

A Zero-or-Infinity Modulus Dichotomy on the Real Line

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Abstract

Let $E \subset [0, 1]$ be compact, regarded as a subset of the real axis in the plane. We prove that the 2-modulus of all curves joining E to its complement in the real axis is infinite whenever E has positive conformal capacity. This gives an affirmative answer to a question of A. A. Gonchar recorded as Problem 7.58 by Hayman and Lingham. If E has zero capacity, the same modulus is zero. The proof combines separated condenser approximations with the classical Sobolev trace theorem and critical rigidity of indicator functions. A strong approximation argument retains the capacity constraint on E even when E has zero length.

1 Statement and conventions

Identify the horizontal line $L = \mathbb{R} \times \{0\}$ with $\mathbb{R}$, and let $B_r = \{z \in \mathbb{R}^2 : |z| < r\}$. For disjoint sets $A, B \subset \mathbb{R}^2$, denote by $\Gamma(A, B)$ the family of rectifiable curves in the plane with one endpoint in A and the other in B . We use the modulus convention

$$M_2(\Gamma) = \inf_{\rho} \int_{\mathbb{R}^2} \rho^2 dz, \quad \int_{\gamma} \rho ds \geq 1 \quad (\gamma \in \Gamma), \quad (1)$$

where $\rho : \mathbb{R}^2 \rightarrow [0, \infty]$ is Borel measurable. In particular, this quantity is not reciprocal extremal length. Using curves parametrized on a compact interval makes their endpoints explicit.

For a bounded domain G and a compact $K \subset G$, write

$$\text{cap}(G, K) = \inf \left\{ \int_G |\nabla v|^2 dz : v \in C_c^\infty(G), v \geq 1 \text{ near } K \right\}. \quad (2)$$

One may restrict to nonnegative tests. One may also require the tests to equal 1 near K : truncate to $[0, 1]$ and mollify. The usual definition requiring $v \geq 1$ only on K has the same infimum, by first multiplying a test by $1 + \delta$ and then letting $\delta \downarrow 0$.

Theorem 1. *Let $E \subset [0, 1]$ be compact and let $F = L \setminus E$. Then*

$$M_2(\Gamma(E, F)) = \begin{cases} 0, & \text{cap}(B_2, E) = 0, \\ \infty, & \text{cap}(B_2, E) > 0. \end{cases} \quad (3)$$

The positive-capacity assertion is Problem 7.58 of Hayman–Lingham [1, p. 178], attributed to A. A. Gonchar and communicated by M. Vuorinen. Its hypothesis is $M_2(\Gamma(E, \partial B_2)) > 0$, which is equivalent to the capacity condition above. Indeed, the classical condenser identity is

$$\text{cap}(G, K) = M_2(\Gamma(K, \partial G; G \setminus K)), \quad (4)$$

where the interior of a connecting curve lies in $G \setminus K$; see [2, (1.1)–(1.2)]. Every plane curve from E to ∂B_2 contains a subcurve of this kind, obtained by its first hit of the circle and its last preceding visit to E . Conversely these subcurves belong to the plane family. Extending densities by zero proves equality of the two moduli.

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No length, smoothness or thickness assumption on E is imposed. In particular, the zero-length case cannot be discarded. The analytic ingredients of the proof are classical. Critical fractional Sobolev rigidity for integer-valued functions is recalled in [3, p. 2083]. We give an elementary proof of the indicator case and an explicit approximation argument for the step where almost-everywhere trace information alone is insufficient.

2 Two trace facts

Use the usual inhomogeneous Sobolev spaces. The seminorm

$$[g]_{1/2}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|g(x) - g(y)|^2}{|x - y|^2} dx dy = 2 \int_0^\infty \frac{\|g(\cdot + h) - g\|_2^2}{h^2} dh \quad (5)$$

is finite for $g \in H^{1/2}(\mathbb{R})$. The second equality follows from Tonelli and translation. Its equivalence to the Fourier definition follows from Plancherel and the change of variable $t = \xi h$ in the integral of $|e^{i\xi h} - 1|^2/h^2$.

Lemma 2. *Restriction to L extends to a bounded operator $T : H^1(\mathbb{R}^2) \rightarrow H^{1/2}(\mathbb{R})$. If $u \in H^1(\mathbb{R}^2)$ has compact support and $Tu = 0$, then there are $w_k \in C_c^\infty(\mathbb{R}^2 \setminus L)$ such that $w_k \rightarrow u$ in H^1 .*

Proof. For a smooth compactly supported u , use unitary Fourier transforms. The Fourier transform of $g(x) = u(x, 0)$ is

$$\widehat{g}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{u}(\xi, \eta) d\eta.$$

Cauchy–Schwarz with weight $1 + \xi^2 + \eta^2$ and $\int_{\mathbb{R}} (1 + \xi^2 + \eta^2)^{-1} d\eta = \pi(1 + \xi^2)^{-1/2}$ gives

$$\int_{\mathbb{R}} (1 + \xi^2)^{1/2} |\widehat{g}(\xi)|^2 d\xi \leq \frac{1}{2} \|u\|_{H^1(\mathbb{R}^2)}^2.$$

Density proves the first assertion.

For the second, almost every vertical slice is a one-dimensional Sobolev function with value zero at $y = 0$. This slice value agrees with the trace, as follows by smooth approximation and the boundedness of evaluation on $H^1(\mathbb{R})$. For $a > 0$, Cauchy–Schwarz and integration on the positive and negative half-lines give

$$\int_{\mathbb{R} \times (-a, a)} |u(x, y)|^2 dx dy \leq \frac{a^2}{2} \int_{\mathbb{R} \times (-a, a)} |\partial_y u(x, y)|^2 dx dy. \quad (6)$$

For example, on $0 < y < a$ use $|u(x, y)|^2 \leq y \int_0^y |\partial_y u(x, t)|^2 dt$ and interchange the integrations.

Take a smooth function η with $\eta = 0$ on $[-1, 1]$ and $\eta = 1$ outside $[-2, 2]$, and set $u_\varepsilon(x, y) = \eta(y/\varepsilon)u(x, y)$. The terms containing u and ∇u converge in L^2 by absolute continuity of the integral. The remaining derivative term is bounded, by (6) with $a = 2\varepsilon$, as follows:

$$\|\varepsilon^{-1} \eta'(y/\varepsilon) u\|_2^2 \leq 2 \|\eta'\|_\infty^2 \int_{\mathbb{R} \times (-2\varepsilon, 2\varepsilon)} |\partial_y u|^2 dx dy \rightarrow 0.$$

Thus $u_\varepsilon \rightarrow u$ in H^1 . Mollify each u_ε at a sufficiently small radius less than $\varepsilon/2$. Compact support is retained and the resulting smooth functions vanish in a strip about L . A diagonal choice proves the assertion. $\square$

Lemma 3. *If a measurable set $E \subset [0, 1]$ satisfies $\mathbf{1}_E \in H^{1/2}(\mathbb{R})$, then $|E| = 0$.*

Proof. Put $g = \mathbf{1}_E$. For $0 < h < 1$,

$$\int_{\mathbb{R}} x(g(x+h) - g(x)) dx = -h|E|.$$

The integrand is supported in $[-h, 1]$, where $|x| \leq 1$. Since the difference of the two indicators has values in $\{-1, 0, 1\}$,

$$\|g(\cdot + h) - g\|_2^2 = \|g(\cdot + h) - g\|_1 \geq h|E|.$$

Equation (5) now gives $[g]_{1/2}^2 \geq 2|E| \int_0^1 dh/h$. Its finiteness forces $|E| = 0$. $\square$

3 Finite modulus forces zero capacity

We prove the contrapositive of the positive-capacity assertion. The case $E = \emptyset$ is immediate, so assume that E is nonempty and that there is an admissible $\rho \in L^2(\mathbb{R}^2)$ for $\Gamma(E, L \setminus E)$.

For $j \geq 1$ define

$$F_j = \{(x, 0) : |x| \leq 3, \text{dist}((x, 0), E) \geq 1/j\}, \quad G_j = B_3 \setminus F_j.$$

The sets F_j are increasing compact subsets of F , separated from E , and their union is $F \cap ([-3, 3] \times \{0\})$. Each G_j is a domain: its upper and lower half-disks are connected through a small ball about any fixed point of E . Its boundary is contained in $F_j \cup \partial B_3$.

Set $\rho_0 = \frac{1}{2} \mathbf{1}_{\{1 < |z| < 3\}}$. Every curve from E to ∂B_3 crosses a radial distance of at least 2 in this annulus, so its ρ_0 -integral is at least 1. The ρ -integral of every curve from E to F_j is at least 1. Therefore $\rho + \rho_0$ is admissible for the condenser in (4), and

$$\text{cap}(G_j, E) \leq \|\rho + \rho_0\|_2^2 \leq 2\|\rho\|_2^2 + 4\pi =: C. \quad (7)$$

Here $\|\rho_0\|_2^2 = 2\pi$.

Choose $u_j \in C_c^\infty(G_j)$ such that $0 \leq u_j \leq 1$, $u_j = 1$ near E , and $\int |\nabla u_j|^2 \leq 4(C + 1)$. For this nonsharp bound, begin with a smooth capacity test of energy at most $C + 1$, double it, truncate to $[0, 1]$, and mollify inside G_j . The truncated function equals 1 on a neighborhood of E ; a sufficiently small mollification preserves a smaller plateau and compact support and does not increase Dirichlet energy.

Extend each u_j by zero. Poincaré's inequality on B_3 bounds their $H^1(\mathbb{R}^2)$ norms. After passing to a subsequence, $u_j \rightharpoonup u$ weakly in H^1 . Mazur's lemma gives finite convex combinations v_k of tails of this subsequence, with all original indices at least k , such that

$$v_k \longrightarrow u \quad \text{strongly in } H^1(\mathbb{R}^2). \quad (8)$$

Every v_k is smooth and compactly supported in B_3 , equals 1 on a neighborhood of E , and vanishes near F_m whenever $k \geq m$. The plateau neighborhood may depend on k ; its existence follows because each combination is finite.

The traces converge strongly in $L^2(L)$. They are 1 on E , eventually 0 on every F_m , and 0 outside $[-3, 3]$. It follows that

$$Tu = \mathbf{1}_E \quad \text{almost everywhere on } L.$$

By Lemmas 2 and 3, $|E| = 0$ and $Tu = 0$. This length conclusion alone does not imply zero capacity.

Apply the second part of Lemma 2, obtaining $w_k \in C_c^\infty(\mathbb{R}^2 \setminus L)$ with $w_k \rightarrow u$ in H^1 . Then $f_k = v_k - w_k \rightarrow 0$ in H^1 . Furthermore, $f_k = 1$ near E : the plateau of v_k meets a neighborhood of L disjoint from the compact support of w_k . Fix $\chi \in C_c^\infty(B_2)$ equal to 1 near $[0, 1]$. The functions χf_k are tests for (2), and

$$\|\nabla(\chi f_k)\|_2 \leq \|\chi\|_\infty \|\nabla f_k\|_2 + \|\nabla \chi\|_\infty \|f_k\|_2 \longrightarrow 0.$$

If nonnegative tests are required, truncate and mollify as before. Consequently $\text{cap}(B_2, E) = 0$. This proves that positive capacity forces infinite modulus.

4 Zero capacity gives zero modulus

Assume $\text{cap}(B_2, E) = 0$, and fix $\delta > 0$. For each positive integer n , choose an open neighborhood U_n of E whose closure is contained in B_2 and such that

$$U_n \subset \{z : \text{dist}(z, E) < 1/n\}.$$

There exists $h_n \in C_c^\infty(U_n)$, equal to 1 near E , with

$$\|\nabla h_n\|_2 \leq \delta 2^{-n}. \quad (9)$$

Indeed, capacity tests in B_2 have arbitrarily small Dirichlet energy and, by Poincaré, arbitrarily small H^1 norm. Multiply them by a fixed cutoff supported in U_n and equal to 1 near E . The multiplication estimate at the end of the preceding section proves (9). The cutoff is fixed before the test's energy is made small; no uniform derivative bound in n is needed.

The Borel function $\sigma = \sum_{n \geq 1} |\nabla h_n|$ satisfies $\|\sigma\|_2 \leq \delta$. For a curve γ from $e \in E$ to $f \in F$, compactness of E implies $\text{dist}(f, E) > 0$. Thus $h_n(e) = 1$ and $h_n(f) = 0$ for all sufficiently large n . The chain rule along a rectifiable curve gives

$$\int_{\gamma} |\nabla h_n| ds \geq 1$$

for all these n . Tonelli implies $\int_{\gamma} \sigma ds = \infty$. Hence σ is admissible for every curve in the family and $M_2(\Gamma(E, F)) \leq \delta^2$. Letting $\delta \downarrow 0$ completes the proof of Theorem 1.

5 Scope and verification

The two cases in (3) use the same modulus convention as the source question. No assertion for other exponents or arbitrary interfaces is needed. The passage from a zero almost-everywhere trace to small capacity tests is essential: replacing that passage by a claim that length zero implies capacity zero would be incorrect.

The proof is analytic. The accompanying script checks the translation moment and inequality on 1,530 exact rational finite-union examples, the annular energy, twelve monomial instances of the strip estimate, and the logarithmic divergence for an interval. These checks support the displayed calculations and do not establish the general theorem.

AI assistance was used in developing the argument, preparing the text, checking calculations and searching the literature. The manuscript has undergone a detailed originating-system self-audit and is an unrefereed preprint, without a claim of independent review or proof-assistant verification. A bounded literature search through 26 October 2026 found no matching resolution of the precise source question; exhaustive priority is not asserted. The capacity, trace and rigidity principles used here are classical.

References

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