] completely; Section 3.3 of [Kam14]; Section 5 of [BF09]; [BK11] except its figures; the EGRES problem page and the EGRES pages “Arborescence” and “Convex node set” in copies saved on 11 October 2026. *Not read:* [Huc95] and [Whi87], cited through [Huc99, FFKK13, BF09]; the dissertation [Hoy19] on independent spanning trees in undirected graphs (abstract only); and two technical-report versions of [FFKK13] from 2009 that are listed in OpenAlex. A. Frank’s book *Connections in Combinatorial*

Optimization (2011) and the lists of citing works in Google Scholar were not consulted.

We make no claim that the method is new. A vector potential chosen by duality of an assignment problem was not found in the texts read, whose proofs order the vertices and remove one arborescence at a time; for two sets it reduces to that ordering argument (Section 5), and its ingredients are standard. Huck’s proof [Huc99], on which the proofs of [FFKK13, Theorem 4] and of [Kam14] are based according to these texts, is of this kind (Section 5). The dissertation [Hoy19], which we did not read, could contain related ideas: by its abstract it proves cases of the independent spanning tree conjectures by embedding vertices or edges in a simplex.

The searches of 11 October 2026 covered the works citing [FFKK13] in OpenAlex, Semantic Scholar, Crossref and OpenCitations (one in each, the survey [Kam14] where it is named); those citing [Huc99] in OpenAlex (10) and Semantic Scholar (19), by titles and available abstracts; queries to the arXiv API, zbMATH Open, OpenAlex and Crossref, repeated in part by run 2; the titles of the EGRES technical reports of 2009–2026; and twelve web searches (not Google Scholar). No answer to the question was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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