

Arithmetic Periodicity of Gonality under Regular Subdivision

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Abstract

For a fixed finite connected loopless multigraph G , let $\sigma_k G$ replace every edge by a path of k edges. We show that, for each degree d , the positive integers k for which $\text{dgon}(\sigma_k G) \leq d$ form a finite union of finitely generated additive submonoids of $\mathbb{N}$, restricted to positive integers. The description is effective. Consequently, for computable integers N, M , the gonality of $\sigma_k G$, for $k \geq N$, depends only on $\gcd(k, M)$. The proof combines finite chip-position types, bounded firing slopes, homogeneous cycle equations, and integer points in rational cones, refining the rational rescaling method used in prior work on discrete and metric gonality. We also determine every regular subdivision of the known tricycle family: its gonality is five for even k , and six for odd k . Thus eventual periodicity need not be eventual constancy. The graph family, its degree-five reduced-divisor restriction, and its original one-to-two subdivision drop are due to van Dobben de Bruyn, Smit, and van der Wegen.

1 Statements and earlier work

Graphs are finite, connected, and loopless; parallel edges are permitted. Put $\mathbb{N} = \{0, 1, 2, \dots\}$. A divisor on H is an integer function on $V(H)$. It is effective if nonnegative, and its degree is the sum of its values. Write

$$(Lf)(x) = \sum_{xy \in E(H)} (f(x) - f(y)), \quad \text{div}(f) = -Lf,$$

counting edges with multiplicity. Divisors are equivalent when their difference is $\text{div}(f)$ for an integer-valued f . An effective divisor has positive rank if, for every prescribed vertex, it is equivalent to an effective divisor containing a chip there. The least degree of such a divisor is the divisorial gonality $\text{dgon}(H)$.

Let $\sigma_k G$ replace every edge by a path of k edges. If subdivision t times means inserting t vertices per edge, our parameter is $k = t + 1$. For finite nonempty $A \subset \mathbb{Z}_{>0}$, put

$$\langle A \rangle = \left\{ \sum_{a \in A} n_a a : n_a \in \mathbb{N} \right\}.$$

Theorem 1. *Given G and $d \geq 1$, one can compute finitely many nonempty finite sets $A_{d,1}, \dots, A_{d,t_d} \subset \mathbb{Z}_{>0}$ such that*

$$\{k \geq 1 : \text{dgon}(\sigma_k G) \leq d\} = \mathbb{Z}_{>0} \cap \bigcup_{j=1}^{t_d} \langle A_{d,j} \rangle. \quad (1)$$

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The union may be empty. There are computable $N, M \geq 1$ and a function F on the positive divisors of M such that

$$\text{dgon}(\sigma_k G) = F(\gcd(k, M)) \quad (k \geq N). \quad (2)$$

No efficiency bound is asserted.

The Egres question on the change of gonality under uniform subdivision appears as AMR-029-0040 in UnsolvedMath 1.6.0 [3, 4]. We give a finite description of the entire sequence and its eventual arithmetic behavior, without claiming small period bounds or a resolution of every related optimization problem.

Van Dobben de Bruyn, Smit, and van der Wegen [1] proved that the minimum over regular subdivisions is the gonality of the unit-length metric graph. Their tricycles have gonality six before subdivision and five after a two-fold subdivision, disproving subdivision invariance. Their Section 3 rational rescaling argument, including its cycle equations and nonnegative-length observation, is the geometric antecedent of our proof. Bodlaender, van der Wegen, and van der Zanden [2] give integer programming certificates for stable divisorial gonality. Here we retain integral lengths for every regular subdivision parameter, giving (1) and (2). The tricycle construction and its degree-five normal-form lemma are explicitly credited to [1].

The minimum over uniform subdivisions is distinct from the minimum over arbitrary edge subdivisions (stable divisorial gonality), and from stable gonality defined by harmonic morphisms. Only divisorial gonality under σ_k is studied here.

2 Finite certificates and rational cones

We use [1, Corollary 2.5]: if H subdivides a loopless graph G , an effective divisor on H has positive rank as soon as it reaches every vertex of G . This is a rank-one consequence of rank-determining sets. The loopless hypothesis ensures the required separation of edge interiors.

Lemma 2. *If D, E are effective of degree d and $E - D = \text{div}(f)$, then $|f(x) - f(y)| \leq d$ on every edge xy .*

Proof. For a nonconstant edge, choose a real c strictly between its endpoint values and unequal to any vertex value. Set $U = \{x : f(x) > c\}$. Summing over U gives

$$\sum_{\substack{xy \in E \\ x \in U, y \notin U}} (f(x) - f(y)) = D(U) - E(U) \leq d.$$

All summands are positive and include the chosen edge difference. $\square$

Fix d , put $V = V(G)$, and $n = |V|$. Positive rank in degree at most d is equivalent to positive rank in degree exactly d , by adding chips. Choose effective degree- d divisors D and $D_v \sim D$, for $v \in V$, with $D_v(v) \geq 1$. These have $d(n+1)$ labelled chips.

A *position type* records which chips coincide, which blocks are on original vertices, and the ordered list of other blocks inside each oriented original edge. There are finitely many types. Coincidence blocks may contain chips from several divisors; labels are not artificially separated. Require a chip from each D_v to lie on the original vertex v . The type gives an abstract refinement H of G . For every v and oriented refined edge a , choose $s_a^v \in \mathbb{Z} \cap [-d, d]$. Retain choices satisfying at each node x

$$\sum_{\text{tail}(a)=x} s_a^v - \sum_{\text{head}(a)=x} s_a^v = D_v(x) - D(x). \quad (3)$$

The right side is fixed by the chip labels. Thus all position and slope types form a finite list independent of k .

For a retained type introduce $\ell_a \geq 0$, $k \geq 0$, and impose

$$\sum_{a \in P_e} \ell_a = k \quad (e \in E(G)), \quad (4)$$

$$\sum_{a \in C} \epsilon_{C,a} s_a^v \ell_a = 0 \quad (v \in V, C \text{ in a cycle basis of } H). \quad (5)$$

Here P_e refines e ; signs compare cycle and edge orientations. These homogeneous integer equations define a pointed rational cone C_τ in the nonnegative orthant.

Lemma 3. *For integer $k > 0$, $\text{dgon}(\sigma_k G) \leq d$ if and only if some C_τ has an integer point with last coordinate k .*

Proof. Given positive rank, choose the D_v and integer firing potentials. Superimpose their supports and V , using actual coincidence blocks. At any intervening degree-two vertex the vanishing divergence means that consecutive slopes agree. Thus every potential is linear between successive support nodes. Lemma 2 bounds its integer slopes by d ; integer gap lengths satisfy (4), and the potentials satisfy (3) and (5).

Conversely, take an integer point with $k > 0$. Contract zero-length segments and add chips at merged nodes. No original edge contracts completely, by (4), so distinct original vertices remain distinct. Equation (3) survives by summing over contracted nodes: internal edge slopes cancel. The products $s_a^v \ell_a$ integrate to a potential, by integrating along a spanning tree and checking the other edges with (5). Zero-length segments have zero potential difference, so the potential descends after contraction. Normalize it to zero at one original vertex. Integer slopes and lengths give integer values on every unit subdivision vertex. The divisor is $D_v - D$; new vertices on constant-slope segments have zero divergence. Every D_v is effective and contains v , so the cited subdivision criterion proves positive rank. $\square$

Here is the finite generation step explicitly. Triangulate a pointed rational cone into finitely many simplicial rational cones, with primitive integer vectors on their rays. Every lattice point in a simplicial cone is a nonnegative integer combination of these vectors plus a lattice point in their bounded half-open fundamental parallelepiped. There are finitely many such remainder points. Taking all rays and all remainders from the triangulation therefore generates the integer points of the original cone. The procedure uses finite rational arithmetic and bounded lattice enumeration.

If G has an edge, (4) implies that $k = 0$ forces all lengths to be zero. Thus nonzero integer generators of C_τ have positive last coordinate. Projecting them to this coordinate gives $A_{d,\tau}$; a cone consisting only of the origin contributes nothing. Lemma 3 proves (1). A single vertex with no edges has gonality one for every k and is handled separately.

3 The effective gcd rule

Let A be finite, nonempty, and positive. Set $g = \gcd(A)$, $B = \{a/g : a \in A\}$, and $m = \min B$. For $0 \leq j < m$, let w_j be the least element of $\langle B \rangle$ congruent to $j \bmod m$. These exist since the residues of B generate $\mathbb{Z}/m\mathbb{Z}$ and every submonoid of a finite group is a subgroup. Compute them by shortest paths on residues modulo m , with edges $u \rightarrow u + b$ of weight b , for $b \in B$. For $k \geq 0$,

$$k \in \langle A \rangle \iff g \mid k \text{ and } k/g \geq w_{(k/g) \bmod m}. \quad (6)$$

Indeed, a difference from the corresponding minimum is a nonnegative multiple of the generator m . For $k \geq g \max_j w_j$, membership is therefore equivalent to $g \mid k$.

The divisor with one chip at each original vertex reaches $V(G)$, so it has positive rank on every subdivision. Hence gonality lies between one and n . Construct (1) for $d = 1, \dots, n$. Let M be the least common multiple of all generator-set gcds $g_{d,j}$, and choose $N \geq 1$ at least all their thresholds $g_{d,j} \max w_j$. For $k \geq N$,

$$\text{dgon}(\sigma_k G) \leq d \iff \text{some } g_{d,j} \mid k \iff \text{some } g_{d,j} \mid \gcd(k, M).$$

Taking the least such d defines F and proves (2). A choice always exists at degree n . Equation (6) also decides all earlier values exactly.

Corollary 4. *The sequence $\text{dgon}(\sigma_k G)$ is eventually periodic. It has an ordinary limit exactly when F is constant; this is decidable from the finite construction.*

Proof. The gcd with M is periodic modulo M , and every positive divisor of M occurs infinitely often as such a gcd. $\square$

4 Exact parity for tricycles

Use the tricycle definition from [1, Definition 4.1]. Take three disjoint cycles C_i , with distinct marked vertices a_i, b_i on each, and join b_i to a_{i+1} by a single transition edge (indices modulo three). A cycle may consist of two parallel edges. Join a central vertex r to all six marked vertices by spokes, each of integer length at least two. Cycle arcs have arbitrary positive integer lengths. Denote this tricycle by T .

Theorem 5. *For every tricycle T and every $k \geq 1$,*

$$\text{dgon}(\sigma_k T) = \begin{cases} 5, & k \text{ even}, \\ 6, & k \text{ odd}. \end{cases}$$

Every positive-rank r -reduced degree-five divisor has two chips at r and one at the midpoint of each length- k transition path.

Dhar's criterion burns from q , then burns a vertex when its number of incident edges from burned vertices exceeds its chip count. If every vertex burns, an effective divisor is q -reduced. If it has zero chips at q , it has no positive rank. Every effective divisor admits an equivalent r -reduced representative.

Our substantial normal-form input is [1, Lemma 4.5 and Corollary 4.6]: on any subdivision of the minimal tricycle, a positive-rank r -reduced divisor of degree at most five has exactly two chips at r and one on each transition path, including endpoints. All such subdivisions have gonality at least five.

Lemma 6. *Suppose D has two chips at r and one on each transition path of $\sigma_k T$. If it has positive rank, the two transition chips adjacent to each cycle are equally far from that cycle.*

Proof. Fix a cycle with unequal adjacent distances $a < b$, where $0 \leq a < b \leq k$. At each integer time $0 \leq t < a$, fire the complement of the cycle and the chip-free portions of its four incident paths, extending those portions up to, but excluding, their boundary chips. At time zero the two spoke chips coincide at r , paying its two boundary edges. Thereafter there is one chip on each spoke and each adjacent transition path. Each firing is legal and moves these four chips one step toward the cycle. The third transition chip stays fixed. Spoke lengths are at least $2k > a$, so no spoke chip reaches the cycle during these a steps.

In the equivalent effective divisor D' , exactly one transition chip has reached the cycle, at a marked vertex u . Its other marked vertex q has no chip. Burn from q . The two empty arcs

toward u give two burning edges at its one chip; the entire cycle burns. The transition path vacated by this chip then burns through to a second cycle. Its entry vertex has no chip, and its cycle has at most one chip elsewhere, from the third transition path. Its two arcs therefore allow the entire second cycle to burn.

The two spokes from the second cycle are empty. If $a > 0$, r has no chips and burns when reached. If $a = 0$, it has two chips, but four empty spokes from the two burned cycles supply four burning incident edges. Thus r burns in both cases. When $a > 0$, the two spoke chips are now attacked from both ends.

The last cycle is reached along its two empty spokes. It has at most two chips, both at marked vertices. If one marked vertex is empty, it burns and its two cycle arcs, together with the other spoke, burn the other marked vertex, even if that vertex carries both chips. If both marked vertices have one chip, both remaining transition paths have empty interiors. They also reach this cycle from burned cycles, giving two burning edges at a marked vertex. The last cycle therefore burns in all cases. Any remaining transition chip is then attacked from both ends. Every vertex has burned while $D'(q) = 0$, contradicting positive rank. $\square$

Proof of Theorem 5. For a degree-five positive-rank divisor take its r -reduced representative and use the cited normal form. Let $p_i \in \{0, \dots, k\}$ be the chip's distance from b_i on the transition path to a_{i+1} . Lemma 6 gives $p_i = k - p_{i-1}$ for all $i \bmod 3$. Thus $p_1 = p_2 = p_3 = k/2$, impossible for odd k .

For even k , put the chips at these three midpoints and two at r . For any cycle perform the legal wave for $k/2$ steps. Both transition chips arrive together, reaching both marked vertices. The divisor also reaches r . The seven vertices r, a_i, b_i are a loopless model's vertex set, so the subdivision rank criterion proves positive rank. The cited lower bound gives gonality five.

For every k , put one chip at each of the six marked vertices. To reach r , move all six chips toward it along their spokes: fire the outer ring together with the terminal spoke portions on its side of the moving chips, until the shortest spoke first delivers a chip to r . This is legal and proves that the degree-six divisor reaches all seven model vertices. Hence gonality is at most six also for odd k , proving the formula. $\square$

The family, its metric gonality five, and the cases $k = 1, 2$ are from [1]. The additional assertion here is midpoint necessity and the resulting formula for every k . This does not resolve the separate existence question about odd-fold drops on other graphs in [1, Section 6].

5 Checks and scope

The cone argument is a written proof, not a claim that all its cones have been enumerated in software. An exact-integer checker verifies the firing wave and full burn order for every unbalanced placement on the minimal tricycle with $1 \leq k \leq 12$, testing every unequal adjacent pair. It checks 22,386 obstruction certificates by direct Laplacian replay, including transition endpoints and the zero-step case. A separately implemented Dhar search for $1 \leq k \leq 7$ returns only the midpoint placement for even k and none for odd k . These finite checks supplement the general proofs; they do not constitute formal verification.

This manuscript was developed with AI assistance for proof exploration, source comparison, exact computations, and text preparation. The arguments were audited within that workflow. The preprint has not received independent peer review and makes no absolute priority certification. Classical inputs are attributed above; the broader Egres research question is not recorded as exhaustively closed by this note.

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