

Scales of Convergent Series and the Null Ideal: An Answer to Hinkkanen's Problem 7.62

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Abstract

Problem 7.62 in Hayman and Lingham's *Research Problems in Function Theory*, posed by A. Hinkkanen, asks whether one can, without assuming the Continuum Hypothesis, associate with every countable ordinal α a convergent series $\sum_n x_{n,\alpha}$ with non-negative non-increasing terms such that $x_{n,\alpha}/x_{n,\beta} \rightarrow 0$ whenever $\alpha < \beta$, and such that every convergent series with positive terms converges faster, in this sense, than one of them. As printed, the last condition concerns all positive summable sequences, and then no such family exists, even under CH. Reading it for positive non-increasing sequences, as the first sentence of the problem suggests, we show that such a family exists if and only if $\text{cof}(\mathcal{N}) = \aleph_1$, where $\text{cof}(\mathcal{N})$ is the cofinality of the ideal of Lebesgue null sets. The main step is that the positive non-increasing summable sequences, ordered by $x \prec y$ if $x_n/y_n \rightarrow 0$, are Tukey equivalent to the null ideal. Via Kraft's inequality this confirms a conjecture of Todorćević, recorded by Filmus, on the partial order of monotone codes of the natural numbers. Consequently the family exists under CH and in the Sacks model, and it does not exist under Martin's Axiom with $\neg\text{CH}$ or in the Cohen and random models. That it cannot be constructed in ZFC alone also follows from earlier results of Filmus and Geschke and of Vojtáš. We also show that a scale of non-increasing convergent series, of some ordinal length, exists if and only if $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N})$; this sharpens Filmus's independence theorem for scales of codes. This is an unrefereed note.

1 Introduction

Problem 7.62 of Hayman and Lingham's *Research Problems in Function Theory* [HL19], posed by A. Hinkkanen, starts from the fact that for countably many convergent series with positive decreasing terms there is such a series converging more slowly than each of them; results of this kind go back to du Bois-Reymond [dBR73] and Hadamard [Had94]. It asks whether, without any assumption about the Continuum Hypothesis (CH), one can associate with every countable ordinal α a convergent series $\sum_{n \geq 1} x_{n,\alpha}$ with $0 \leq x_{n+1,\alpha} \leq x_{n,\alpha}$ such that

- (a) if $\alpha < \beta$, then $x_{n,\alpha}/x_{n,\beta} \rightarrow 0$ as $n \rightarrow \infty$, and
- (b) if $x_n > 0$ and $\sum_n x_n < \infty$, then $x_n/x_{n,\alpha} \rightarrow 0$ for some α .

Update 7.62 of [HL19] reports that no progress has been communicated. The problem is record AMR-022-7062 of the UnsolvedMath dataset [ula26]. Problem 2.66 of [HL19] asks the same question for the maximum modulus of entire functions, and Update 2.66 reports unpublished work of

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Hinkkanen showing that the answer there is independent of the axioms of set theory. We do not treat Problem 2.66.

Under CH a transfinite recursion of length ω_1 , which uses the first sentence of the problem at each step, produces such a family (Lemma 3.3). So the question is whether ZFC alone suffices. Condition (b) as printed concerns *all* positive summable sequences. Then no family exists, not even under CH, because a summable sequence with sparse spikes is not $o(z_n)$ for any non-increasing summable z (Remark 7.1). The first sentence of the problem speaks only of series with positive decreasing terms, and we read (b) accordingly:

(b') if $x_n > 0$, $x_{n+1} \leq x_n$ for all n , and $\sum_n x_n < \infty$, then $x_n/x_{n,\alpha} \rightarrow 0$ for some α .

A sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ of non-negative, non-increasing, summable sequences $x_\alpha = (x_{n,\alpha})_{n \geq 1}$ satisfying (a) and (b') will be called a *Hinkkanen family*. If instead (b) is kept as printed and the monotonicity of the members is dropped, the answer is the same (Remark 7.2).

Let $\mathcal{D}$ be the set of positive, non-increasing, summable sequences, and write $x \prec y$ if $x_n/y_n \rightarrow 0$. Apart from its first member, which may lie outside $\mathcal{D}$, a Hinkkanen family is the same as a $\prec$ -increasing sequence of length ω_1 that is cofinal in $(\mathcal{D}, \prec)$ (Lemma 3.3). Our main result determines the cofinal structure of $(\mathcal{D}, \prec)$. Let $\mathcal{N}$ be the ideal of Lebesgue null sets, and recall that $\text{cof}(\mathcal{N})$ is the least size of a family of null sets such that every null set is contained in one of them, and $\text{add}(\mathcal{N})$ is the least size of a family of null sets whose union is not null. Tukey equivalence, in the sense of generalized Galois–Tukey connections, is recalled in Section 2.

Theorem 1.1. *$(\mathcal{D}, \prec)$ is Tukey equivalent to $(\ell^1, \leq^*)$, and hence to the null ideal $(\mathcal{N}, \subseteq)$. In particular $\text{cof}(\mathcal{D}, \prec) = \text{cof}(\mathcal{N})$ and $\text{add}(\mathcal{D}, \prec) = \text{add}(\mathcal{N})$.*

Here $\text{cof}(\mathcal{D}, \prec)$ is the least size of a set $F \subseteq \mathcal{D}$ such that every $x \in \mathcal{D}$ satisfies $x \prec y$ for some $y \in F$, and $\text{add}(\mathcal{D}, \prec)$ is the least size of a subset of $\mathcal{D}$ that has no common $\prec$ -upper bound in $\mathcal{D}$. So the second identity gives the exact extent of the first sentence of the problem: fewer than $\text{add}(\mathcal{N})$ series with positive decreasing terms always admit one such series that converges more slowly than each of them, and some family of $\text{add}(\mathcal{N})$ such series admits none.

Corollary 1.2. *A Hinkkanen family exists if and only if $\text{cof}(\mathcal{N}) = \aleph_1$. In particular:*

- (i) *a Hinkkanen family exists under CH, and in the Sacks model, where $\mathfrak{c} = \aleph_2$;*
- (ii) *no Hinkkanen family exists under Martin's Axiom with $\mathfrak{c} > \aleph_1$, in the Cohen model, or in the random model.*

Hence, if ZFC is consistent, the existence of a Hinkkanen family is neither provable nor refutable in $\text{ZFC} + \neg \text{CH}$.

So the answer to the question as posed is negative: a Hinkkanen family cannot be constructed in ZFC alone. In the random model $\mathfrak{b} = \mathfrak{d} = \aleph_1$, so these two cardinals being $\aleph_1$ does not suffice.

Corollary 1.3. *Call a sequence $\langle x_\alpha : \alpha < \lambda \rangle$ in $\mathcal{D}$, where λ is an ordinal, a scale if $x_\alpha \prec x_\beta$ for all $\alpha < \beta < \lambda$ and every $x \in \mathcal{D}$ satisfies $x \prec x_\alpha$ for some α . A scale exists if and only if $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N})$, and then every scale has length of cofinality $\text{cof}(\mathcal{N})$. In particular, scales exist under Martin's Axiom and do not exist in the Cohen and random models.*

Earlier work. Filmus [Fil13] studied codes of the natural numbers: non-decreasing $c: \omega \rightarrow \omega$ with $\sum_n 2^{-c(n)} = 1$, where c is *better* than d if $c(n) - d(n) \rightarrow -\infty$. He observes that, via $c \mapsto (2^{-c(n)})_n$, this is the ratio comparison of convergent series. He proved that every countable set of codes has a better code, that CH implies the existence of a scale of codes, and that there is no scale in the model obtained by adding $\aleph_2$ Cohen generic codes [Fil13, Theorems 3.1, 4.2 and 4.8]. The last two arguments had been outlined by Geschke in an answer to Filmus’s MathOverflow question [Fil10]. In the concluding discussion of [Fil13], Filmus records a conjecture of Todorćević that the partial order of codes is Tukey equivalent to the null ideal, as is known for codes that are not required to be monotone. Kraft’s inequality and Filmus’s completion lemma show that the codes and $(\mathcal{D}, \prec)$ are Tukey equivalent (Proposition 5.1). With this equivalence, the impossibility of a construction in ZFC for Problem 7.62 already follows from [Fil13, Theorem 4.8] (see also [Fil10]), although we found no source that connects these works with Hinkkanen’s problem. Theorem 1.1 confirms Todorćević’s conjecture (Corollary 5.2), and Corollaries 1.2 and 1.3 sharpen Filmus’s independence theorem to exact conditions.

Kholshchevnikova [Kho83] and Vojtáš [Voj87] studied convergent series compared by the ratio of their remainders. Vojtáš proved that for this order the unbounding number of convergent series is $\mathfrak{b}$, and he noted that the dominating number is $\mathfrak{d}$. The same holds for non-increasing series, and since the remainder order is weaker than $\prec$, this gives $\mathfrak{d} \leq \text{cof}(\mathcal{D}, \prec)$, and hence the negative answer in ZFC (Remark 7.3). For non-negative summable sequences ordered by eventual domination, Vojtáš [Voj88] proved that the least size of a dominating family equals $\text{cof}(\mathcal{N})$. The deep ingredient of Theorem 1.1 is Bartoszyński’s characterization of the null ideal [Bar84, Bar10]. The new step is the passage between arbitrary and non-increasing summable sequences (Lemmas 4.1 and 4.2); one direction uses Cauchy condensation.

Section 6 gives direct proofs of $\mathfrak{d} \leq \text{cof}(\mathcal{D}, \prec)$ and $\text{cov}(\mathcal{N}) \leq \text{cof}(\mathcal{D}, \prec)$, without Bartoszyński’s theorem; they suffice for the negative half of Corollary 1.2.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$; sequences are indexed by $\mathbb{N}$ unless stated otherwise. For positive sequences we write $x \prec y$ if $x_n/y_n \rightarrow 0$. This relation is irreflexive and transitive. If $x \prec y$, $z_n \leq Mx_n$ and $y_n \leq Mw_n$ for all large n , then $z \prec w$. We use the following structures.

- $\mathcal{D} = \{x \in (0, \infty)^{\mathbb{N}} : x_{n+1} \leq x_n \text{ for all } n, \text{ and } \sum_n x_n < \infty\}$, with the relation $\prec$.
- $\ell^1 = \{f \in [0, \infty)^{\mathbb{N}} : \sum_n f(n) < \infty\}$, with $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n .
- $\mathcal{C} = \{S: \mathbb{N} \rightarrow [\omega]^{<\omega} : \sum_n |S(n)| 2^{-n} < \infty\}$, with $S \subseteq^* T$ if $S(n) \subseteq T(n)$ for all but finitely many n ; here $[\omega]^{<\omega}$ is the set of finite subsets of ω .
- $\mathcal{N}$, the ideal of null subsets of the Cantor space 2^ω with the usual product measure, ordered by inclusion. (The cardinal invariants below are the same for Lebesgue measure on $\mathbb{R}$.)

Relational systems. We use the framework of Vojtáš [Voj93], in the notation of [Bar10, Section 2]; section and lemma numbers of [Bar10] refer to its arXiv version. A relational system is a triple $\mathbf{A} = (A_-, A_+, R)$ with $R \subseteq A_- \times A_+$. Put

$$\begin{aligned} \text{cof}(\mathbf{A}) &= \min\{|F| : F \subseteq A_+, \forall a \in A_- \exists f \in F (a R f)\}, \\ \text{add}(\mathbf{A}) &= \min\{|U| : U \subseteq A_-, \forall f \in A_+ \exists a \in U \neg(a R f)\}. \end{aligned}$$

For a relation R on a set P we identify (P, R) with the relational system (P, P, R) and write $\text{cof}(P, R)$ and $\text{add}(P, R)$. Thus $\text{cof}(\mathcal{N}, \subseteq)$ and $\text{add}(\mathcal{N}, \subseteq)$ are the usual $\text{cof}(\mathcal{N})$ and $\text{add}(\mathcal{N})$. A *morphism* from $\mathbf{A}$ to $\mathbf{B} = (B_-, B_+, S)$ is a pair of maps $\varphi_-: A_- \rightarrow B_-$ and $\varphi_+: B_+ \rightarrow A_+$ such that $\varphi_-(a) S b$ implies $a R \varphi_+(b)$ for all $a \in A_-$ and $b \in B_+$. We write $\mathbf{A} \leq_T \mathbf{B}$ if there is a morphism, and $\mathbf{A} \equiv_T \mathbf{B}$ (Tukey equivalence) if $\mathbf{A} \leq_T \mathbf{B} \leq_T \mathbf{A}$. Morphisms compose. For directed partial orders $(P, P, \leq)$ and $(Q, Q, \leq)$ a morphism consists of a Tukey map and a cofinal map, so $\equiv_T$ is Tukey equivalence in the classical sense [Tuk40]; see also [Fre91].

Lemma 2.1 ([Voj93]; [Bar10, Lemma 2.5]). *If $\mathbf{A} \leq_T \mathbf{B}$, then $\text{cof}(\mathbf{A}) \leq \text{cof}(\mathbf{B})$ and $\text{add}(\mathbf{A}) \geq \text{add}(\mathbf{B})$.*

Proof. If $F \subseteq B_+$ witnesses $\text{cof}(\mathbf{B})$, then $\varphi_+[F]$ has the property required in $\text{cof}(\mathbf{A})$. If $U \subseteq A_-$ is unbounded in $\mathbf{A}$, so is $\varphi_-[U]$ in $\mathbf{B}$: if $\varphi_-(a) S b$ for all $a \in U$, then $a R \varphi_+(b)$ for all $a \in U$. $\square$

Cardinal invariants. Let $\mathfrak{b}$ and $\mathfrak{d}$ be the unbounding and dominating numbers of $(\omega^\omega, \leq^*)$, let $\text{cov}(\mathcal{N})$ be the least number of null sets covering 2^ω , and let $\mathfrak{c} = 2^{\aleph_0}$. By Cichoń's diagram [BJ95, Bla10], $\aleph_1 \leq \text{add}(\mathcal{N}) \leq \mathfrak{b} \leq \mathfrak{d} \leq \text{cof}(\mathcal{N}) \leq \mathfrak{c}$ and $\text{add}(\mathcal{N}) \leq \text{cov}(\mathcal{N}) \leq \text{cof}(\mathcal{N})$, and every cardinal of the diagram other than $\aleph_1$ and $\mathfrak{c}$ lies between $\text{add}(\mathcal{N})$ and $\text{cof}(\mathcal{N})$. Table 1 lists the values we use. Martin's Axiom implies $\text{add}(\mathcal{N}) = \mathfrak{c}$ [MS70]. The Cohen, random and Sacks models are obtained from a model of CH by adding $\aleph_2$ Cohen reals, by adding $\aleph_2$ random reals, and by a countable support iteration of Sacks forcing of length ω_2 , respectively; the values in these models are standard [BJ95, Bla10].

	$\text{add}(\mathcal{N})$	$\text{cov}(\mathcal{N})$	$\mathfrak{b}$	$\mathfrak{d}$	$\text{cof}(\mathcal{N})$	$\mathfrak{c}$	Hinkkanen family / scale
CH	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	yes / yes
Sacks model	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_2$	yes / yes
MA + $\mathfrak{c} > \aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	no / yes
Cohen model	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_2$	$\aleph_2$	$\aleph_2$	no / no
Random model	$\aleph_1$	$\aleph_2$	$\aleph_1$	$\aleph_1$	$\aleph_2$	$\aleph_2$	no / no

Table 1: Values of some cardinal invariants [BJ95, Bla10, MS70], and the existence of Hinkkanen families and of scales in $(\mathcal{D}, \prec)$ by Corollaries 1.2 and 1.3.

3 Strict upper bounds and scales

The following lemma is the first sentence of Problem 7.62; we include a proof that keeps the bound non-increasing.

Lemma 3.1. *For every countable $C \subseteq \mathcal{D}$ there is $\bar{u} \in \mathcal{D}$ with $c \prec \bar{u}$ for all $c \in C$.*

Proof. We may assume that $C = \{y^{(i)} : i \geq 1\}$ is non-empty. Put $s = \sum_i 2^{-i} y^{(i)} / \|y^{(i)}\|_1$. Then $s \in \mathcal{D}$, $\sum_n s_n = 1$ and $y^{(i)} \leq 2^i \|y^{(i)}\|_1 s$, so it suffices to find $\bar{u} \in \mathcal{D}$ with $s \prec \bar{u}$. Let $T(m) = \sum_{n \geq m} s_n$, and choose integers $1 = N_0 < N_1 < N_2 < \dots$ with $N_k \geq 2N_{k-1}$ and $T(N_k) \leq 4^{-k}$ for $k \geq 1$. Put $u_n = 2^k s_n$ for $N_k \leq n < N_{k+1}$, and $\bar{u}_n = \sup_{m \geq n} u_m$.

For $j \geq 1$, monotonicity of s gives $(N_j - N_{j-1})s_{N_j} \leq T(N_{j-1}) \leq 4^{1-j}$, and $N_j - N_{j-1} \geq N_j/2$; hence $2^j s_{N_j} \leq 8 \cdot 2^{-j}/N_j$, which is at most $4 \cdot 2^{-k}/N_{k+1}$ when $j > k$. Since u is non-increasing on each interval $[N_j, N_{j+1})$, for $N_k \leq n < N_{k+1}$ we obtain

$$\bar{u}_n = \max \left\{ 2^k s_n, \sup_{j > k} 2^j s_{N_j} \right\} \leq 2^k s_n + \frac{4 \cdot 2^{-k}}{N_{k+1}}.$$

So $\bar{u}$ is finite, non-increasing and positive, and for $k \geq 1$

$$\sum_{N_k \leq n < N_{k+1}} \bar{u}_n \leq 2^k T(N_k) + 4 \cdot 2^{-k} \leq 5 \cdot 2^{-k}.$$

Thus $\bar{u} \in \mathcal{D}$. Finally $s_n/\bar{u}_n \leq s_n/u_n = 2^{-k}$ for $N_k \leq n < N_{k+1}$, so $s \prec \bar{u}$. $\square$

For a set P with a strict partial order $<$ (irreflexive and transitive), call $B \subseteq P$ *bounded* if some $p \in P$ satisfies $b < p$ for all $b \in B$. A *scale* of length λ , an ordinal, is a sequence $\langle p_\alpha : \alpha < \lambda \rangle$ with $p_\alpha < p_\beta$ for $\alpha < \beta$ whose range is cofinal: every $p \in P$ satisfies $p < p_\alpha$ for some α .

Lemma 3.2. *Let $(P, <)$ be a strict partial order in which every countable set is bounded. Then $\aleph_1 \leq \text{add}(P, <) \leq \text{cof}(P, <)$, and:*

- (i) *if there is a scale of length λ , then $\text{cf}(\lambda) = \text{add}(P, <) = \text{cof}(P, <)$;*
- (ii) *if $\text{add}(P, <) = \text{cof}(P, <) = \kappa$, then there is a scale of length κ .*

Proof. The first inequality is the hypothesis. A cofinal set F is unbounded: if $f < p$ for all $f \in F$, then $p < f$ for some $f \in F$, and $f < f$. Hence $\text{add}(P, <) \leq \text{cof}(P, <)$.

(i) λ is a limit ordinal: if $\lambda = \nu + 1$, cofinality gives $p_\nu < p_\alpha$ for some $\alpha \leq \nu$, which is impossible. Let $\mu = \text{cf}(\lambda)$ and let $\langle \alpha_\xi : \xi < \mu \rangle$ be increasing and cofinal in λ . Every p_α is equal to or below some p_{α_ξ} , so $\{p_{\alpha_\xi} : \xi < \mu\}$ is cofinal in P and $\text{cof}(P, <) \leq \mu$. If $B \subseteq P$ and $|B| < \mu$, choose for each $b \in B$ some ξ_b with $b < p_{\alpha_{\xi_b}}$. Since μ is regular, some $\xi < \mu$ exceeds all ξ_b , and then $b < p_{\alpha_\xi}$ for all $b \in B$. So $\text{add}(P, <) \geq \mu$, and the three cardinals coincide.

(ii) Let $\{f_\xi : \xi < \kappa\}$ be cofinal. Recursively choose p_ξ with $p_\eta < p_\xi$ for $\eta < \xi$ and $f_\xi < p_\xi$; this is possible because the set $\{p_\eta : \eta < \xi\} \cup \{f_\xi\}$ has fewer than $\kappa = \text{add}(P, <)$ elements. Every $p \in P$ satisfies $p < f_\xi < p_\xi$ for some ξ . $\square$

Lemma 3.3. *A Hinkkanen family exists if and only if $\text{cof}(\mathcal{D}, \prec) = \aleph_1$. In particular a Hinkkanen family exists under CH.*

Proof. By Lemma 3.1, Lemma 3.2 applies to $(\mathcal{D}, \prec)$. Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a Hinkkanen family. For $\beta \geq 1$, condition (a) with $\alpha = 0$ requires $x_{n,\beta} > 0$ for all large n , hence for all n , as x_β is non-negative and non-increasing. So $x_\beta \in \mathcal{D}$. For the same reason (b') can use $\alpha = 0$ only if $x_0 \in \mathcal{D}$, and then $x \prec x_0 \prec x_1$. Hence $\langle x_\alpha : 1 \leq \alpha < \omega_1 \rangle$ is a scale of length ω_1 in $(\mathcal{D}, \prec)$, and Lemma 3.2(i) gives $\text{cof}(\mathcal{D}, \prec) = \aleph_1$. Conversely, if $\text{cof}(\mathcal{D}, \prec) = \aleph_1$, then $\text{add}(\mathcal{D}, \prec) = \aleph_1$ as well, and Lemma 3.2(ii) gives a scale of length ω_1 . Its members are positive, non-increasing and summable; (a) is the monotonicity of the scale, and (b') is its cofinality. Under CH, $\text{cof}(\mathcal{D}, \prec) \leq |\mathcal{D}| = \mathfrak{c} = \aleph_1$. $\square$

4 Tukey equivalence with the null ideal

Lemma 4.1. $(\mathcal{D}, \mathcal{D}, \prec) \leq_T (\ell^1, \ell^1, \leq^*)$.

Proof. Let $\varphi_-(y) = y$. For $w \in \ell^1$ and $N \in \mathbb{N}$ define $m^{w,N}$ by $m_n^{w,N} = \min_{N \leq k \leq n} w(k) + 2^{-n}$ for $n \geq N$, and $m_n^{w,N} = m_N^{w,N}$ for $n < N$. Each $m^{w,N}$ is positive and non-increasing, and $\sum_{n \geq N} m_n^{w,N} \leq \sum_n w(n) + 1$, so $m^{w,N} \in \mathcal{D}$. By Lemma 3.1 choose $\varphi_+(w) \in \mathcal{D}$ with $m^{w,N} \prec \varphi_+(w)$ for every N . Now let $y \in \mathcal{D}$ satisfy $y_n \leq w(n)$ for all $n \geq N$. For $n \geq N$ and $N \leq k \leq n$ we have $y_n \leq y_k \leq w(k)$, so $y_n < m_n^{w,N}$. Hence $y \prec \varphi_+(w)$. $\square$

Lemma 4.2. $(\ell^1, \ell^1, \leq^*) \leq_T (\mathcal{D}, \mathcal{D}, \prec)$.

Proof. For $x \in \ell^1$ let $\varphi_-(x) = y^x$, where

$$y_k^x = 2^{-k} + \sup\{x(n)2^{-n} : n \in \mathbb{N}, 2^n \geq k\} \quad (k \in \mathbb{N}).$$

The index set shrinks as k grows, so y^x is positive and non-increasing. Bounding the supremum by the sum gives $\sum_k y_k^x \leq 1 + \sum_n x(n)2^{-n} \cdot \#\{k : k \leq 2^n\} = 1 + \sum_n x(n)$, so $y^x \in \mathcal{D}$; moreover $y_{2^n}^x \geq x(n)2^{-n}$. For $z \in \mathcal{D}$ let $\varphi_+(z)(n) = 2^n z_{2^n}$. By Cauchy condensation, $2^{n-1} z_{2^n} \leq \sum_{2^{n-1} < k \leq 2^n} z_k$, so $\sum_n \varphi_+(z)(n) \leq 2 \sum_k z_k$ and $\varphi_+(z) \in \ell^1$. If $y^x \prec z$, then $y_k^x \leq z_k$ for all $k \geq K$, and therefore $x(n) \leq 2^n y_{2^n}^x \leq 2^n z_{2^n} = \varphi_+(z)(n)$ whenever $2^n \geq K$. So $x \leq^* \varphi_+(z)$. $\square$

Lemma 4.3 (Bartoszyński). (i) $(\mathcal{N}, \mathcal{N}, \subseteq) \equiv_T (\mathcal{C}, \mathcal{C}, \subseteq^*)$.

(ii) $(\mathcal{C}, \mathcal{C}, \subseteq^*) \equiv_T (\ell^1, \ell^1, \leq^*)$.

Part (i) goes back to Bartoszyński's combinatorial characterization of the null ideal [Bar84]; in the form stated it is [Bar10, Lemma 3.13], where $\mathcal{N}$ is represented by a cofinal family of null sets indexed by ω^ω , which does not change its Tukey type. Part (ii) is [Bar10, Lemma 4.12]. We include the short proof of (ii) for completeness.

Proof of (ii). For $\ell^1 \leq_T \mathcal{C}$, let $\varphi_-(f)(n) = \{k \in \mathbb{N} : 2^{-n-1} \leq f(k) < 2^{-n}\}$. This set is finite, and $\sum_n |\varphi_-(f)(n)|2^{-n} \leq 2 \sum_k f(k)$, because these sets are disjoint and $2^{-n} \leq 2f(k)$ for $k \in \varphi_-(f)(n)$. Let $\varphi_+(S)(k) = \max\{2^{-n} : k \in S(n)\}$, or 0 if there is no such n ; then $\sum_k \varphi_+(S)(k) \leq \sum_n |S(n)|2^{-n}$. If $\varphi_-(f)(n) \subseteq S(n)$ for all $n \geq n_0$, then every k with $0 < f(k) < 2^{-n_0}$ lies in $S(n)$ for the $n \geq n_0$ with $2^{-n-1} \leq f(k) < 2^{-n}$, so $\varphi_+(S)(k) \geq 2^{-n} > f(k)$. Only finitely many k have $f(k) \geq 2^{-n_0}$, so $f \leq^* \varphi_+(S)$.

For $\mathcal{C} \leq_T \ell^1$, fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N} \times \omega \rightarrow \mathbb{N}$. Let $\varphi_-(S)(\langle n, m \rangle) = 2^{-n}$ if $m \in S(n)$ and 0 otherwise, so that $\sum \varphi_-(S) = \sum_n |S(n)|2^{-n}$. Let $\varphi_+(g)(n) = \{m : g(\langle n, m \rangle) \geq 2^{-n}\}$; this set is finite, and $\sum_n |\varphi_+(g)(n)|2^{-n} \leq \sum g$. If $\varphi_-(S) \leq g$ outside a finite set $E \subseteq \mathbb{N}$, let n_1 exceed the first coordinates of the elements of E . For $n \geq n_1$ and $m \in S(n)$ we get $g(\langle n, m \rangle) \geq 2^{-n}$, that is, $S(n) \subseteq \varphi_+(g)(n)$. $\square$

Proof of Theorem 1.1. By Lemmas 4.1 and 4.2, $(\mathcal{D}, \prec) \equiv_T (\ell^1, \leq^*)$, and by Lemma 4.3, $(\ell^1, \leq^*) \equiv_T (\mathcal{C}, \subseteq^*) \equiv_T (\mathcal{N}, \subseteq)$. Morphisms compose, and Lemma 2.1 gives the two identities. $\square$

If $x \sqsubseteq y$ means $x_n = O(y_n)$, then the identity maps, and the identity together with $y \mapsto$ (a strict upper bound from Lemma 3.1), show $(\mathcal{D}, \sqsubseteq) \equiv_T (\mathcal{D}, \prec)$. As $(\mathcal{D}, \sqsubseteq)$ is a directed preorder, Theorem 1.1 is also a Tukey equivalence in the classical sense.

5 Consequences, and codes

Proof of Corollary 1.2. By Lemma 3.3 and Theorem 1.1, a Hinkkanen family exists if and only if $\text{cof}(\mathcal{N}) = \aleph_1$. The values of $\text{cof}(\mathcal{N})$ in Table 1 give (i) and (ii). The Sacks model satisfies $\neg\text{CH}$ and has a Hinkkanen family; the Cohen model satisfies $\neg\text{CH}$ and has none. $\square$

Proof of Corollary 1.3. By Lemma 3.1, Lemma 3.2 applies to $(\mathcal{D}, \prec)$: a scale exists if and only if $\text{add}(\mathcal{D}, \prec) = \text{cof}(\mathcal{D}, \prec)$, which equals the cofinality of the length of every scale. By Theorem 1.1 this condition reads $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N})$. The examples follow from Table 1. $\square$

If a Hinkkanen family exists, then all cardinals of Cichoń's diagram other than $\aleph_1$, since they lie between $\text{add}(\mathcal{N})$ and $\text{cof}(\mathcal{N})$.

Codes. Following [Fil13, Definitions 2.3 and 2.4], a *precode* is a non-decreasing $c: \omega \rightarrow \omega$ with $\sigma(c) = \sum_n 2^{-c(n)} \leq 1$; it is a *code* if $\sigma(c) = 1$ and *proper* if $\sigma(c) < 1$. A code d is better than c if $d(n) - c(n) \rightarrow -\infty$. Let $\mathcal{K}$ be the set of codes, and write $c \triangleleft d$ if d is better than c . This is a strict partial order, and a scale of codes in the sense of [Fil13, Definition 4.1] is a scale in $(\mathcal{K}, \triangleleft)$ in the sense of Section 3. For $c \in \mathcal{K}$ let $\iota(c) \in \mathcal{D}$ be the sequence $\iota(c)_n = 2^{-c(n-1)}$ ($n \in \mathbb{N}$). Then

$$c \triangleleft d \iff \iota(c) \prec \iota(d).$$

Proposition 5.1. *For every $z \in \mathcal{D}$ there is a code e with $z_n \leq 4\sigma_z \iota(e)_n$ for all n , where $\sigma_z = \sum_k z_k$. Consequently $(\mathcal{K}, \triangleleft) \equiv_T (\mathcal{D}, \prec)$.*

Proof. Let $e'(n) = \lceil \log_2(2\sigma_z/z_{n+1}) \rceil$ for $n \in \omega$. Since z is non-increasing, e' is non-decreasing; since $z_{n+1} < \sigma_z$, the values are positive integers; and $2^{-e'(n)} \leq z_{n+1}/(2\sigma_z)$, so $\sigma(e') \leq 1/2$. Thus e' is a proper precode. By Filmus's completion lemma [Fil13, Lemma 3.4] there is a code e with $e(n) \leq e'(n)$ for all n . Then $2^{-e(n)} \geq 2^{-e'(n)} > z_{n+1}/(4\sigma_z)$, which is the first claim. Write e_z for such a code. If $\iota(c) \prec z$, then $\iota(c) \prec \iota(e_z)$, so $(\iota, z \mapsto e_z)$ is a morphism from $(\mathcal{K}, \triangleleft)$ to $(\mathcal{D}, \prec)$. If $e_z \triangleleft d$, then $\iota(e_z) \prec \iota(d)$ and hence $z \prec \iota(d)$; so $(z \mapsto e_z, \iota)$ is a morphism from $(\mathcal{D}, \prec)$ to $(\mathcal{K}, \triangleleft)$. $\square$

Corollary 5.2. *The partial order of codes is Tukey equivalent to the null ideal, that is,*

$$(\mathcal{K}, \triangleleft) \equiv_T (\mathcal{N}, \subseteq).$$

This confirms Todorćević's conjecture recorded in [Fil13, Section 5]. A scale of codes exists if and only if $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N})$, and a scale of codes of length ω_1 exists if and only if $\text{cof}(\mathcal{N}) = \aleph_1$.

Proof. Combine Proposition 5.1 with Theorem 1.1. By Lemma 2.1, $\text{add}(\mathcal{K}, \triangleleft) = \text{add}(\mathcal{N}) \geq \aleph_1$ and $\text{cof}(\mathcal{K}, \triangleleft) = \text{cof}(\mathcal{N})$, so Lemma 3.2 applies to $(\mathcal{K}, \triangleleft)$. $\square$

As in the remark after Theorem 1.1, with strict upper bounds given by [Fil13, Theorem 3.1], the relation $\triangleleft$ may be replaced by the directed preorder $\iota(c) = O(\iota(d))$ without changing the Tukey type, so Corollary 5.2 also holds in the classical sense.

Thus [Fil13, Theorem 4.2] (CH gives a scale of codes) is the case $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \aleph_1$. In the Cohen model $\text{add}(\mathcal{N}) = \aleph_1 < \text{cof}(\mathcal{N})$, which recovers the conclusion of [Fil13, Theorem 4.8] for that model. Under Martin's Axiom scales of codes exist, of length $\mathfrak{c}$, while in the random model they do not.

6 Direct lower bounds

The next two bounds follow from Theorem 1.1 and Cichoń's diagram, but they have short direct proofs that do not use Lemma 4.3. With Lemma 3.3 they show that no Hinkkanen family exists when $\mathfrak{d} > \aleph_1$ or $\text{cov}(\mathcal{N}) > \aleph_1$, in particular under $\text{MA} + \neg\text{CH}$ and in the Cohen and random models.

Proposition 6.1. $\mathfrak{d} \leq \text{cof}(\mathcal{D}, \prec)$.

Proof. Call $g: \omega \rightarrow \mathbb{N}$ *convex* if $g(0) = 1$ and the gaps $L_k = g(k+1) - g(k) \geq 1$ are non-decreasing. Every $f \in \omega^\omega$ satisfies $f(k) < g(k)$ for all $k \geq 1$ for some convex g : take $L_{-1} = 1$, $L_k = \max\{L_{k-1}, f(k+1)\}$ and $g(k+1) = g(k) + L_k$. For convex g let $x_n^g = 2^{-k}/L_k$ for $g(k) \leq n < g(k+1)$. Then $x^g \in \mathcal{D}$, and $\sum_{n \geq g(k)} x_n^g = 2^{1-k}$. For $y \in \mathcal{D}$ let $h_y(k) = \min\{m : \sum_{n \geq m} y_n < 2^{-k}\}$. If $x^g \prec y$, then $y_n \geq 2x_n^g$ for all $n \geq N$, so for every k with $g(k) \geq N$ we get $\sum_{n \geq g(k)} y_n \geq 2^{2-k}$, and hence $h_y(k) > g(k)$. Therefore, if $F \subseteq \mathcal{D}$ is $\prec$ -cofinal, then $\{h_y : y \in F\}$ is dominating. $\square$

Proposition 6.2. $\text{cov}(\mathcal{N}) \leq \text{cof}(\mathcal{D}, \prec)$.

Proof. For $k \in \omega$ let $b(k) = 2^{k+5}$ and $c_k = (k+1)^{-2}$, let $Q_0 = 4$ and $Q_{k+1} = Q_k 4^{b(k)+1}$, and let $P_{k,i} = Q_k 4^i$ for $i < b(k)$. Listed in increasing order, consecutive numbers $P_{k,i}$ have ratio at least 4. So the sets $J_{k,i} = \{n \in \mathbb{N} : P_{k,i}/4 < n \leq P_{k,i}\}$ are pairwise disjoint, $|J_{k,i}| = 3P_{k,i}/4$, and $J_{k,i} \subseteq W_k = \{n : Q_k/4 < n \leq Q_{k+1}/4\}$, where the W_k are pairwise disjoint. Let $X = \prod_k \{0, \dots, b(k) - 1\}$ with the product μ of the uniform probability measures; writing each coordinate in binary identifies (X, μ) with 2^ω and its usual measure.

For $x \in X$ let $p_k = P_{k,x(k)}$ and $p_{-1} = 0$, and let $y_n^x = c_k/p_k$ for $p_{k-1} < n \leq p_k$. Then y^x is positive and non-increasing, and $\sum_n y_n^x \leq \sum_k c_k$, so $y^x \in \mathcal{D}$. For $z \in \mathcal{D}$ let

$$S_z(k) = \{i < b(k) : z_{P_{k,i}} \geq c_k/P_{k,i}\}.$$

If $i \in S_z(k)$, then $z_n \geq c_k/P_{k,i}$ on $J_{k,i}$, so $\sum_{n \in J_{k,i}} z_n \geq 3c_k/4$. Summing over $i \in S_z(k)$ gives $|S_z(k)| \leq \frac{4}{3}(k+1)^2 \sum_{n \in W_k} z_n$, and therefore $|S_z(k)| \leq (k+1)^2$ for all $k \geq K_z$. Since $(k+1)^2 \leq b(k)/2$, the set $\{x \in X : x(k) \in S_z(k) \text{ for all } k \geq K\}$ has measure $\prod_{k \geq K} |S_z(k)|/b(k) = 0$ for every $K \geq K_z$, so the set $A_z = \{x \in X : x(k) \in S_z(k) \text{ for all large } k\}$ is μ -null. If $y^x \prec z$, then $z_{p_k} > y_{p_k}^x = c_k/p_k$ for all large k , that is, $x \in A_z$. Hence, if $F \subseteq \mathcal{D}$ is $\prec$ -cofinal, the null sets A_z ($z \in F$) cover X , and $\text{cov}(\mathcal{N}) \leq |F|$. $\square$

The proof shows why monotone series behave like the null ideal rather than like $(\omega^\omega, \leq^*)$: a non-increasing z can exceed c/P at a point P only by spending mass at least $3c/4$ on $(P/4, P]$, so being large at prescribed points is a localization problem.

7 Other readings and other orders

Remark 7.1 (Condition (b) as printed). Let $y_{4^k} = k^{-2}$ for $k \geq 1$ and $y_n = 2^{-n}$ for all other n . Then y is positive and summable. If z is non-increasing and $y_n \leq z_n$ for all $n \geq N$, then for $4^{k-1} \geq N$

$$\sum_{4^{k-1} < n \leq 4^k} z_n \geq 3 \cdot 4^{k-1} z_{4^k} \geq 3 \cdot 4^{k-1}/k^2,$$

which is unbounded; so z is not summable. Since $y_n/x_{n,\alpha} \rightarrow 0$ implies $y_n \leq x_{n,\alpha}$ for large n , condition (b) as printed fails for y and every family of non-increasing summable members, in ZFC and in particular under CH. For codes, Filmus gives a similar example [Fil13, Lemma 2.5], and the need for monotonicity is also noted in [Fil10].

Remark 7.2 (Members that are not monotone). Keep (a) and (b) as printed, but allow members that are not monotone. For $\beta \geq 1$, (a) forces x_β to be eventually positive, and changing finitely many terms affects neither (a) nor (b). So such a family exists if and only if there is a scale of length ω_1 in $(\ell_+^1, \prec)$, where ℓ_+^1 is the set of positive summable sequences. The proof of Lemma 3.1 works in ℓ_+^1 (omit the envelope $\bar{u}$, as $\sum_n u_n < \infty$). The pairs $(x \mapsto x, w \mapsto \text{a strict } \prec\text{-bound of } w + (2^{-n})_n)$ and $(x \mapsto x + (2^{-n})_n, z \mapsto z)$ are morphisms showing $(\ell_+^1, \prec) \equiv_T (\ell_+^1, \leq^*)$. Hence such a family exists if and only if $\text{cof}(\mathcal{N}) = \aleph_1$. The identity $\text{cof}(\ell_+^1, \leq^*) = \text{cof}(\mathcal{N})$ is due to Vojtáš [Voj88].

Remark 7.3 (Comparison of remainders). For $x, y \in \mathcal{D}$ write $x \triangleleft_r y$ if $\sum_{k \geq n} x_k / \sum_{k \geq n} y_k \rightarrow 0$. This comparison of convergent series by their remainders was studied set-theoretically by Kholshchevnikova [Kho83] and Vojtáš [Voj87]. For arbitrary convergent series with non-negative terms, Vojtáš showed that the unbounding number of this order is $\mathfrak{b}$ and noted that its dominating number is $\mathfrak{d}$ [Voj87]. The same holds on $\mathcal{D}$, where in fact $(\mathcal{D}, \triangleleft_r) \equiv_T (\omega^\omega, \leq^*)$. For $f \in \omega^\omega$ let g be the convex function from the proof of Proposition 6.1, with $f(k) < g(k)$ for $k \geq 1$, and let $z_n^g = 4^{-k}/L_k$ for $g(k) \leq n < g(k+1)$. For $y \in \mathcal{D}$ let $f_y(k) = \min\{m : \sum_{n \geq m} y_n < 8^{-k}\}$. The proof of Proposition 6.1 uses only remainders, so $(f \mapsto x^g, y \mapsto h_y)$ is a morphism from $(\omega^\omega, \leq^*)$ to $(\mathcal{D}, \triangleleft_r)$. If $f_y \leq^* f$, then $\sum_{n \geq g(k)} y_n < 8^{-k}$

for all large k ; the remainder of z^g at any n with $g(k) \leq n < g(k+1)$ is at least $4^{-k}/3$, so the ratio of remainders is below $3 \cdot 2^{-k}$ there, and $y \triangleleft_r z^g$. Hence $(y \mapsto f_y, f \mapsto z^g)$ is a morphism in the other direction. Since $x \prec y$ implies $x \triangleleft_r y$, every $\prec$ -cofinal subset of $\mathcal{D}$ is $\triangleleft_r$ -cofinal, and $\mathfrak{d} = \text{cof}(\mathcal{D}, \triangleleft_r) \leq \text{cof}(\mathcal{D}, \prec)$; this already gives the negative answer in ZFC. Moreover, Lemma 3.1 holds for $\triangleleft_r$, so by Lemma 3.2 the analogue of Problem 7.62 in which (a) and (b') compare remainders has a solution if and only if $\mathfrak{d} = \aleph_1$. For the ratio of terms, which is what Problem 7.62 asks about, the relevant invariant is $\text{cof}(\mathcal{N})$, and in the random model $\mathfrak{d} = \aleph_1 < \text{cof}(\mathcal{N})$.

Verification. The proofs above do not depend on computation. The elementary inequalities used in them were also checked numerically, as follows.

1. A program by the author, `verify_note.py` (standard library only, exact rational arithmetic, a few seconds), checks finite instances of the inequalities in Lemmas 3.1, 4.1 and 4.2, Lemma 4.3(ii), Propositions 5.1, 6.1 and 6.2 and Remarks 7.1 and 7.3: 9,732 checks on explicit and random sequences, all passed. A variant with deliberately tightened bounds fails, so the checks are not vacuous.
2. An earlier program, written separately, checks inequalities of the same kind in exact arithmetic, for Lemmas 3.1, 4.1 and 4.2, Propositions 6.1 and 6.2 and Remark 7.1.
3. An independent verification run, AI-assisted, re-derived every proof, read the problem in the source of [HL19] and the relevant parts of [Bar10, Fil13] in their arXiv sources, re-ran the earlier program, and wrote its own exact-arithmetic checks together with a negative control. It found no mathematical error. It identified the prior work of Filmus, Geschke and Vojtáš discussed in the introduction and asked for the corresponding credit and for corrections to the presentation, which have been made.
4. A second independent verification run, also AI-assisted, again re-derived every proof. It re-read the problem and the cited results of [Bar10, Fil13], including the numbering of the cited lemmas, in their arXiv sources, read [Voj87] in the journal scan and [Fil10], checked the DOIs of the references, and re-ran all programs from the source files, which reproduced the recorded outputs exactly. Its own exact-arithmetic program, `run2_checks.py`, makes 785,958 checks, among them Lemma 2.1 on small finite relational systems, Filmus's completion procedure [Fil13, Lemma 3.4] on finite data and the values in Table 1; all passed, and a negative control with eleven deliberately broken claims fails as expected. It found no mathematical error and asked for small corrections to notation, attributions and wording, which have been made.

These computations test finite instances only; the theorems rest on the proofs. All programs and outputs are in the source files.

Scope and priority. Corollary 1.2 answers Problem 7.62 of [HL19] in the reading (b'). As printed, condition (b) cannot be satisfied (Remark 7.1), and the variant with non-monotone members has the same answer as (b') (Remark 7.2). The negative answer in ZFC is not new in substance: it follows from [Fil13, Theorem 4.8], after Geschke's answer in [Fil10], together with Proposition 5.1, and also from Vojtáš's results on the remainder order [Voj87] (Remark 7.3). We found no source that connects these results with Hinkkanen's problem. As far as we could determine, Theorem 1.1, which confirms Todorčević's conjecture recorded in [Fil13], and the exact characterizations in Corollaries 1.2 and 1.3 are new. The deep ingredient, Bartoszyński's characterization of the null ideal, is classical and the remaining steps are short, so these facts may be known to experts. We read the problem in the arXiv source of [HL19], Filmus's paper and Bartoszyński's survey in their arXiv sources, and [Voj87] in a

scan of the journal; we know [Kho83, Voj88, Voj93, Fre91] only through their zbMATH reviews. We searched arXiv, Crossref, zbMATH, OpenAlex, Semantic Scholar and MathOverflow, and ran four general web searches. We found no later work on Problem 7.62 and no later resolution of Todorčević’s conjecture. MathSciNet and Google Scholar were not searched, and earlier printed versions of the problem list were not consulted. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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