

A Cauchy Kernel without the Fixed-Coefficient Clifford Relations

Alper Ferudun*

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Abstract

For the commutative complex algebra $\mathbb{C} \oplus \mathbb{C}$, we construct a real-analytic Cauchy kernel for a first-order operator whose coefficients admit no Clifford-dual family. The operator is the direct sum of two differently scaled Cauchy–Riemann operators; its reproducing formula is the ordinary Cauchy formula in two real-linear coordinates. The kernel is right regular away from the origin, homogeneous of degree -1 , and normalized by unit boundary flux. No real change of coordinates supplies a common scalar quadratic factorization of the symbol. Consequently, existence of a Cauchy kernel does not imply the fixed-coefficient algebraic relations printed in Hayman–Lingham Problem 7.69, even with real-analytic rather than merely C^1 regularity. This gives a negative answer to part (a) in that existence formulation. The nonassociative question in part (b) is not addressed.

1 The precise question

Let A be a unital associative algebra, and fix linearly independent coefficients $e_1, \dots, e_n$. On their real span, consider

$$D = \sum_{j=1}^n e_j \partial_{x_j}, \quad d\sigma = \sum_{j=1}^n (-1)^j e_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

The existence formulation in Hayman–Lingham [1, Problem 7.69, pp. 181–182] links a real-analytic reproducing kernel for D to elements $p_1, \dots, p_n \in A$ satisfying

$$p_j e_\ell + p_\ell e_j = 2\delta_{j\ell} \mathbf{1}. \quad (1)$$

Part (a) asks whether the criterion remains valid for a C^1 kernel. We show that its necessity fails already for a real-analytic kernel.

We use real coordinate slices, as in the cited original [2, Theorem 1, p. 71]. That theorem begins with a complex subspace V and integrates over compact real n -manifolds in a translate of the real span $V_{\mathbb{R}}$ of the chosen coefficients. It also states uniqueness, whereas the problem book retains existence. Our conclusion concerns that printed existence criterion. In particular, we do not claim to settle every possible additional normalization or intended formulation of the original theorem.

2 The algebra and the reproducing formula

Equip

$$A = \mathbb{C} \oplus \mathbb{C}$$

with componentwise addition and multiplication, and set

$$\mathbf{1} = e_1 = (1, 1), \quad e_2 = (i, 2i). \quad (2)$$

The determinant of these columns is i , so they are linearly independent over $\mathbb{C}$. Thus $V = \text{span}_{\mathbb{C}}\{e_1, e_2\} = A$ and $V_{\mathbb{R}} = \{xe_1 + ye_2 : x, y \in \mathbb{R}\}$ is a real two-plane. We identify that plane with $\mathbb{R}^2$ using (x, y) . For a C^1 function $f = (f_1, f_2)$,

$$Df = (\partial_x f_1 + i\partial_y f_1, \partial_x f_2 + 2i\partial_y f_2), \quad d\sigma = -e_1 dy + e_2 dx. \quad (3)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

A function with $Df = 0$ will be called D -regular.

Theorem 1. For $(u, v) \neq (0, 0)$ define

$$G(u, v) = -\frac{1}{2\pi} \left(\frac{1}{u + iv}, \frac{1}{2u + iv} \right). \quad (4)$$

Let $\Omega \subset \mathbb{R}^2$ be bounded with C^1 boundary, let f be C^1 and D -regular on a neighborhood of $\overline{\Omega}$, and let $(x_0, y_0) \in \Omega$. With the boundary orientation induced by $dx \wedge dy$,

$$f(x_0, y_0) = \int_{\partial\Omega} G(x - x_0, y - y_0) d\sigma f(x, y). \quad (5)$$

The kernel is real analytic off the origin, right regular there, homogeneous of degree -1 under positive real dilations, and $O((u^2 + v^2)^{-1/2})$.

Proof. Use the real-linear complex coordinates

$$\zeta_1 = x + iy, \quad \zeta_2 = 2x + iy.$$

Their real Jacobian determinants are 1 and 2, respectively. If $h_j = f_j \circ \zeta_j^{-1}$, the chain rule gives

$$(\partial_x + i\partial_y)f_1 = 2\partial_{\bar{\zeta}_1}h_1, \quad (\partial_x + 2i\partial_y)f_2 = 4\partial_{\bar{\zeta}_2}h_2.$$

Thus h_j is holomorphic on the image of the relevant neighborhood. The components of the boundary form satisfy

$$(d\sigma)_1 = -dy + i dx = i d\zeta_1, \quad (d\sigma)_2 = -dy + 2i dx = i d\zeta_2.$$

Writing $\zeta_{j,0} = \zeta_j(x_0, y_0)$, we obtain

$$G_j(x - x_0, y - y_0)(d\sigma)_j = \frac{1}{2\pi i} \frac{d\zeta_j}{\zeta_j - \zeta_{j,0}}.$$

Each component of (5) is therefore the ordinary Cauchy integral formula on the orientation-preserving image $\zeta_j(\Omega)$. The usual formula for the full oriented boundary also covers multiply connected domains.

Neither denominator in (4) vanishes at a nonzero real pair (u, v) . This proves real analyticity. Direct differentiation gives

$$\partial_u G + (\partial_v G)e_2 = 0.$$

Homogeneity follows directly from the formula. Finally, $|u + iv|^2 = u^2 + v^2$ and $|2u + iv|^2 = 4u^2 + v^2$, which prove the decay estimate. $\square$

The same argument applies to compact C^1 two-manifolds with boundary embedded in the coordinate plane, component by component. Taking $f = \mathbf{1}$ gives the unit-flux normalization

$$\int_{\partial\Omega} G(x - x_0, y - y_0) d\sigma = \mathbf{1}. \quad (6)$$

If instead F is holomorphic on a complex open subset of V and left regular in the sense of [2, Definition 4], then its restriction to each relevant translated real slice is D -regular. Indeed, holomorphy identifies its derivatives in the real coordinate directions with the complex coordinate derivatives. Thus (5) applies to that original function class as well.

3 Failure of the algebraic necessity

Theorem 2. The coefficients (2) admit no $p_1, p_2 \in A$ satisfying (1). In particular, real-analytic kernel existence does not imply those relations.

Proof. The diagonal relations force

$$p_1 e_1 = \mathbf{1}, \quad p_2 e_2 = \mathbf{1}, \quad p_1 = (1, 1), \quad p_2 = (-i, -i/2).$$

There is no freedom in these choices because both e_1 and e_2 are invertible. Their cross relation is

$$p_1 e_2 + p_2 e_1 = (i, 2i) + (-i, -i/2) = (0, 3i/2) \neq (0, 0).$$

The kernel in Theorem 1 supplies the opposing existence statement. $\square$

The obstruction is not that A lacks Clifford subalgebras. It concerns the prescribed coefficients of D . Nor can the mismatch of the two scales be removed by one real change of coordinates, as the following algebraic statement makes precise.

Proposition 3. *There are no $a, b \in A$ and nonzero homogeneous real quadratic polynomial $Q(x, y)$ such that*

$$(ax + by)(e_1x + e_2y) = Q(x, y)\mathbf{1}. \quad (7)$$

Consequently no real invertible coordinate change makes the transformed symbol admit a Clifford-dual family.

Proof. The two scalar components of $e_1x + e_2y$ are $x + iy$ and $x + 2iy$. If (7) holds, then both linear polynomials divide Q in $\mathbb{C}[x, y]$. Write $Q = rx^2 + 2sxy + ty^2$ with $r, s, t \in \mathbb{R}$. Substituting $x = -iy$ gives $t - r - 2is = 0$, and substituting $x = -2iy$ gives $t - 4r - 4is = 0$. Hence $s = 0$, $t = r = 4r$, and $r = s = t = 0$, a contradiction.

For the last assertion, the dual relations for a symbol $\tilde{e}_1\xi + \tilde{e}_2\eta$ would give

$$(\tilde{p}_1\xi + \tilde{p}_2\eta)(\tilde{e}_1\xi + \tilde{e}_2\eta) = (\xi^2 + \eta^2)\mathbf{1}.$$

Pulling this identity back by the inverse real coordinate change yields (7) with a nonzero positive definite real quadratic Q . The first assertion excludes it. $\square$

4 Normalization and scope

Uniqueness is separate from the existence issue. In this example, adding a constant $c \in A$ to a reproducing kernel leaves (5) unchanged: by Stokes' theorem and (3),

$$\int_{\partial\Omega} c \, d\sigma \, f = -c \int_{\Omega} Df \, dx \, dy = 0.$$

Our counterexample already satisfies the usual homogeneity and right-regularity conditions. In fact, these conditions together with unit flux determine it uniquely.

Proposition 4. *Among C^1 kernels $K : \mathbb{R}^2 \setminus \{0\} \rightarrow A$ that are right D -regular and homogeneous of degree -1 under positive dilations, G is the unique one with unit flux around the origin.*

Proof. In each coordinate ζ_j , right regularity says that the corresponding scalar function k_j is holomorphic on $\mathbb{C} \setminus \{0\}$. Homogeneity gives $k_j(tz) = t^{-1}k_j(z)$ for $t > 0$. Differentiating in t at 1 gives $zk'_j(z) = -k_j(z)$, so $(zk_j(z))' = 0$. Connectedness implies $k_j(z) = c_j/z$. The integral of $(c_j/\zeta_j)i \, d\zeta_j$ around the positively oriented image of a circle is $-2\pi c_j$. Unit flux forces $c_1 = c_2 = -1/(2\pi)$. $\square$

These calculations use only the classical one-variable Cauchy formula. Cauchy formulas over commutative algebras have a substantial earlier literature; for a recent treatment of A -holomorphic functions and references to that literature, see [3]. We do not claim a new general calculus. The point is the explicit failure of the fixed-coefficient necessity in the stated source question.

There is also a family of examples: replace e_2 by (i, iq) for any $q > 0$, $q \neq 1$, replace ζ_2 by $qx + iy$, and use $-(2\pi)^{-1}(1/(u + iv), 1/(qu + iv))$. The cross relation becomes $(0, i(q - q^{-1}))$, whereas the same Cauchy proof applies. The single choice $q = 2$ suffices.

Thus part (a) of Problem 7.69, understood as the printed equivalence between kernel existence and (1), has a negative answer. Its real-analytic version already fails, so weakening regularity to C^1 cannot repair it. This does not classify all admissible algebras, does not rule out other correctly normalized classification theorems, and does not answer the nonassociative part (b).

A companion symbolic script verifies the coefficient obstruction, right regularity, one-form identity, unit-circle flux, and common quadratic obstruction. These checks supplement the written proofs. AI assistance was used in development, source comparison, computation, and drafting. This is an author-self-audited preprint; no independent peer review or formal proof-assistant verification is claimed.

References

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