

Martin Kernels without Green-Line Contact on an Annulus

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Abstract

Arsove and Johnson asked whether every normalized Martin kernel on a multiply connected Green domain agrees with the Harnack envelope along a Green line issuing from the normalization point. We give a negative answer on the circular annulus $e^{-\pi/2} < |z| < e^{\pi/2}$, with normalization point 1. For every outer boundary point whose argument differs from π by at most $1/40$, its Martin kernel is strictly smaller than the envelope throughout one common punctured neighborhood of 1. The gap is at least $|\log z|/25$ after the neighborhood is made small enough. The proof periodizes the elementary strip Poisson kernel and compares four normalized kernel gradients. The target gradients lie strictly inside their convex hull. Uniform Taylor estimates then rule out contact along any curve approaching the base point.

1 The question and the counterexample

For a Green domain D and $z_0 \in D$, write

$$H_{z_0}(z) = \sup\{h(z) : h > 0 \text{ is harmonic on } D, h(z_0) \leq 1\}.$$

This supremum is finite at each interior point by Harnack's inequality. Normalize each Martin kernel by $K_\zeta(z_0) = 1$. For the unit disk with base point 0, the kernel belonging to a boundary point agrees with H_0 along the radius toward that point. Problem 7.42 of Hayman–Lingham [1, pp. 173–174], attributed to M. Arsove and G. Johnson, asks whether the corresponding Green-line contact property persists on multiply connected domains. We use the usual normalization just stated; the repeated words and inconsistent subscripts in the printed definition do not affect this interpretation.

Set

$$A = \{z \in \mathbb{C} : e^{-\pi/2} < |z| < e^{\pi/2}\}, \quad z_0 = 1, \quad I = [\pi - 1/40, \pi + 1/40].$$

Let K_θ^+ denote the normalized Martin kernel at the outer boundary point $\zeta_\theta = e^{\pi/2+i\theta}$.

Theorem 1. *There exists $\varepsilon > 0$ such that, simultaneously for every $\theta \in I$ and every $w \in \mathbb{C}$ with $0 < |w| < \varepsilon$,*

$$H_1(e^w) - K_\theta^+(e^w) \geq \frac{|w|}{25} > 0. \tag{1}$$

Consequently, for these boundary points no nonconstant curve issuing from 1 can be a curve of equality $K_\theta^+ = H_1$. In particular the Green-line contact property in Problem 7.42 fails.

The counterexample uses an annulus with two analytic boundary components, rather than a puncture or an irregular boundary. It also supplies an arc of boundary points of positive harmonic measure for which contact fails locally. The classical Harnack metric, its infinitesimal description, and extremal Poisson-kernel ratios are discussed by Herron [2], including an exact annulus distance calculation. Our argument uses these elementary kernel ideas to exclude the particular contact property in the question.

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2 The normalized kernels

Use logarithmic coordinates $z = e^{u+iv}$, with $-\pi/2 < u < \pi/2$ and v taken modulo 2π . For $\sigma \in \{+1, -1\}$ define the angular Poisson densities

$$P^\sigma(u, v; \theta) = \frac{\cos u}{2\pi} \sum_{m \in \mathbb{Z}} \frac{1}{\cosh(v - \theta + 2\pi m) - \sigma \sin u}. \quad (2)$$

They are densities with respect to $d\theta$ on the outer ($\sigma = +1$) and inner ($\sigma = -1$) boundary components. Conversion to arclength multiplies each density by a constant on its component; that constant cancels upon normalization.

Lemma 2. *The functions in (2) are positive harmonic functions on A . Their normalizations*

$$K_\theta^\sigma(e^{u+iv}) = \frac{P^\sigma(u, v; \theta)}{P^\sigma(0, 0; \theta)} \quad (3)$$

are the minimal Martin kernels at $e^{\sigma\pi/2+i\theta}$. All derivatives in u, v, θ converge locally uniformly.

Proof. On a closed substrip, each summand and each of its derivatives of any fixed order is bounded, for large $|m|$, by a constant times $e^{-2\pi|m|}$. This gives the convergence assertions. Every summand is positive, harmonic, and periodic after summation.

For completeness, the strip Green function in coordinates $0 < x, a < \pi$, $y, b \in \mathbb{R}$, is

$$G_S(x, y; a, b) = \frac{1}{2} \log \frac{\cosh(y - b) - \cos(x + a)}{\cosh(y - b) - \cos(x - a)}. \quad (4)$$

It follows by mapping the strip to a half-plane, or by checking its zero boundary values and logarithmic pole. Summing (4) over $b + 2\pi m$ gives the Green function on the cylinder, hence on A . Differentiating at its two boundary components gives

$$\lim_{a \uparrow \pi} \frac{G_S(x, y; a, b)}{\pi - a} = \frac{\sin x}{\cosh(y - b) + \cos x}, \quad \lim_{a \downarrow 0} \frac{G_S(x, y; a, b)}{a} = \frac{\sin x}{\cosh(y - b) - \cos x}.$$

Substitution $x = u + \pi/2$ yields $2\pi P^+$ and $2\pi P^-$ after periodization. The normal derivative formula therefore gives (2); the Green-function quotient at e^{u+iv} and at 1 tends to (3) when its pole approaches the corresponding boundary point. These limits are locally uniform in the interior, also when the angular coordinate of the pole varies.

We recall directly why these boundary kernels are minimal. Every nonnegative harmonic function on the cylinder has a unique representation

$$h(u, v) = \sum_{\sigma=\pm 1} \int_0^{2\pi} P^\sigma(u, v; \theta) d\mu_\sigma(\theta) \quad (5)$$

by finite positive measures on its two boundary circles. One elementary construction is to exhaust the cylinder by smaller concentric cylinders. The mean of h over each circle is affine in u , so the total masses of their boundary-value measures stay bounded. Take weak limits of these measures and pass to the limit in the Poisson formula on each fixed compact interior subcylinder. The kernels of the exhausting cylinders converge uniformly there, giving (5). For uniqueness, take Fourier coefficients in v : for nonzero frequency n the coefficient satisfies $h_n'' - n^2 h_n = 0$, and for frequency zero it is affine. Its two endpoint limits recover the Fourier coefficients of the two measures. All Fourier coefficients determine a finite measure.

The representation of $P^\sigma(\cdot; \theta)$ has unit point mass at θ on component σ , and zero measure on the other component. If $0 \leq h \leq P^\sigma(\cdot; \theta)$, apply (5) to h and to $P^\sigma - h$. Uniqueness forces both representing measures to be supported at that same point. Thus h is a constant multiple of P^σ . Together with the Green quotient limit this proves the minimal Martin-kernel assertion. $\square$

3 Four competing gradients

Introduce the positive, even, 2π -periodic functions

$$S(t) = \sum_{m \in \mathbb{Z}} \operatorname{sech}(t + 2\pi m), \quad T(t) = \sum_{m \in \mathbb{Z}} \operatorname{sech}^2(t + 2\pi m).$$

Differentiation of (2) at $(u, v) = (0, 0)$ gives

$$\nabla_{u,v} K_\theta^\sigma(1) = \left(\sigma \frac{T(\theta)}{S(\theta)}, -\frac{S'(\theta)}{S(\theta)} \right). \quad (6)$$

In this formula and below, derivatives of a function on A mean derivatives after composing it with $(u, v) \mapsto e^{u+iv}$. In particular the four kernels with $\theta = 1, -1$ and $\sigma = +1, -1$ have gradients $(\pm a, \pm b)$, where

$$a = \frac{T(1)}{S(1)}, \quad b = -\frac{S'(1)}{S(1)}.$$

Lemma 3. *Both $a > 1/5$ and $b > 1/5$.*

Proof. The elementary estimates $2 < e < 3$, $e^3 > 20$, and $\pi > 3$ imply

$$s := \operatorname{sech} 1 > \frac{1}{2}, \quad \tanh 1 > \frac{1}{2}, \quad e^{2\pi} > 400.$$

For example $e^3 > 20$ follows already by summing its exponential series through degree eight. Using $\operatorname{sech} x < 2e^{-|x|}$, the remaining terms satisfy

$$R := \sum_{m \neq 0} \operatorname{sech}(1 + 2\pi m) < \frac{2(e + e^{-1})}{e^{2\pi} - 1} < \frac{8}{399} < \frac{1}{25}. \quad (7)$$

Thus $S(1) < 26/25$ and $T(1) > s^2 > 1/4$, whence $a > 25/104 > 1/5$. Also $|\tanh x| \leq 1$ gives

$$-S'(1) = s \tanh 1 + \sum_{m \neq 0} \operatorname{sech}(1 + 2\pi m) \tanh(1 + 2\pi m) > \frac{1}{4} - R > \frac{21}{100}.$$

It follows that $b > 21/104 > 1/5$. □

The kernel at the antipodal outer boundary point has gradient $(c, 0)$, with

$$c = \frac{T(\pi)}{S(\pi)} \leq \operatorname{sech} \pi < \frac{1}{10}.$$

Here $S'(\pi) = 0$ by symmetry, and T/S is a weighted average of the positive numbers $\operatorname{sech}(\pi + 2\pi m)$. This already places its gradient strictly inside the rectangle of Lemma 3. The following estimates make the conclusion uniform on an arc.

Lemma 4. *For every $\theta \in I$, write*

$$c_\theta = \frac{T(\theta)}{S(\theta)}, \quad d_\theta = -\frac{S'(\theta)}{S(\theta)}.$$

Then

$$0 < c_\theta < \frac{4}{39}, \quad |d_\theta| \leq \frac{1}{20}. \quad (8)$$

Proof. For all integers m one has $|\theta + 2\pi m| \geq \pi - 1/40$. The weighted-average observation yields

$$c_\theta \leq \operatorname{sech}(\pi - 1/40) < 2e^{-\pi} e^{1/40} < \frac{1}{10} \frac{40}{39} = \frac{4}{39}.$$

We used $e^\pi > 20$ and $e^x \leq (1-x)^{-1}$ for $0 \leq x < 1$, which follows by comparing power series. For $s(t) = \operatorname{sech} t$, the identities $s' = -s \tanh t$ and $s'' = s(1 - 2s^2)$ imply $|S'| \leq S$ and $|S''| \leq S$. Consequently

$$|(\log S)''| \leq \frac{|S''|}{S} + \left(\frac{S'}{S} \right)^2 \leq 2.$$

Since $S'(\pi) = 0$, the mean value theorem gives $|d_\theta| \leq 2|\theta - \pi| \leq 1/20$. □

4 Strict local separation

We state the elementary comparison principle used to turn the gradient estimates into a genuine gap.

Lemma 5. *Let $f_1, \dots, f_N$ and a family g_θ be real C^2 functions near $0 \in \mathbb{R}^d$, all taking value 1 there. Assume their second derivatives are uniformly bounded near 0 and*

$$\max_j \langle \nabla f_j(0), x \rangle - \langle \nabla g_\theta(0), x \rangle \geq \delta |x|$$

for all x and all parameters θ , with $\delta > 0$. Then for every $0 < \delta' < \delta$ there is a common neighborhood in which

$$\max_j f_j(x) - g_\theta(x) \geq \delta' |x|.$$

Proof. Taylor's theorem bounds the difference of the two remainders by $C|x|^2$, uniformly in j, θ . Choose j that maximizes its linear term at x . The left-hand side is at least $\delta|x| - C|x|^2$, which is at least $\delta'|x|$ for $|x|$ small. $\square$

Proof of Theorem 1. Take the four competitors $f_j = K_{\pm 1}^\sigma$ in logarithmic coordinates and let $g_\theta = K_\theta^+$, $\theta \in I$. They are positive harmonic functions, equal to 1 at the base point. For $x = (u, v)$, Lemmas 3 and 4 imply

$$\begin{aligned} \max_j \langle \nabla f_j(0), x \rangle - \langle \nabla g_\theta(0), x \rangle &= a|u| + b|v| - c_\theta u - d_\theta v \\ &\geq (a - c_\theta)|u| + (b - |d_\theta|)|v| \\ &\geq \frac{19}{195}|u| + \frac{3}{20}|v| \\ &\geq \frac{9}{100}\sqrt{u^2 + v^2}. \end{aligned}$$

The denominators in (3) are positive and continuous in θ . On the compact set I , their minimum is positive. Lemma 2 therefore gives uniformly bounded second derivatives on a fixed closed neighborhood of $(0, 0)$. Apply Lemma 5 with $\delta = 9/100$ and $\delta' = 1/25$. Since every competitor is admissible in the definition of H_1 , we have $H_1 \geq \max_j f_j$, and (1) follows.

Choose ε small enough that the exponential map is injective on $|w| < \varepsilon$. Any curve issuing from 1 and containing distinct points arbitrarily close to 1 enters this punctured neighborhood. At all of those points the gap is positive. It cannot be a curve of equality, regardless of whether it is a Green line. Finally, the angular Poisson density at 1 is positive on I , so the corresponding boundary arc has positive harmonic measure. $\square$

5 Scope and reproducibility

The construction answers the multiply connected extension question negatively; it does not classify which other boundary points or domains admit contact. In particular, minimality of a positive harmonic function does not force its gradient at the base point to lie on the boundary of the convex hull of all such gradients. The four gradients above make that distinction explicit.

The accompanying script checks the harmonicity and derivative identities symbolically and verifies the rational bounds used in the proof. As an independent numerical cross-check it compares the periodized kernels with the Fourier formulas

$$\begin{aligned} S(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos(nt)}{\cosh(n\pi/2)}, \\ T(t) &= \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{n \cos(nt)}{\sinh(n\pi/2)}, \\ 2\pi P^+(u, v; \theta) &= \frac{u + \pi/2}{\pi} + 2 \sum_{n=1}^{\infty} \frac{\sinh(n(u + \pi/2))}{\sinh(n\pi)} \cos(n(v - \theta)). \end{aligned}$$

It also checks Green-function quotient limits and sampled directional gaps. These finite-precision calculations support the normalization and algebra; the proof of the uniform continuum statement is the analytic argument above. No numerical radius is asserted.

The author used AI assistance in exploring the construction, checking the derivations, and preparing the manuscript. The accompanying records include a detailed self-audit of the proof and source scope; this preprint has not undergone independent peer review or formal proof-assistant verification. The literature search located no matching resolution of this contact question, but does not establish exhaustive priority. The kernel formulas and the comparison principle are classical ingredients.

References

- [1] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory (New Edition)*, arXiv:1809.07200v2 (2018), Problem 7.42, pp. 173–174. [arXiv:1809.07200](#).
- [2] D. A. Herron, The Harnack and other conformally invariant metrics, *Kodai Mathematical Journal* **10** (1987), 9–19. [doi:10.2996/kmj/1138037356](#).