

Positive Functions Equimeasurable with Their Derivatives: Construction and Rigidity

Alper Ferudun*

26 October 2026

Abstract

Rubel asked for functions whose first derivative has the same value distribution as the function. We parametrize all strictly positive C^1 solutions on a compact interval by continuous Lebesgue-measure-preserving maps satisfying a spectral-radius condition. If the function is twice differentiable in the interior, the only possibilities are exponentials and an explicitly phased sine family. The tent map yields strictly positive C^1 solutions precisely for interval lengths below 2, with an exact spectral radius, endpoint value, and uniform approximation error. These solutions have prescribed losses of differentiability at a dense set of dyadic points. At the critical length, a rescaling of the classical Fabius distribution gives a nonnegative C^∞ example. The construction also gives positive functions equimeasurable with any prescribed higher derivative. This answers the request for additional examples and gives a classification in the strictly positive class; the unrestricted classification of functions that vanish or change sign remains outside the results.

1 The question and positive solutions

Throughout, $I = [0, L]$, $L > 0$, and m denotes Lebesgue measure. For real-valued measurable functions on I , write $u \sim v$ if

$$m\{x : u(x) < y\} = m\{x : v(x) < y\} \quad (y \in \mathbb{R}).$$

Equivalently, the push-forward measures u_*m and v_*m agree. Problem 7.77 of Hayman and Lingham [1, p. 185], attributed to L. A. Rubel, asks which functions satisfy $f' \sim f$, requests examples beyond exponentials and sines, and asks about other differential rearrangements. We give complete results for strictly positive C^1 functions, their twice-differentiable subclass, and explicit constructions. No claim to classify all sign-changing or vanishing functions is made.

A continuous map $T : I \rightarrow I$ is measure preserving if $m(T^{-1}E) = m(E)$ for every Borel set $E \subseteq I$. Define the positive bounded operator on $C(I)$ by

$$(\mathcal{K}_T h)(x) = \int_0^x h(T(t)) dt. \quad (1)$$

Its norm in the supremum norm is L . Write $r(\mathcal{K}_T)$ for its spectral radius (using the complexification if necessary).

Theorem 1. *Strictly positive functions $f \in C^1(I)$ satisfying $f' \sim f$ are in bijection with pairs (a, T) , where $a > 0$, $T : I \rightarrow I$ is continuous and measure preserving, and $r(\mathcal{K}_T) < 1$. The corresponding function is*

$$f = a \sum_{j=0}^{\infty} \mathcal{K}_T^j \mathbf{1}, \quad f' = f \circ T, \quad f(0) = a. \quad (2)$$

The series converges uniformly. Conversely, the unique map associated with f is

$$T = f^{-1} \circ f'. \quad (3)$$

In particular, every continuous measure-preserving map is admissible when $L < 1$.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Proof. Let $f > 0$ and $f' \sim f$. Continuous equimeasurable functions on a compact interval have the same minimum and maximum: a value outside the other function's range would, by continuity, give a positive-measure level set absent from its distribution. Thus $f' \geq \min_I f > 0$. The function f is a C^1 increasing bijection onto its range, and (3) defines a continuous map into I . For $0 \leq y \leq L$,

$$m\{T < y\} = m\{f' < f(y)\} = m\{f < f(y)\} = y.$$

Therefore $T_*m = m$. The fundamental theorem of calculus gives $f = a\mathbf{1} + \mathcal{K}_T f$ with $a = f(0) > 0$.

Let $M = \|f\|_\infty$ and use the equivalent weighted norm $\|h\|_f = \sup_I |h|/f$. Positivity yields

$$\|\mathcal{K}_T h\| \leq \|h\|_f \mathcal{K}_T f = \|h\|_f (f - a) \leq (1 - a/M) \|h\|_f f.$$

Consequently $r(\mathcal{K}_T) \leq 1 - a/M < 1$.

Conversely, if $r(\mathcal{K}_T) < 1$, the spectral-radius formula implies that $\sum_{j \geq 0} \mathcal{K}_T^j$ converges in operator norm. Hence (2) defines the unique continuous solution of $f = a\mathbf{1} + \mathcal{K}_T f$. Every term is nonnegative, so $f \geq a$. Differentiating this integral identity gives $f \in C^1(I)$ and $f' = f \circ T$. Measure preservation gives $f' \sim f$. The recovered map is unique because f is strictly increasing. Finally $r(\mathcal{K}_T) \leq \|\mathcal{K}_T\| = L$. $\square$

The operator criterion is an exact parametrization, not a claim that the spectral radius of an arbitrary map is available in closed form. Section 3 computes it exactly for the tent map.

2 Rigidity with a second derivative

Lemma 2. *A continuous measure-preserving map $T : [0, L] \rightarrow [0, L]$ that is differentiable at every interior point is either $T(x) = x$ or $T(x) = L - x$.*

Proof. For every closed interval $J \subseteq [0, L]$, continuity makes $T(J)$ an interval, and

$$m(J) \leq m(T^{-1}(T(J))) = m(T(J)).$$

If $|T'(x)| < 1$ at an interior point, differentiability implies, for all sufficiently small $h > 0$, that $T([x - h, x + h])$ lies in an interval of length strictly less than $2h$, a contradiction. Thus $|T'| \geq 1$. The Darboux property of derivatives implies that T' has a constant sign, and the mean value theorem makes T strictly monotone. The image is all of $[0, L]$, since a missing open interval would contradict measure preservation. An increasing measure-preserving bijection of an interval is its identity; a decreasing one is its reflection, as follows by evaluating the measures of inverse images of initial intervals. $\square$

Theorem 3. *Let $f \in C^1([0, L])$ be strictly positive and twice differentiable on $(0, L)$. Then $f' \sim f$ if and only if either*

$$f(x) = ae^x \quad (a > 0),$$

or $0 < L < \pi/2$ and

$$f(x) = c \sin \left(x + \frac{\pi}{4} - \frac{L}{2} \right) \quad (c > 0). \quad (4)$$

Proof. The map $T = f^{-1} \circ f'$ of Theorem 1 is differentiable in the interior, because $f' > 0$ and f'' exists there. Lemma 2 leaves two cases. The identity gives $f' = f$. Reflection gives $f'(x) = f(L - x)$ and hence $f'' = -f$. At the midpoint, $f'(L/2) = f(L/2) > 0$, so this solution is exactly (4). The simultaneous strict positivity of f and f' on the whole closed interval is equivalent to $L < \pi/2$. Conversely, the exponential has $f' = f$, and (4) has $f' = f \circ T$ for the measure-preserving reflection. $\square$

3 The tent map: exact threshold and regularity

Consider

$$T_L(x) = \begin{cases} 2x, & 0 \leq x \leq L/2, \\ 2L - 2x, & L/2 \leq x \leq L. \end{cases} \quad (5)$$

The two inverse branches each scale length by $1/2$, so T_L preserves Lebesgue measure. Put $\mathcal{K} = \mathcal{K}_{T_L}$.

Theorem 4. For every $L > 0$ and every integer $n \geq 1$,

$$\|\mathcal{K}^n\| = \frac{L^n}{2^{n-1}}, \quad r(\mathcal{K}) = \frac{L}{2}. \quad (6)$$

For prescribed $a > 0$, a strictly positive solution of $f' = f \circ T_L$, $f(0) = a$, exists if and only if $L < 2$. It is unique and is given by (2). Its endpoint value is

$$f(L) = a \frac{2+L}{2-L}. \quad (7)$$

Writing $q = L/2$ and $S_N = a \sum_{j=0}^N \mathcal{K}^j \mathbf{1}$, its errors satisfy the exact formulas

$$\|f - S_N\|_\infty = \frac{2aq^{N+1}}{1-q} \quad (N \geq 0), \quad \|f' - S'_N\|_\infty = \frac{2aq^N}{1-q} \quad (N \geq 1). \quad (8)$$

Proof. Set $h_n = \mathcal{K}^n \mathbf{1}$. For $n \geq 1$, h_n is nonnegative, increasing, and zero at 0. Since $T_L(L-x) = T_L(x)$, the derivative $h'_n = h_{n-1} \circ T_L$ is symmetric about $L/2$. Therefore

$$h_n(x) + h_n(L-x) = h_n(L).$$

Measure preservation and integration of this symmetry give, for $n \geq 2$,

$$h_n(L) = \int_0^L h_{n-1}(T_L(t)) dt = \int_0^L h_{n-1}(t) dt = \frac{L}{2} h_{n-1}(L).$$

Since $h_1(L) = L$, the endpoint is $L^n/2^{n-1}$. For a positive operator on $C(I)$, $|h| \leq \mathbf{1}$ implies $|\mathcal{K}^n h| \leq \mathcal{K}^n \mathbf{1}$, so its norm is $\|\mathcal{K}^n \mathbf{1}\|_\infty$. This proves (6); taking n th roots gives the spectral radius.

Theorem 1 now gives existence, uniqueness and the sharp threshold. Summing the endpoint series gives (7). Each term in its remainder is nonnegative and increasing, so the supremum of the remainder occurs at L . Summation of its geometric endpoint series proves the first equality in (8). For $N \geq 1$, $S'_N = S_{N-1} \circ T_L$. Surjectivity of T_L therefore makes the derivative error equal to $\|f - S_{N-1}\|_\infty$. $\square$

For example, $a = 1$ and $L = 1/2$ give a fully specified solution with $f(1/2) = 5/3$. The rational piecewise-polynomial approximant S_8 has endpoint 54613/32768, with exact uniform errors 1/98304 for the function and 1/24576 for its first derivative.

Proposition 5. For $0 < L < 2$, the solution in Theorem 4 is not twice differentiable at $L/2$; its one-sided second derivatives are $2a$ and $-2a$. More generally, if $x_0 \in (0, L)$ first reaches $L/2$ after $m \geq 0$ iterates of T_L , then f is C^{m+1} near x_0 and is not $(m+2)$ times differentiable there. The absolute jump between the one-sided derivatives of order $m+2$ is

$$2a 2^{(m+1)(m+2)/2}.$$

Such points are dense. In particular, f is not analytic on any nonempty open subinterval.

Proof. The first assertion follows from $f' = f \circ T_L$ and $f'(L) = f(0) = a$. For the general assertion choose a neighborhood of x_0 on which the first m branches before the turning point are fixed. Repeated chain differentiation, using the affine branches, gives there

$$f^{(m+1)}(x) = C_m f(T_L^{m+1}(x)), \quad C_m = \prod_{j=0}^{m-1} \left(T'_L(T_L^j(x)) \right)^{m-j}. \quad (9)$$

For $m = 0$ the product is 1. In this neighborhood C_m is constant, with absolute value $2^{m(m+1)/2}$. The preceding derivatives exist continuously by the same chain rule. The map T_L^{m+1} has a corner at x_0 , takes the value L there, and has one-sided slopes with absolute value 2^{m+1} and opposite signs. Applying the one-sided chain rule in (9) proves the stated nonzero jump. The first-hit points are precisely the interior dyadic multiples of L , with a suitable finite first-hit time, and are dense. $\square$

4 The critical case and the classical Fabius distribution

The strict positivity assumption cannot be dropped from Theorem 3. The critical tent interval supplies a smooth example from Fabius's classical construction [2]. We recall the construction and verify the differential rearrangement directly; neither the Fabius distribution nor its smoothness is claimed as new.

Let $U_1, U_2, \dots$ be independent uniform random variables on $[0, 1]$, and let F be the distribution function of

$$X = \sum_{j=1}^{\infty} 2^{-j} U_j.$$

Extend F by 0 to the left of 0 and by 1 to the right of 1. The random variable X has no atoms, since its first uniform summand is independent of the remainder. Symmetry of the summands gives $F(1-x) = 1 - F(x)$ on $[0, 1]$. The distributional identity $X \stackrel{d}{=} (U_1 + X')/2$, with X' an independent copy, gives

$$F(x) = \int_0^1 F(2x - u) du, \quad F'(x) = 2(F(2x) - F(2x - 1)).$$

Starting with continuity, the last identity proves inductively that the extended F is C^∞ .

Proposition 6. *The nonnegative function $f(x) = F(x/2)$ on $[0, 2]$ is C^∞ , is strictly positive for $x > 0$, and satisfies*

$$f' = f \circ T_2, \quad f' \sim f, \quad f(0) = f'(0) = 0, \quad f(2) = 1.$$

It is neither an exponential nor a sine of the forms above.

Proof. For $x \leq 1$, $f'(x) = F(x) = f(2x)$. For $x \geq 1$, symmetry gives $f'(x) = 1 - F(x-1) = F(2-x) = f(4-2x)$. Thus the tent identity holds, and measure preservation proves equimeasurability. To see $F(t) > 0$ for $t > 0$, choose a finite tail bound below $t/2$ and require the finitely many remaining summands to sum to less than $t/2$, an event of positive probability. Finally the two zero initial data, together with nontriviality, exclude exponential and sine solutions. $\square$

5 Higher differential rearrangements

Theorem 7. *Let $r \geq 1$, let $T : [0, L] \rightarrow [0, L]$ be continuous and measure preserving, and suppose $q = L^r/r! < 1$. For each $a > 0$ there is a unique solution $f \in C^r([0, L])$ of*

$$f^{(r)} = f \circ T, \quad f(0) = a, \quad f^{(j)}(0) = 0 \quad (1 \leq j < r).$$

It is strictly positive and satisfies $f^{(r)} \sim f$. For the tent map this solution is not $(r+1)$ times differentiable at $L/2$.

Proof. Define

$$(\mathcal{K}_{T,r}h)(x) = \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} h(T(t)) dt.$$

This positive operator has norm q . The norm-convergent Neumann series $f = a \sum_{n \geq 0} \mathcal{K}_{T,r}^n \mathbf{1}$ solves $f = a\mathbf{1} + \mathcal{K}_{T,r}f$ uniquely and is bounded below by a . Repeated differentiation gives the initial data and the differential identity. Equimeasurability follows from measure preservation. For the tent map, the one-sided derivatives of $f^{(r)} = f \circ T_L$ at $L/2$ are $2f'(L)$ and $-2f'(L)$. Here $f'(L) > 0$: for $r = 1$ it equals a , and for $r \geq 2$ it follows by differentiating the positive integral representation once. Thus the one-sided derivatives disagree. $\square$

For $S_N = a \sum_{n=0}^N \mathcal{K}_{T,r}^n \mathbf{1}$ the same argument gives

$$\|f - S_N\|_\infty \leq \frac{aq^{N+1}}{1-q}, \quad \|f^{(r)} - S_N^{(r)}\|_\infty \leq \frac{aq^N}{1-q} \quad (N \geq 1).$$

These bounds certify approximation of both quantities, rather than asserting that a finite truncation itself has exactly equal distributions.

Scope and verification. The positive parametrization, rigidity theorem, tent threshold and higher-order constructions are the complete assertions proved here. They supply further examples requested in Rubel’s problem; functions that vanish or change sign are not classified. The accompanying standard-library Python code checks rational piecewise-polynomial recurrences, initial data, continuity, tent norm powers and finite moment-error bounds. The proofs above do not extrapolate from those finite checks. The tent map, Neumann series and Fabius distribution are established tools. AI tools assisted the research, calculations and manuscript preparation; the author checked the arguments. This preprint has not undergone independent peer review or formal verification, and no exhaustive priority claim is made.

References

- [1] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, arXiv:1809.07200v2 (2018), Problem 7.77, p. 185. <https://arxiv.org/abs/1809.07200>.
- [2] J. Fabius, A probabilistic example of a nowhere analytic C^∞ -function, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **5** (1966), 173–174. <https://doi.org/10.1007/BF00536652>. Original report and published text: <https://ir.cwi.nl/pub/8160>.