

On Hayman's Problem on the Series $\sum f(c_n/a_n)$ Formed with Second Partial Sums: A Comparison Theorem and a Characterization

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Abstract

Problem 7.31 of Hayman and Lingham's collection *Research Problems in Function Theory*, proposed by W. K. Hayman, concerns sequences with $a_1 > 0$ and $0 \leq a_n \leq n$ and their partial sums $b_n = a_1 + \dots + a_n$, $c_n = b_1 + \dots + b_n$. It states that $\sum (a_n/c_n)^\alpha < \infty$ for $\alpha > \frac{1}{2}$, and asks for which functions f the series $\sum f(c_n/a_n)$ converges, and whether, under a regularity condition on f , it converges whenever $\sum f(n^2)$ does. We prove: if $\sqrt{x} f(x)$ is non-negative and non-increasing and $\sum f(n^2) < \infty$, then $\sum f(c_n/a_n) \leq 6 \sum f(n^2) + (2 \log(1/a_1) + 2 + 5 \log 2) f(1)$ for every such sequence, and if $\sum f(n^2) = \infty$, the series diverges for $a_n = n$. For an arbitrary real function f , the series converges for all such sequences if and only if $\int^\infty \sup_{t \geq u} \sqrt{t} |f(t)| du/u < \infty$; the answer is the same under the normalizations $a_n \leq 1$ and $b_n \leq n$. The proofs rest on the bound $\sum_{c_n \leq X a_n} (a_n/c_n)^{1/2} < 3 \log X + 2 \log(1/a_1) + 2 + 2 \log 2$ and on explicit sequences along which c_n/a_n is piecewise constant. Without a condition the comparison fails: there are a non-increasing C^∞ function, and a non-negative C^∞ function vanishing at all squares, with $\sum f(n^2) < \infty$ and $\sum f(c_n/a_n) = \infty$; the exponent $\frac{1}{2}$ cannot be lowered, and $a_n \leq n$ cannot be relaxed to $a_n \leq n\psi(n)$ with ψ unbounded. The problem quotes as known the case $f(x) = x^{-\alpha}$ and an analogue for c_n/b_n attributed to D. Borwein. Borwein's note of 1965, which we read, concerns positive non-decreasing sequences $b_n \leq nk_n$ with $k_n \geq 1$: if $xf(x)$ is positive and non-increasing and $\int^\infty f < \infty$, then $\sum k_n f(k_n c_n/b_n) < \infty$, hence $\sum f(c_n/b_n) < \infty$ if $b_n \leq Kn$. We extend the latter statement to $0 \leq a_n \leq Kn$. We did not read Monthly problem 5167 of Barry and Hayman (1964–65), the problem list of Anderson, Barth and Brannan (1977), in which the problem was proposed according to the tables of the collection, the later lists, or Update 7.31 of the 2019 edition of the collection (those we tried were not accessible to us), and the origin of the statement for $\alpha > \frac{1}{2}$ is not identified. Our literature search found no earlier answer; a search that finds nothing is not a proof of novelty, no priority is claimed, and the arguments are elementary and may be known to specialists. The computations reported are tests and are not used in the proofs. This is an unrefereed note.

1 Introduction

1.1 The problem

Problem 7.31 of the collection of Hayman and Lingham [HL18], proposed by W. K. Hayman, concerns real sequences $(a_n)_{n \geq 1}$ with

$$a_1 > 0, \quad 0 \leq a_n \leq n \quad (n \geq 1), \quad (1)$$

and their first and second partial sums

$$b_n = \sum_{\nu=1}^n a_\nu, \quad c_n = \sum_{\nu=1}^n b_\nu.$$

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The problem states that then $\sum (a_n/c_n)^\alpha < \infty$ if $\alpha > \frac{1}{2}$, and it asks two questions. For what other functions f is $\sum f(c_n/a_n) < \infty$? And is it true, for instance, that under some smoothness condition on f the series $\sum f(c_n/a_n)$ converges with $\sum f(n^2)$? It adds that the analogous result for c_n/b_n was obtained by Borwein [Bor65]: if $xf(x)$ is positive and non-increasing for $x \geq a > 0$ and $\sum f(n) < \infty$, then $\sum f(c_n/b_n) < \infty$. The update printed with the problem in [HL18] reads: “No progress on this problem has been reported to us.” The problem is record AMR-022-7031 of the UnsolvedMath collection [ula26], where it is listed as open.

We call a sequence satisfying (1) *admissible* and write $\mathcal{A}$ for the set of admissible sequences. For $a_n > 0$ we put

$$r_n = \frac{c_n}{a_n}.$$

Then $r_1 = 1$ and $r_n \geq 1 + a_1$ for $n \geq 2$ (Lemma 2.1), so only the values of f on $[1, \infty)$ matter. Four points have to be settled before the questions can be answered.

Terms with $a_n = 0$. The hypothesis allows $a_n = 0$ for $n \geq 2$. Then c_n/a_n is not defined, and in the series of the problem the term $(a_n/c_n)^\alpha$ is 0. Throughout, $\sum f(c_n/a_n)$ denotes the sum over the indices n with $a_n > 0$, in their natural order. (Any value $v \neq 0$ given to the other terms would make the series diverge for every sequence with infinitely many zeros, whatever f is.) The admissible sequences for which divergence is proved in Theorems 1.1 and 1.2 and in Theorem 1.3(a), (b) have $a_n > 0$ for every n .

The quantifier. The statement for $\alpha > \frac{1}{2}$ concerns all admissible sequences, and it is in this sense that the exponent $\frac{1}{2}$ is the threshold: for $a_n = n$ we have $r_n = \frac{1}{6}(n+1)(n+2)$. We therefore ask for which f the series converges for *every* admissible sequence.

“Converges with”. The sentence on Borwein’s theorem shows the intended direction: the convergence of $\sum f(n^2)$ should imply that of $\sum f(c_n/a_n)$. The converse cannot hold for each sequence separately. For the admissible sequence $a_n = 2^{-n}$ we have $b_n = 1 - 2^{-n}$, $c_n = n - 1 + 2^{-n}$ and $r_n = (n-1)2^n + 1$, so that $\sum (a_n/c_n)^{1/2} < \infty$ although $\sum 1/n = \infty$. The converse does hold in the sense that convergence for all admissible sequences forces $\sum |f(n^2)| < \infty$ (Theorem 1.2).

The normalization. Borwein’s note has a bound for b_n , in the simplest case $b_n \leq n$, where the problem has $a_n \leq n$ (see Section 1.3). Besides $\mathcal{A}$ we therefore consider the classes

$$\mathcal{A}_1 = \{(a_n) : 0 < a_n \leq 1 \text{ for all } n\} \subset \mathcal{A}_b = \{(a_n) : a_1 > 0, a_n \geq 0, b_n \leq n \text{ for all } n\} \subset \mathcal{A}.$$

(If $a_n \leq 1$ for all n , then $b_n \leq n$; if $b_n \leq n$, then $a_n \leq b_n \leq n$.) The answers below are the same for the three classes: the convergence statements are proved for $\mathcal{A}$, and the divergent sequences lie in $\mathcal{A}_1$.

1.2 Results

The first theorem answers the second question in the form suggested by the statement quoted from Borwein, with $\sqrt{x}f(x)$ in the place of $xf(x)$ and $\sum f(n^2)$ in the place of $\sum f(n)$.

Theorem 1.1 (comparison with $\sum f(n^2)$). *Let $f : [1, \infty) \rightarrow \mathbb{R}$ and $x_0 \geq 1$, and suppose that on $[x_0, \infty)$ the function f is non-negative and $\sqrt{x}f(x)$ is non-increasing.*

- (a) *If $\sum_n f(n^2) < \infty$, then $\sum f(c_n/a_n)$ converges for every admissible sequence. If $x_0 = 1$, then for every admissible sequence*

$$\sum_{n: a_n > 0} f\left(\frac{c_n}{a_n}\right) \leq 6 \sum_{n=1}^{\infty} f(n^2) + \left(2 \log \frac{1}{a_1} + 2 + 5 \log 2\right) f(1). \quad (2)$$

- (b) *If $\sum_n f(n^2) = \infty$, then $\sum f(c_n/a_n) = \infty$ for the admissible sequences $a_n = n$ and $a_n = 1$.*

In particular, $\sum (a_n/c_n)^\alpha < \infty$ for all admissible sequences if and only if $\alpha > \frac{1}{2}$.

With $h(y) = f(y^2)$ the theorem reads: if $yh(y)$ is non-negative and non-increasing for $y \geq y_0$ and $\sum h(n) < \infty$, then $\sum h((c_n/a_n)^{1/2}) < \infty$. This is the statement quoted from Borwein with c_n/b_n replaced by $(c_n/a_n)^{1/2}$.

The second theorem answers the first question. No regularity of f is assumed.

Theorem 1.2 (characterization). *Let $f: [1, \infty) \rightarrow \mathbb{R}$ be an arbitrary function and put*

$$\varphi^*(u) = \sup_{t \geq u} \sqrt{t} |f(t)| \in [0, \infty] \quad (u \geq 1).$$

The following statements are equivalent.

- (a) $\sum f(c_n/a_n)$ converges for every sequence in $\mathcal{A}$.
- (a') $\sum f(c_n/a_n)$ converges for every sequence in $\mathcal{A}_b$.
- (a'') $\sum f(c_n/a_n)$ converges for every sequence in $\mathcal{A}_1$.
- (b) $\sum |f(c_n/a_n)| < \infty$ for every sequence in $\mathcal{A}$.
- (c) There are $x_0 \geq 1$ and a function $F: [x_0, \infty) \rightarrow [0, \infty)$ such that $|f| \leq F$ on $[x_0, \infty)$, $\sqrt{x} F(x)$ is non-increasing, and $\sum_{n \geq \sqrt{x_0}} F(n^2) < \infty$.
- (d) $\int_{x_0}^{\infty} \varphi^*(u) \frac{du}{u} < \infty$ for some $x_0 \geq 1$.

Each of them implies $\sum_n |f(n^2)| < \infty$. Condition (d) holds if and only if $\sum_{j \geq j_0} \varphi^(4^j) < \infty$ for some j_0 .*

Condition (c) says that $|f|$ is dominated near infinity by a function to which Theorem 1.1(a) applies; so the monotonicity condition of Theorem 1.1 is, up to domination, also necessary. The function φ^* is non-increasing, so the integral in (d) is defined even if f is not measurable. In (a) the series is only required to converge. By (b), if it converges for every admissible sequence, then it converges absolutely for every admissible sequence; for a single sequence the convergence can be conditional (Remark 5.4(4)). Corollary 5.3 states the result for the functions for which $\sqrt{x} f(x)$ is almost decreasing: for these, convergence for all admissible sequences is equivalent to $\sum f(n^2) < \infty$.

The third theorem shows that some condition of this kind is needed in the second question, and that the exponent $\frac{1}{2}$ and the hypothesis $a_n \leq n$ cannot be weakened.

Theorem 1.3 (necessity of the conditions). *The following statements hold.*

- (a) For every $\theta \in [0, \frac{1}{2})$ there are a positive function f_θ on $[1, \infty)$ such that $x^\theta f_\theta(x)$ is non-increasing and $\sum_n f_\theta(n^2) < \infty$, and a sequence in $\mathcal{A}_1$ with $\sum f_\theta(c_n/a_n) = \infty$. For $\theta = 0$ the function can be chosen non-increasing and of class C^∞ .
- (b) There are a function $f \geq 0$ of class C^∞ on $[1, \infty)$ with $f(n^2) = 0$ for every integer $n \geq 1$, and a sequence in $\mathcal{A}_1$ with $\sum f(c_n/a_n) = \infty$.
- (c) Let $\psi: \mathbb{N} \rightarrow [1, \infty)$ be unbounded. There is a sequence with $a_1 > 0$ and $0 \leq a_n \leq n\psi(n)$ for all n such that $c_n/a_n \leq 3$ for infinitely many n . For this sequence $\sum (a_n/c_n)^\alpha = \infty$ for every real α .

So neither monotonicity nor smoothness of f alone makes the comparison with $\sum f(n^2)$ true; “ $x^\theta f(x)$ non-increasing” with $\theta < \frac{1}{2}$ is not enough; and the hypothesis $a_n \leq n$ cannot be replaced by $a_n \leq n^p$ with $p > 1$, nor by $a_n \leq n(1 + \log n)$. The sequences and the functions in (a) and (b) are explicit (Proposition 6.1).

The key estimate. Both convergence theorems rest on one inequality. For $X \geq 1$ let G_X be the set of the indices n with $a_n > 0$ and $c_n \leq Xa_n$. Lemma 2.2 shows that for every admissible sequence

$$W(X) := \sum_{n \in G_X} \left(\frac{a_n}{c_n} \right)^{1/2} < 3 \log X + 2 \log \frac{1}{a_1} + 2 + 2 \log 2. \quad (3)$$

The set G_X is finite, but its largest element is not bounded in terms of X and a_1 , and it can have more than $(\sqrt{X} - 1) \log(\frac{1}{2}\sqrt{X})$ elements. We do not know the best constants in (3). Proposition 7.1 shows what we can prove: in any inequality $W(X) \leq A \log X + B \log(1/a_1) + C$ valid for all admissible sequences and all $X \geq 1$ one must have $A \geq \frac{1}{2}\sqrt{6}$ and $B \geq 1$.

The divergent sequences. They follow a greedy rule: if $a_n = (b_{n-1} + c_{n-1})/(Y - 1)$, then $c_n/a_n = Y$ exactly. Proposition 4.2 shows that after a phase with the value $Y' \leq \frac{1}{4}Y$ the rule can be followed with the value Y for at least $\frac{3}{16}\sqrt{Y} \log(Y/Y')$ consecutive indices without leaving the class $\mathcal{A}_1$. Together with a selection lemma (Lemma 5.1) this gives the implication from (a'') to (d) in Theorem 1.2.

The ratio c_n/b_n . The method also gives, for all admissible sequences, the statement that the problem quotes for c_n/b_n (Theorem 8.2).

Organisation. Section 2 contains the key lemma, Sections 3, 5 and 6 the proofs of Theorems 1.1 to 1.3, Section 4 the greedy sequences, Section 7 the remarks on the constants of (3), and Section 8 the ratio c_n/b_n . Section 9 describes the computations, which are tests, and how the note was verified. Logarithms are natural, and $b_0 = c_0 = 0$.

1.3 Sources, and what is not claimed

The problem is Hayman’s. The convergence of $\sum (a_n/c_n)^\alpha$ for $\alpha > \frac{1}{2}$ and the result for c_n/b_n are quoted in the problem as known; the first is given there without proof or reference, and we could not identify its origin. Both are proved again here, as special cases of Theorems 1.1 and 8.2.

What Borwein’s note proves. We read the note [Bor65] (two pages) in the scan that is linked from the bibliography page “David Borwein at Ninety” of the CARMA centre. It considers a sequence $0 < u_1 \leq u_2 \leq \dots$ with $s_n = u_1 + \dots + u_n$; in the notation of Problem 7.31, $u_n = b_n$ and $s_n = c_n$. It recalls that Hayman and Barry had posed in the Monthly [BH64] the problem of showing that $\sum (u_n/s_n)^\alpha < \infty$ if $u_n \leq n$ and $\alpha > 1$, and it proves the following. *If $u_n \leq nk_n$ with $k_n \geq 1$, if $xf(x)$ is positive and non-increasing for $x \geq a > 0$, and if $\int_a^\infty f(x) dx < \infty$, then $\sum_{n \geq n_0} k_n f(k_n s_n / u_n) < \infty$ for a suitable n_0 .* A remark adds that for bounded (k_n) the conclusion can be replaced by $\sum_{n \geq n_0} f(s_n / u_n) < \infty$, and that for every unbounded (k_n) with nk_n non-decreasing there is such a sequence (u_n) with $s_n / u_n \leq 2$ for infinitely many n . The note contains nothing on a ratio of the type c_n/a_n .

Hence, for the unweighted conclusion, Borwein’s hypothesis is $b_n \leq Kn$, not $a_n \leq n$. Under (1) one only has $b_n \leq \frac{1}{2}n(n+1)$, and the sentence of the problem on c_n/b_n , read under (1), is not the theorem of [Bor65]. It is true nevertheless: Theorem 8.2 proves it, and thereby extends the unweighted form of Borwein’s theorem from non-decreasing sequences $b_n \leq Kn$ to non-decreasing sequences with increments $0 \leq b_n - b_{n-1} \leq Kn$ (Remark 8.3). Theorem 1.3(c) is the analogue for c_n/a_n of the counterexample in Borwein’s remark.

Where the problem was proposed, and what was not read. According to the tables at the end of the collection [HL18, HL19], Problems 7.23–7.32 were proposed in the problem list of Anderson, Barth and Brannan [ABB77] of 1977, and no progress on Problem 7.31 is recorded from the later lists. We could not read [ABB77], nor the later list [BBH84]; nor the Monthly problem of Barry and Hayman

[BH64] and its solution [BHB65] (here the names are in the order of the Crossref record; Borwein writes “Hayman and Barry”); nor Update 7.31 of the Springer edition [HL19] of the collection. Details, and the searches that were made, are in the paragraph “Scope and priority” at the end. No earlier answer to the two questions was found. A search that finds nothing is not a proof of novelty; we claim no priority. The arguments are elementary (the inequality of the arithmetic and geometric means, $1 - 1/x \leq \log x$, telescoping, summation by parts) and may be known to specialists.

2 The key lemma

Lemma 2.1 (elementary facts). *Let $(a_n) \in \mathcal{A}$.*

- (a) $0 < a_1 = b_1 \leq b_2 \leq \dots$; $c_n - c_{n-1} = b_n > 0$ and $c_n \geq na_1$ for $n \geq 1$; and $a_1 \leq 1$.
- (b) $a_n \leq b_n \leq c_n$ for $n \geq 1$, and $c_1 = b_1 = a_1$. If $n \geq 2$ and $a_n > 0$, then $r_n \geq 1 + a_1$.
- (c) $c_m \geq (m - n + 1)b_n$ for $1 \leq n \leq m$.
- (d) For $n \geq 2$,

$$\left(\frac{a_n}{c_n}\right)^{1/2} \leq \frac{1}{2} \log \frac{b_n}{b_{n-1}} + \frac{1}{2} \log \frac{c_n}{c_{n-1}},$$

and both logarithms are non-negative.

Proof. (a) The sequence (b_n) is non-decreasing because $a_n \geq 0$, and $b_1 = a_1 > 0$; hence $c_n = b_1 + \dots + b_n \geq nb_1$. The bound $a_1 \leq 1$ is (1) for $n = 1$. (c) $c_m \geq b_n + b_{n+1} + \dots + b_m \geq (m - n + 1)b_n$. (b) $b_n = b_{n-1} + a_n \geq a_n$ and $c_n = c_{n-1} + b_n \geq b_n$. For $n \geq 2$, by (a), $c_n = c_{n-1} + b_{n-1} + a_n \geq (n - 1)a_1 + a_1 + a_n$, so that $r_n \geq 1 + na_1/a_n \geq 1 + a_1$, because $a_n \leq n$. (d) By (a), $b_{n-1} > 0$ and $c_{n-1} > 0$ for $n \geq 2$, and $b_n \geq b_{n-1}$, $c_n \geq c_{n-1}$. By the inequality of the arithmetic and geometric means and by $1 - 1/x \leq \log x$ ($x > 0$),

$$\left(\frac{a_n}{c_n}\right)^{1/2} = \left(\frac{a_n}{b_n} \cdot \frac{b_n}{c_n}\right)^{1/2} \leq \frac{1}{2} \left(\frac{a_n}{b_n} + \frac{b_n}{c_n}\right) = \frac{1}{2} \left(1 - \frac{b_{n-1}}{b_n}\right) + \frac{1}{2} \left(1 - \frac{c_{n-1}}{c_n}\right) \leq \frac{1}{2} \log \frac{b_n}{b_{n-1}} + \frac{1}{2} \log \frac{c_n}{c_{n-1}}. \quad \square$$

For $(a_n) \in \mathcal{A}$ and $X \geq 1$ let

$$G_X = \{n \geq 1 : a_n > 0 \text{ and } c_n \leq Xa_n\}, \quad W(X) = \sum_{n \in G_X} \left(\frac{a_n}{c_n}\right)^{1/2}.$$

Lemma 2.2 (key lemma). *Let $(a_n) \in \mathcal{A}$ and $X \geq 1$.*

- (i) If $m \in G_X$ and $1 \leq n \leq m/2$, then $b_n < 2X$.
- (ii) If $n, m \in G_X$ and $m \geq 2n$, then $na_1 < 2X^2$.
- (iii) G_X is finite and contains 1.
- (iv) $W(X) < 3 \log X + 2 \log(1/a_1) + 2 + 2 \log 2$.
- (v) $|G_X| \leq \sqrt{X} W(X)$.

Proof. (i) By Lemma 2.1(c) and $n \leq m/2$, $c_m \geq (m - n + 1)b_n \geq (\frac{m}{2} + 1)b_n > \frac{m}{2}b_n$. Since $m \in G_X$ and $a_m \leq m$, we have $c_m \leq Xa_m \leq Xm$. Hence $\frac{m}{2}b_n < Xm$.

(ii) By (i), $a_n \leq b_n < 2X$. Since $n \in G_X$, Lemma 2.1(a) gives $na_1 \leq c_n \leq Xa_n < 2X^2$.

- (iii) $1 \in G_X$ because $c_1 = a_1 \leq Xa_1$. If G_X were infinite, every $n \in G_X$ would have some $m \in G_X$ with $m \geq 2n$, and (ii) would give $n < 2X^2/a_1$ for all $n \in G_X$; this is impossible for an infinite set.
(iv) Let $N = \max G_X$ and split G_X into

$$E = \{n \in G_X : 2n \leq N\}, \quad T = \{n \in G_X : 2n > N\}.$$

Every term of $W(X)$ is at most 1, by Lemma 2.1(b).

The sum over T . $N \in T$. Let $n_0 = \min T$; its term is at most 1. For the other elements of T we use Lemma 2.1(d). All logarithms there are non-negative, so the sum over the elements of T that are larger than n_0 is at most the full telescoping sum from $n_0 + 1$ to N :

$$\sum_{n \in T, n > n_0} \left(\frac{a_n}{c_n} \right)^{1/2} \leq \frac{1}{2} \sum_{n=n_0+1}^N \left(\log \frac{b_n}{b_{n-1}} + \log \frac{c_n}{c_{n-1}} \right) = \frac{1}{2} \log \frac{b_N}{b_{n_0}} + \frac{1}{2} \log \frac{c_N}{c_{n_0}}.$$

Here $b_N \leq c_N \leq Xa_N \leq XN$, because $N \in G_X$ and $a_N \leq N$. Further $c_{n_0} \geq n_0 a_1 > \frac{1}{2}Na_1$, because $2n_0 > N$, and $b_{n_0} \geq a_{n_0} \geq c_{n_0}/X > Na_1/(2X)$, because $n_0 \in G_X$. Therefore $c_N/c_{n_0} < 2X/a_1$ and $b_N/b_{n_0} < 2X^2/a_1$, and

$$\sum_{n \in T} \left(\frac{a_n}{c_n} \right)^{1/2} < 1 + \frac{1}{2} \log \frac{4X^3}{a_1^2}. \quad (4)$$

(If $T = \{N\}$, the sum is at most 1, and (4) holds because $4X^3/a_1^2 \geq 4$.)

The sum over E . Let $E \neq \emptyset$. Then $N \geq 2$, hence $1 \in E$; its term is $(a_1/c_1)^{1/2} = 1$. Let $n_E = \max E$, so that $n_E \leq N/2$. As before, by Lemma 2.1(d) and telescoping from 2 to n_E ,

$$\sum_{n \in E, n \geq 2} \left(\frac{a_n}{c_n} \right)^{1/2} \leq \frac{1}{2} \log \frac{b_{n_E}}{b_1} + \frac{1}{2} \log \frac{c_{n_E}}{c_1} = \frac{1}{2} \log \frac{b_{n_E}}{a_1} + \frac{1}{2} \log \frac{c_{n_E}}{a_1}.$$

By (i) with $m = N$ and $n = n_E$ we have $b_{n_E} < 2X$, and then $c_{n_E} \leq Xa_{n_E} \leq Xb_{n_E} < 2X^2$, because $n_E \in G_X$. Hence

$$\sum_{n \in E} \left(\frac{a_n}{c_n} \right)^{1/2} < 1 + \frac{1}{2} \log \frac{4X^3}{a_1^2}, \quad (5)$$

and this also holds if $E = \{1\}$ or $E = \emptyset$.

Adding (4) and (5) we obtain $W(X) < 2 + \log(4X^3/a_1^2) = 3 \log X + 2 \log(1/a_1) + 2 + 2 \log 2$.

(v) Every term of $W(X)$ is at least $X^{-1/2}$. $\square$

Remark 2.3. (1) The hypothesis $a_n \leq n$ is used in (i), in the form $c_m \leq Xm$, and for $b_N \leq c_N \leq XN$ in (iv); the hypothesis $a_1 > 0$ is used for $c_n \geq na_1$ and for $b_{n-1}, c_{n-1} > 0$. No monotonicity or regularity of (a_n) is needed, and zeros are allowed.

(2) The largest element of G_X is not bounded in terms of X and a_1 . For $M \geq 2$ let $a_1 = 1$, $a_n = 0$ for $1 < n < M$, and $a_M = M$. Then $b_M = M + 1$ and $c_M = (M - 1) + (M + 1) = 2M$, so that $M \in G_2$. Thus a single telescoping sum over all indices up to $\max G_X$ cannot give (iv). What is bounded, by (ii), is every element of G_X that has another element at least twice as large; all other elements lie in $(N/2, N]$. This is the reason for the splitting into E and T .

(3) By (iv) and (v), $|G_X| = O(\sqrt{X} \log X)$ for fixed a_1 . This order is attained: for every sequence of Proposition 4.2 with $s_0 = 2$, $s_1 = \sqrt{X} \geq 4$ and m_1 as in part (c) there, $G_X = \{1, \dots, N_1\}$ has more than $(\sqrt{X} - 1) \log(\frac{1}{2}\sqrt{X})$ elements.

3 Sufficiency

Theorem 3.1. *Let $(a_n) \in \mathcal{A}$, let $x_0 \geq 1$, and let $g: [x_0, \infty) \rightarrow [0, \infty)$ be non-increasing. Put $A = 3$ and $B = 2 \log(1/a_1) + 2 + 2 \log 2$. Then*

$$\sum_{n: a_n > 0, r_n \geq x_0} \frac{g(r_n)}{\sqrt{r_n}} \leq (A \log(2x_0) + B) g(x_0) + A \int_{x_0}^{\infty} g(u) \frac{du}{u}. \quad (6)$$

Proof. For $k \geq 0$ let

$$D_k = \{n : a_n > 0, 2^k x_0 \leq r_n < 2^{k+1} x_0\}, \quad w_k = \sum_{n \in D_k} r_n^{-1/2}, \quad g_k = g(2^k x_0),$$

and $V_K = w_0 + \dots + w_K$, $V_{-1} = 0$. The sets D_k are pairwise disjoint, their union is $\{n : a_n > 0, r_n \geq x_0\}$, and $D_0 \cup \dots \cup D_K \subset G_{2^{K+1}x_0}$; in particular they are finite. Since $r_n^{-1/2} = (a_n/c_n)^{1/2}$, Lemma 2.2(iv) gives

$$V_K \leq W(2^{K+1}x_0) \leq \alpha(K+1) + \beta, \quad \alpha = A \log 2, \quad \beta = A \log x_0 + B.$$

For $n \in D_k$ we have $g(r_n) \leq g_k$. Hence, by summation by parts and because $g_k - g_{k+1} \geq 0$ and $g_K \geq 0$,

$$\begin{aligned} \sum_{k=0}^K \sum_{n \in D_k} \frac{g(r_n)}{\sqrt{r_n}} &\leq \sum_{k=0}^K g_k (V_k - V_{k-1}) = \sum_{k=0}^{K-1} (g_k - g_{k+1}) V_k + g_K V_K \\ &\leq \sum_{k=0}^{K-1} (g_k - g_{k+1}) (\alpha(k+1) + \beta) + g_K (\alpha(K+1) + \beta) = \beta g_0 + \alpha \sum_{k=0}^K g_k. \end{aligned}$$

(In the last step the coefficient of g_k is $\alpha(k+1) - \alpha k = \alpha$ for $k \geq 1$ and $\alpha + \beta$ for $k = 0$.) For $k \geq 1$ the monotonicity of g gives

$$g_k \log 2 = g_k \int_{2^{k-1}x_0}^{2^k x_0} \frac{du}{u} \leq \int_{2^{k-1}x_0}^{2^k x_0} g(u) \frac{du}{u},$$

so that $\alpha \sum_{k=1}^K g_k \leq A \int_{x_0}^{\infty} g(u) \frac{du}{u}$, while $\alpha g_0 = A \log 2 \cdot g(x_0)$. All terms are non-negative; letting $K \rightarrow \infty$ we obtain (6), since $\beta + \alpha = A \log(2x_0) + B$. $\square$

Proof of Theorem 1.1. Put $g(x) = \sqrt{x} f(x)$ for $x \geq x_0$; this function is non-negative and non-increasing. The function $t \mapsto f(t^2) = g(t^2)/t$ is non-increasing on $[\sqrt{x_0}, \infty)$, as a product of two non-negative non-increasing functions, and the substitution $u = t^2$ gives

$$\int_{x_0}^{\infty} g(u) \frac{du}{u} = 2 \int_{\sqrt{x_0}}^{\infty} f(t^2) dt.$$

Let $n_0 = \lceil \sqrt{x_0} \rceil$. Then $\int_{\sqrt{x_0}}^{n_0} f(t^2) dt \leq f(x_0)$ and $\int_n^{n+1} f(t^2) dt \leq f(n^2)$ for $n \geq n_0$, hence

$$\int_{x_0}^{\infty} g(u) \frac{du}{u} \leq 2f(x_0) + 2 \sum_{n \geq n_0} f(n^2). \quad (7)$$

(a) Let $\sum f(n^2) < \infty$ and $(a_n) \in \mathcal{A}$. For the indices with $a_n > 0$ and $r_n \geq x_0$ we have $f(r_n) = g(r_n)/\sqrt{r_n} \geq 0$, and by Theorem 3.1 and (7) the sum of these terms is finite. The remaining indices

with $a_n > 0$ have $r_n < x_0$; they lie in the finite set G_{x_0} (Lemma 2.2(iii)), and f is finite-valued. So the series converges. If $x_0 = 1$, then $r_n \geq 1 = x_0$ for every n with $a_n > 0$, $g(1) = f(1)$, $n_0 = 1$ and $\int_1^\infty g(u) du/u \leq 2 \sum_{n \geq 1} f(n^2)$, so that (6) becomes

$$\sum_{n: a_n > 0} f(r_n) \leq (A \log 2 + B)f(1) + 2A \sum_{n \geq 1} f(n^2),$$

which is (2), because $A \log 2 + B = 2 \log(1/a_1) + 2 + 5 \log 2$.

(b) For $a_n = n$ we have $b_n = \frac{1}{2}n(n+1)$, $c_n = \frac{1}{6}n(n+1)(n+2)$ and $r_n = \frac{1}{6}(n+1)(n+2) \leq n^2$, the last inequality being $(5n+2)(n-1) \geq 0$. For $a_n = 1$ we have $b_n = n$ and $r_n = c_n = \frac{1}{2}n(n+1) \leq n^2$. In both cases $r_n \rightarrow \infty$, and for all n with $r_n \geq x_0$

$$f(r_n) = \frac{g(r_n)}{\sqrt{r_n}} \geq \frac{g(n^2)}{n} = f(n^2),$$

because g is non-negative and non-increasing and $x_0 \leq r_n \leq n^2$. So $\sum f(n^2) = \infty$ implies $\sum f(r_n) = \infty$.

Powers. For $\alpha > \frac{1}{2}$ the function $f(x) = x^{-\alpha}$ satisfies the hypothesis with $x_0 = 1$, and $\sum n^{-2\alpha} < \infty$; so (a) applies. For $\alpha \leq \frac{1}{2}$ we have $(a_n/c_n)^\alpha \geq (a_n/c_n)^{1/2}$, because $0 < a_n/c_n \leq 1$, and by (b) with $f(x) = x^{-1/2}$ the series $\sum (a_n/c_n)^{1/2}$ diverges for $a_n = n$. $\square$

4 Greedy sequences

Lemma 4.1 (one greedy step). *Let $s > 1$, $Y = s^2$, $b > 0$ and $c > 0$, and put*

$$a = \frac{b+c}{Y-1}, \quad b' = b+a, \quad c' = c+b'.$$

Then $c' = Ya$ and

$$c' + sb' = \frac{s}{s-1} (c + sb), \quad c' - sb' = \frac{s}{s+1} (c - sb). \quad (8)$$

Proof. $c' = c + b + a = (Y-1)a + a = Ya$. Further $b' = (Yb+c)/(Y-1)$ and $c' = Y(b+c)/(Y-1)$, hence

$$c' \pm sb' = \frac{(Y \pm s)c + Y(1 \pm s)b}{Y-1} = \frac{s(s \pm 1)(c \pm sb)}{(s-1)(s+1)},$$

because $Y \pm s = s(s \pm 1)$ and $Y(1 \pm s) = \pm s \cdot s(s \pm 1)$. $\square$

Proposition 4.2 (piecewise greedy sequences). *Let $s_0 \geq 2$ and let $s_1, s_2, \dots$ be real numbers with $s_l \geq 2s_{l-1}$ for all $l \geq 1$; put $Y_l = s_l^2$. Let $m_1, m_2, \dots$ be integers with*

$$1 \leq m_l \leq (s_l - 1) \log \frac{s_l}{s_{l-1}}, \quad (9)$$

and put $N_0 = 1$, $N_l = N_{l-1} + m_l$. Define $a_1 = 1$ and

$$a_n = \frac{b_{n-1} + c_{n-1}}{Y_l - 1} \quad \text{for } N_{l-1} < n \leq N_l, \quad l = 1, 2, \dots$$

(a) $a_n > 0$ and $c_n = Y_l a_n$ for $N_{l-1} < n \leq N_l$.

(b) $a_n \leq \frac{5}{16}$ for all $n \geq 2$. Hence $0 < a_n \leq 1$ and $b_n \leq n$ for all n , that is, $(a_n) \in \mathcal{A}_1$.

(c) The integer $m_l = \lfloor (s_l - 1) \log(s_l/s_{l-1}) \rfloor$ satisfies (9), and for it $m_l \geq \frac{3}{16} s_l \log(Y_l/Y_{l-1})$.

(d) For every function f on $[1, \infty)$ and every $L \geq 1$,

$$\sum_{n=1}^{N_L} f\left(\frac{c_n}{a_n}\right) = f(1) + \sum_{l=1}^L m_l f(Y_l). \quad (10)$$

Proof. (a) By induction $b_{n-1} > 0$ and $c_{n-1} > 0$, and $Y_l > 1$; so $a_n > 0$, and $c_n = Y_l a_n$ by Lemma 4.1. We call the indices with $N_{l-1} < n \leq N_l$ the *phase* l , and write $\lambda_l = s_l/(s_l - 1)$.

Claim 1: $c_n \leq s_l b_n$ for $N_{l-1} \leq n \leq N_l$ and every $l \geq 1$. We use induction on l . At $n = N_{l-1}$ the claim holds: for $l = 1$ because $c_1 = b_1 = 1 \leq s_1 b_1$, and for $l \geq 2$ by the claim for $l - 1$ at its last index and $s_{l-1} < s_l$. Inside the phase, the second identity in (8) with $s = s_l$ shows that $c_n - s_l b_n$ keeps its sign.

Claim 2: $\Psi_l := c_{N_l} + s_l b_{N_l} \leq (1 + 1/s_1) s_l^2/s_0$ for every $l \geq 1$. By the first identity in (8), $\Psi_l = \lambda_l^{m_l} (c_{N_{l-1}} + s_l b_{N_{l-1}})$. For $l = 1$ the bracket is $1 + s_1$. For $l \geq 2$ it is at most $(s_l/s_{l-1})\Psi_{l-1}$, because $s_l/s_{l-1} \geq 1$. By (9),

$$\lambda_l^{m_l} = \left(1 + \frac{1}{s_l - 1}\right)^{m_l} \leq \exp \frac{m_l}{s_l - 1} \leq \frac{s_l}{s_{l-1}}.$$

Hence $\Psi_1 \leq (s_1/s_0)(1 + s_1) = (1 + 1/s_1)s_1^2/s_0$ and $\Psi_l \leq (s_l/s_{l-1})^2 \Psi_{l-1}$ for $l \geq 2$, and the claim follows by induction.

(b) Let n be in phase l . Since $\lambda_l > 1$, the first identity in (8) gives $c_n + s_l b_n \leq \Psi_l$, and by Claim 1, $2c_n \leq c_n + s_l b_n$. By (a) and Claim 2,

$$a_n = \frac{c_n}{s_l^2} \leq \frac{\Psi_l}{2s_l^2} \leq \frac{1 + 1/s_1}{2s_0} \leq \frac{5/4}{4} = \frac{5}{16},$$

because $s_0 \geq 2$ and $s_1 \geq 4$. With $a_1 = 1$ this gives $0 < a_n \leq 1$ for all n , and then $b_n \leq n$.

(c) Let $x = (s_l - 1) \log(s_l/s_{l-1})$. Since $s_l \geq 4$ and $s_l/s_{l-1} \geq 2$, we have $x \geq 3 \log 2 > 2$. So $\lfloor x \rfloor \geq 1$ and $\lfloor x \rfloor \geq x - 1 \geq \frac{1}{2}x$. As $s_l - 1 \geq \frac{3}{4}s_l$, we obtain $\lfloor x \rfloor \geq \frac{3}{8}s_l \log(s_l/s_{l-1}) = \frac{3}{16}s_l \log(Y_l/Y_{l-1})$.

(d) Since $m_l \geq 1$, every $n \geq 2$ lies in exactly one phase, and $r_n = Y_l$ in phase l by (a); moreover $r_1 = 1$. $\square$

Remark 4.3. By (9), $\sum_{l \leq L} m_l/s_l \leq \log(s_L/s_0)$ for every sequence of Proposition 4.2, and $W(Y_L) = 1 + \sum_{l \leq L} m_l/s_l$; with the choice of (c) this is at least $1 + \frac{3}{16} \log(Y_L/Y_0)$. So $W(X)$ is of the exact order $\log X$ along $X = Y_L$, in accordance with Lemma 2.2(iv).

5 The characterization

Lemma 5.1 (selection). *Let $\varphi: [1, \infty) \rightarrow [0, \infty)$, and let $\varphi^*(u) = \sup_{t \geq u} \varphi(t) \in [0, \infty]$ for $u \geq 1$. Suppose that $\int_x^\infty \varphi^*(u) du/u = \infty$ for every $x \geq 1$. Then there are numbers $t_0 = 4 < t_1 < t_2 < \dots$ such that $t_k \geq 4t_{k-1}$ and $\varphi(t_k) > 0$ for all $k \geq 1$, and*

$$\sum_{k \geq 1} \varphi(t_k) \log \frac{t_k}{t_{k-1}} = \infty.$$

Proof. We have $\varphi^*(v) > 0$ for every $v \geq 1$; otherwise $\varphi^* = 0$ on $[v, \infty)$ and the integral from v vanishes. Put $t_0 = 4$ and choose recursively, with $v_{k-1} = 4t_{k-1}$, a number $t_k \geq v_{k-1}$ such that

$$\varphi(t_k) \geq \min\left(1, \frac{1}{2}\varphi^*(v_{k-1})\right) > 0.$$

This is possible because $\varphi^*(v_{k-1})$ is a supremum over $t \geq v_{k-1}$: if it is infinite, take t with $\varphi(t) \geq 1$; if it is finite, take t with $\varphi(t) > \frac{1}{2}\varphi^*(v_{k-1})$. Then $t_k \geq 4t_{k-1}$, $v_k \geq 4v_{k-1}$, and $v_k/v_{k-1} = t_k/t_{k-1}$. The function φ^* is non-increasing. If $\varphi^*(v_{k-1}) > 2$ for all k , then $\varphi(t_k) \geq 1$ and $\log(t_k/t_{k-1}) \geq \log 4$ for all k , and the series diverges. Otherwise there is k_0 with $\varphi^*(v_{k-1}) \leq 2$ for all $k \geq k_0$. For these k we have $\varphi^* \leq \varphi^*(v_{k-1}) \leq 2\varphi(t_k)$ on $[v_{k-1}, v_k]$, hence

$$\int_{v_{k-1}}^{v_k} \varphi^*(u) \frac{du}{u} \leq 2\varphi(t_k) \log \frac{v_k}{v_{k-1}} = 2\varphi(t_k) \log \frac{t_k}{t_{k-1}},$$

and summing over $k \geq k_0$ gives $\infty = \int_{v_{k_0-1}}^{\infty} \varphi^*(u) \frac{du}{u} \leq 2 \sum_{k \geq k_0} \varphi(t_k) \log(t_k/t_{k-1})$. $\square$

Proof of Theorem 1.2. Let $\varphi(t) = \sqrt{t}|f(t)|$, so that $\varphi^*(u) = \sup_{t \geq u} \varphi(t)$. We prove the implications

$$(d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) \Rightarrow (a') \Rightarrow (a'') \Rightarrow (d).$$

(d) $\Rightarrow$ (c). If $\int_{x_0}^{\infty} \varphi^*(u) \frac{du}{u} < \infty$, then $\varphi^*(u) < \infty$ for all $u > x_0$, because φ^* is non-increasing. Put $x_1 = x_0 + 1$ and $F(x) = \varphi^*(x)/\sqrt{x}$ for $x \geq x_1$. Then $|f| \leq F$, the function $\sqrt{x}F(x) = \varphi^*(x)$ is non-increasing, and $t \mapsto F(t^2) = \varphi^*(t^2)/t$ is non-increasing on $[\sqrt{x_1}, \infty)$. For integers $n \geq \sqrt{x_1} + 1$ this gives $F(n^2) \leq \int_{n-1}^n F(t^2) dt$, hence

$$\sum_{n \geq \sqrt{x_1} + 1} F(n^2) \leq \int_{\sqrt{x_1}}^{\infty} F(t^2) dt = \frac{1}{2} \int_{x_1}^{\infty} \varphi^*(u) \frac{du}{u} < \infty,$$

and at most one integer lies in $[\sqrt{x_1}, \sqrt{x_1} + 1)$. This is (c) with x_1 in the place of x_0 .

(c) $\Rightarrow$ (b). Let $(a_n) \in \mathcal{A}$ and $G(x) = \sqrt{x}F(x)$. As in (7),

$$\int_{x_0}^{\infty} G(u) \frac{du}{u} = 2 \int_{\sqrt{x_0}}^{\infty} F(t^2) dt \leq 2F(x_0) + 2 \sum_{n \geq \sqrt{x_0}} F(n^2) < \infty. \quad (11)$$

The indices with $a_n > 0$ and $r_n < x_0$ lie in the finite set G_{x_0} . For the others, $|f(r_n)| \leq F(r_n) = G(r_n)/\sqrt{r_n}$, and Theorem 3.1 with $g = G$ bounds their sum by the right-hand side of (6), which is finite by (11).

(b) $\Rightarrow$ (a) $\Rightarrow$ (a') $\Rightarrow$ (a''). Absolute convergence implies convergence, and $\mathcal{A}_1 \subset \mathcal{A}_b \subset \mathcal{A}$.

(a'') $\Rightarrow$ (d). Suppose that (d) fails, that is, $\int_x^{\infty} \varphi^*(u) \frac{du}{u} = \infty$ for every $x \geq 1$. Let $f_{\pm} = \max(\pm f, 0)$ and $\varphi_{\pm}(t) = \sqrt{t}f_{\pm}(t)$. Then $\varphi = \max(\varphi_+, \varphi_-)$ and $\varphi^* = \max(\varphi_+^*, \varphi_-^*) \leq \varphi_+^* + \varphi_-^*$. For each integer $j \geq 1$ at least one of the two integrals $\int_j^{\infty} \varphi_{\pm}^*(u) \frac{du}{u}$ is infinite; one sign $\sigma \in \{+, -\}$, which we also read as the number ± 1 , occurs for infinitely many j , and then $\int_x^{\infty} \varphi_{\sigma}^*(u) \frac{du}{u} = \infty$ for every $x \geq 1$, because this integral is non-increasing in x . Lemma 5.1, applied to φ_{σ} , gives numbers $t_0 = 4 < t_1 < \dots$ with $t_k \geq 4t_{k-1}$, $\varphi_{\sigma}(t_k) > 0$ and $\sum_k \varphi_{\sigma}(t_k) \log(t_k/t_{k-1}) = \infty$. The inequality $\varphi_{\sigma}(t_k) > 0$ means that $\sigma f(t_k) = |f(t_k)| > 0$.

Now apply Proposition 4.2 with $s_0 = 2$ and $s_k = \sqrt{t_k}$, so that $s_k \geq 2s_{k-1}$ and $Y_k = t_k$, and with the numbers m_k of part (c). The sequence obtained lies in $\mathcal{A}_1$, and by (10) and Proposition 4.2(c), for every L ,

$$\sigma \sum_{n=1}^{N_L} f\left(\frac{c_n}{a_n}\right) = \sigma f(1) + \sum_{k=1}^L m_k \frac{\varphi_{\sigma}(t_k)}{s_k} \geq \sigma f(1) + \frac{3}{16} \sum_{k=1}^L \varphi_{\sigma}(t_k) \log \frac{t_k}{t_{k-1}}.$$

The right-hand side tends to ∞ as $L \rightarrow \infty$. So the partial sums of $\sum f(c_n/a_n)$ are unbounded, and (a'') fails.

The remaining statements. By (c), $\sum_{n \geq \sqrt{x_0}} |f(n^2)| \leq \sum_{n \geq \sqrt{x_0}} F(n^2) < \infty$. For $u \in [4^j, 4^{j+1}]$ we have $\varphi^*(4^{j+1}) \leq \varphi^*(u) \leq \varphi^*(4^j)$, so that $\int_{4^j}^{4^{j+1}} \varphi^*(u) \frac{du}{u}$ lies between $\log 4 \cdot \varphi^*(4^{j+1})$ and $\log 4 \cdot \varphi^*(4^j)$; this gives the dyadic form of (d). $\square$

Remark 5.2. The equivalence of (c) and (d) does not depend on the sequences: besides (d) $\Rightarrow$ (c), proved above, (c) $\Rightarrow$ (d) follows directly. Indeed, if F is as in (c) and $G(x) = \sqrt{x} F(x)$, then $\varphi \leq G$ on $[x_0, \infty)$ and G is non-increasing, hence $\varphi^* \leq G$ there, and $\int_{x_0}^{\infty} \varphi^*(u) du/u < \infty$ by (11).

Corollary 5.3. *Let $f: [1, \infty) \rightarrow \mathbb{R}$, $x_0 \geq 1$ and $C \geq 1$, and suppose that $f \geq 0$ on $[x_0, \infty)$ and that $\sqrt{t} f(t) \leq C\sqrt{x} f(x)$ whenever $t \geq x \geq x_0$. Then $\sum f(c_n/a_n)$ converges for every admissible sequence if and only if $\sum_n f(n^2) < \infty$.*

Proof. If $\sum f(n^2) = \infty$, then (a) of Theorem 1.2 fails, by the last statement of that theorem. Conversely, let $\sum f(n^2) < \infty$ and put $F(x) = x^{-1/2} \sup_{t \geq x} \sqrt{t} f(t)$ for $x \geq x_0$. Then $f \leq F \leq Cf$, the function $\sqrt{x} F(x)$ is non-increasing, and $\sum F(n^2) \leq C \sum f(n^2) < \infty$. This is (c) of Theorem 1.2. $\square$

Remark 5.4. (1) *Scaling.* Let $K > 0$, and let (a_n) satisfy $a_1 > 0$ and $0 \leq a_n \leq Kn$. Then (a_n/K) is admissible and has the same ratios c_n/a_n and the same sets G_X . So Lemma 2.2(ii)–(v), Theorems 1.1, 1.2 and 3.1 and Corollary 5.3 hold for this class, with a_1 replaced by a_1/K in the constants; for the implication (a'') $\Rightarrow$ (d) one multiplies the sequences of Proposition 4.2 by $\min(K, 1)$.

(2) *The supremum in (d).* A single point t carries the weight $\log(t/t')$ of the whole gap between t and the previous point t' of the support. For instance, let $f(x) = \gamma_j x^{-1/2}$ for $x = 2^{2^j}$, $j \geq 1$, and $f(x) = 0$ otherwise. Then $\varphi^*(u) = \sup_{i \geq j} \gamma_i$ for $2^{2^{j-1}} < u \leq 2^{2^j}$. For $\gamma_j = 2^{-j}$ the integral in (d) is $\sum_j 2^{-j} \cdot 2^{j-1} \log 2 = \infty$, and $\sum f(c_n/a_n)$ diverges for some sequence in $\mathcal{A}_1$, although $\sum f(n^2) < \infty$; for $\gamma_j = 2^{-j} j^{-2}$ the integral is finite, and the series converges for every admissible sequence.

(3) *Subclasses.* The sequences of Proposition 4.2 are not monotone: a_n drops at the beginning of every phase. For subclasses of $\mathcal{A}$, for instance for non-decreasing admissible sequences, the convergence statements hold a fortiori, but we do not know whether (d) is still necessary.

(4) *Conditional convergence.* Let $f(x) = (-1)^j/(j\sqrt{x})$ for $x = T_j = 4 \cdot 16^j$, $j \geq 1$, and $f(x) = 0$ otherwise. Then $\varphi^*(u) = 1/j$ for $T_{j-1} < u \leq T_j$ (with $T_0 = 4$), the integral in (d) is $\sum_j j^{-1} \log 16 = \infty$, and by Theorem 1.2 the series $\sum f(c_n/a_n)$ diverges for some sequence in $\mathcal{A}_1$. For the sequence of Proposition 4.2 with $s_0 = 2$, $s_j = \sqrt{T_j} = 2 \cdot 4^j$ and $m_j = \lfloor (s_j - 1) \log 4 \rfloor$ it converges, but not absolutely. Indeed, by (10) the partial sums at the indices N_L are $\sum_{j \leq L} (-1)^j m_j / (j s_j)$, and $\log 4 - (1 + \log 4)/s_j < m_j/s_j \leq \log 4$; so these partial sums converge, while $\sum_j m_j / (j s_j) = \infty$. For $N_{L-1} < n \leq N_L$ all terms have the sign $(-1)^L$, so every partial sum lies between two consecutive ones of these.

6 Necessity of the conditions

Theorem 1.3(a) and (b) are contained in Proposition 6.1, and Theorem 1.3(c) is Proposition 6.2.

Proposition 6.1. *The following statements hold.*

(a) *Let $s_l = 2^{l^2+1}$ and $Y_l = s_l^2 = 4^{l^2+1}$ for $l \geq 0$, and for $\theta \in [0, \frac{1}{2})$ let*

$$f_\theta(x) = l^{-2} s_l^{2\theta-1} x^{-\theta} \quad (Y_{l-1} < x \leq Y_l, \quad l \geq 1), \quad f_\theta(x) = s_1^{2\theta-1} x^{-\theta} \quad (1 \leq x \leq Y_0).$$

Then $f_\theta > 0$, $x^\theta f_\theta(x)$ is non-increasing, $\sum_n f_\theta(n^2) \leq 2 + \frac{\pi^2}{6(1-2\theta)}$, and $\sum f_\theta(c_n/a_n) = \infty$ for the sequence of Proposition 4.2 with $s_0 = 2$, these s_l , and $m_l = \lfloor (s_l - 1)(2l - 1) \log 2 \rfloor$.

(b) *Let $\beta \in C^\infty(\mathbb{R})$ with $0 \leq \beta \leq 1$, $\beta(0) = 1$ and $\beta(x) = 0$ for $|x| \geq \frac{1}{2}$, let $Y_l = 3 \cdot 4^{l+1}$ for $l \geq 0$, and*

$$f(x) = x^{-1/2} \sum_{l \geq 1} \beta(x - Y_l) \quad (x \geq 1).$$

Then $f \geq 0$ is of class C^∞ , $f(n^2) = 0$ for every integer $n \geq 1$, and $\sum f(c_n/a_n) = \infty$ for the sequence of Proposition 4.2 with $s_l = \sqrt{Y_l} = \sqrt{12} \cdot 2^l$ and $m_l = \lfloor (s_l - 1) \log 2 \rfloor$.

- (c) Extend f_0 by the value $\frac{1}{4}$ to $(0, 1)$, let $\eta \geq 0$ be a C^∞ function on $\mathbb{R}$ with support in $[0, \log 2]$ and $\int \eta = 1$, and let $\tilde{f}(x) = \int f_0(xe^{-v}) \eta(v) dv$ for $x \geq 1$. Then $\tilde{f}$ is positive, non-increasing and of class C^∞ , $\sum_n \tilde{f}(n^2) < \infty$, and $\sum \tilde{f}(c_n/a_n) = \infty$ for the sequence of (a).

Proof. (a) We have $s_0 = 2$, $s_1 = 4$ and $s_l/s_{l-1} = 2^{2l-1} \geq 2$, so Proposition 4.2 applies, and $m_l = \lfloor (s_l - 1) \log(s_l/s_{l-1}) \rfloor$ is the number of Proposition 4.2(c). The function $x^\theta f_\theta(x)$ equals $s_1^{2\theta-1}$ on $[1, Y_1]$ and $l^{-2} s_l^{2\theta-1}$ on $(Y_{l-1}, Y_l]$; these values decrease with l , because $2\theta - 1 < 0$. So $x^\theta f_\theta(x)$ is non-increasing, and f_θ is positive and non-increasing; f_0 is a step function.

Convergence of $\sum f_\theta(n^2)$. The terms with $n^2 \leq Y_0$ are those with $n \leq 2$, and each is at most 1. For $l \geq 1$ the integers n with $Y_{l-1} < n^2 \leq Y_l$ are those in $(s_{l-1}, s_l]$, and

$$\sum_{n \leq s_l} n^{-2\theta} \leq \int_0^{s_l} x^{-2\theta} dx = \frac{s_l^{1-2\theta}}{1-2\theta}.$$

So these n contribute at most $l^{-2} s_l^{2\theta-1} \cdot s_l^{1-2\theta} / (1-2\theta) = l^{-2} / (1-2\theta)$ to $\sum f_\theta(n^2)$, and the bound follows from $\sum l^{-2} = \pi^2/6$.

Divergence. We have $f_\theta(Y_l) = l^{-2} s_l^{2\theta-1} s_l^{-2\theta} = 1/(l^2 s_l)$, and by the proof of Proposition 4.2(c), $m_l \geq \frac{3}{8} s_l (2l-1) \log 2$. By (10),

$$\sum_{n \leq N_L} f_\theta\left(\frac{c_n}{a_n}\right) \geq \sum_{l=1}^L \frac{m_l}{l^2 s_l} \geq \frac{3 \log 2}{8} \sum_{l=1}^L \frac{2l-1}{l^2} \geq \frac{3 \log 2}{8} \sum_{l=1}^L \frac{1}{l} \rightarrow \infty.$$

(This does not contradict Theorem 1.1: $\sqrt{x} f_\theta(x) = l^{-2} (x/Y_l)^{1/2-\theta}$ increases on each interval $(Y_{l-1}, Y_l]$ up to the value l^{-2} , and $\int \varphi^*(u) du/u = \sum_l l^{-2} (2l-1) \log 4 = \infty$.)

(b) The numbers $Y_l = 3 \cdot (2^{l+1})^2$ are integers, and consecutive ones differ by at least 144; so at most one term of the sum is non-zero near any point, and f is of class C^∞ and non-negative. Since 3 is not the square of a rational number, Y_l is not a perfect square; hence $|n^2 - Y_l| \geq 1$ for all integers n and $l \geq 1$, and $f(n^2) = 0$. Proposition 4.2 applies, because $s_0 = \sqrt{12} \geq 2$ and $s_l = 2s_{l-1}$, and $m_l \geq \frac{3}{8} s_l \log 2$ by the proof of Proposition 4.2(c). Since $f(Y_l) = Y_l^{-1/2} = 1/s_l$ and $f(1) = 0$, (10) gives

$$\sum_{n \leq N_L} f\left(\frac{c_n}{a_n}\right) = \sum_{l=1}^L \frac{m_l}{s_l} \geq \frac{3 \log 2}{8} L \rightarrow \infty.$$

(c) The extended function f_0 is positive and non-increasing on $(0, \infty)$. For $y \in \mathbb{R}$ we have $\tilde{f}(e^y) = \int f_0(e^{y-v}) \eta(v) dv = \int f_0(e^w) \eta(y-w) dw$, a convolution of a bounded measurable function with a C^∞ function of compact support; so $\tilde{f}$ is of class C^∞ . It is non-increasing, as an average of the non-increasing functions $x \mapsto f_0(xe^{-v})$. Since $\frac{1}{2}x \leq xe^{-v} \leq x$ on the support of η ,

$$f_0(x) \leq \tilde{f}(x) \leq f_0(x/2) \quad (x \geq 1).$$

The left inequality gives $\sum \tilde{f}(c_n/a_n) \geq \sum f_0(c_n/a_n) = \infty$ for the sequence of (a). For the right one: $f_0(n^2/2) = \frac{1}{4}$ for $n = 1, 2$, and for $l \geq 1$ at most $\sqrt{2} s_l$ integers n satisfy $n^2/2 \leq Y_l$; so $\sum_n \tilde{f}(n^2) \leq \sum_n f_0(n^2/2) \leq \frac{1}{2} + \sum_{l \geq 1} \sqrt{2} s_l / (l^2 s_l) = \frac{1}{2} + \sqrt{2} \pi^2/6$. $\square$

The numbers m_l of (a) do not depend on θ ; the first values are $m_1 = 2$, $m_2 = 64$, $m_3 = 3545$ and $m_4 = 635960$.

Proposition 6.2. Let $\psi: \mathbb{N} \rightarrow [1, \infty)$ be unbounded. There is a sequence (a_n) with $a_1 > 0$ and $0 \leq a_n \leq n\psi(n)$ for all n such that $c_n/a_n \leq 3$ for infinitely many n . Consequently $\sum (a_n/c_n)^\alpha = \infty$ for every real α .

Proof. Let $M_1 = 1$ and $a_1 = 1 \leq \psi(1)$. If M_{k-1} and $a_1, \dots, a_{M_{k-1}}$ are defined, choose $M_k > M_{k-1}$ with $\psi(M_k) \geq \max(b_{M_{k-1}}, c_{M_{k-1}})$, and put $a_n = 0$ for $M_{k-1} < n < M_k$ and $a_{M_k} = M_k \psi(M_k)$. Then $b_n = b_{M_{k-1}}$ for $M_{k-1} \leq n < M_k$, hence

$$c_{M_k} = c_{M_{k-1}} + (M_k - M_{k-1})b_{M_{k-1}} + a_{M_k} \leq c_{M_{k-1}} + M_k b_{M_{k-1}} + a_{M_k},$$

and

$$\frac{c_{M_k}}{a_{M_k}} \leq \frac{c_{M_{k-1}}}{M_k \psi(M_k)} + \frac{b_{M_{k-1}}}{\psi(M_k)} + 1 \leq 3.$$

So $(a_n/c_n)^\alpha \geq \min(1, 3^{-\alpha})$ for $n = M_k$, $k \geq 2$. $\square$

Proposition 6.2 is the analogue for c_n/a_n of the counterexample in Borwein's remark (Section 1.3), where $s_n/u_n \leq 2$ for infinitely many n .

7 On the constants of the key lemma

Proposition 7.1. *The following statements hold.*

- (a) *The constants 2 in Lemma 2.2(i) and (ii) cannot be replaced by smaller ones: for every integer $X \geq 4$ there are an admissible sequence with $a_1 = 1$ and an index n with $n \in G_X$, $2n \in G_X$,*

$$b_n = 2X \left(1 - \frac{2}{X}\right) \quad \text{and} \quad na_1 = 2X^2 \left(1 - \frac{1}{X}\right) \left(1 - \frac{5}{2X}\right).$$

- (b) *For $a_n = n$ and every $X \geq 1$, $W(X) \geq \frac{1}{2}\sqrt{6} \log X - \frac{3}{2}$.*

- (c) *For every $X > 1$ and every $\varepsilon \in (0, 1]$ there is an admissible sequence with $a_1 = \varepsilon$ and*

$$W(X) \geq \theta(X) \log \frac{1}{\varepsilon}, \quad \theta(X) = \left(\sqrt{X} \log \frac{\sqrt{X}}{\sqrt{X}-1} \right)^{-1} \in (0, 1), \quad \lim_{X \rightarrow \infty} \theta(X) = 1.$$

Consequently, if $W(X) \leq A \log X + B \log(1/a_1) + C$ for all admissible sequences and all $X \geq 1$, then $A \geq \frac{1}{2}\sqrt{6}$ and $B \geq 1$.

Proof. (a) Let $\alpha = 2X - 5$ and $n = (X - 1)\alpha$, and let $a_1 = 1$, $a_n = \alpha$, $a_{2n} = 2n$, and $a_k = 0$ for all other k . The sequence is admissible, because $0 < \alpha \leq n$. We have $b_k = 1$ for $k < n$, hence $b_n = 1 + \alpha$ and $c_n = (n - 1) + (1 + \alpha) = n + \alpha = X\alpha$; so $n \in G_X$. Further $b_k = 1 + \alpha$ for $n \leq k < 2n$ and

$$c_{2n} = c_n + n(1 + \alpha) + 2n = 4n + \alpha + n\alpha \leq 2nX = Xa_{2n},$$

the inequality being equivalent to $\alpha \leq n$; so $2n \in G_X$. Finally $b_n = 2X - 4$ and $na_1 = (X - 1)(2X - 5)$.

(b) Here $r_n = \frac{1}{6}(n + 1)(n + 2)$. Let N be the largest n with $(n + 1)(n + 2) \leq 6X$. Then $(N + 3)^2 > (N + 2)(N + 3) > 6X$ and, since $(n + 1)(n + 2) < (n + 2)^2$,

$$W(X) = \sum_{n=1}^N \left(\frac{6}{(n + 1)(n + 2)} \right)^{1/2} > \sqrt{6} \sum_{k=3}^{N+2} \frac{1}{k} > \sqrt{6} \left(\log(N + 3) - \frac{3}{2} \right) > \frac{\sqrt{6}}{2} \log(6X) - \frac{3\sqrt{6}}{2},$$

and $\frac{1}{2}\sqrt{6} \log 6 - \frac{3}{2}\sqrt{6} > -\frac{3}{2}$.

(c) Let $s = \sqrt{X}$, $\lambda = s/(s - 1)$, and $K = \lfloor \log(1/\varepsilon)/\log \lambda \rfloor$. Let $a_1 = \varepsilon$, $a_n = (b_{n-1} + c_{n-1})/(X - 1)$ for $2 \leq n \leq K + 1$, and $a_n = 0$ for $n > K + 1$. By Lemma 4.1, for $2 \leq n \leq K + 1$ we have $c_n = Xa_n$, $c_n + sb_n = \lambda^{n-1}(1 + s)\varepsilon$ and $c_n - sb_n = (s/(s + 1))^{n-1}(1 - s)\varepsilon \leq 0$. Hence $2c_n \leq c_n + sb_n$ and

$$0 < a_n = \frac{c_n}{s^2} \leq \frac{(1 + s)\lambda^{n-1}\varepsilon}{2s^2} \leq \lambda^K \varepsilon \leq 1 \leq n,$$

because $1 + s \leq 2s^2$. So the sequence is admissible, and $\{1, \dots, K+1\} \subset G_X$ with the terms 1 for $n = 1$ and $1/s$ for $n \geq 2$. Therefore

$$W(X) \geq 1 + \frac{K}{s} > 1 + \frac{1}{s} \left(\frac{\log(1/\varepsilon)}{\log \lambda} - 1 \right) > \frac{\log(1/\varepsilon)}{s \log \lambda} = \theta(X) \log \frac{1}{\varepsilon}.$$

Since $s \log \lambda = -s \log(1 - 1/s) > 1$ and $s \log \lambda \rightarrow 1$ as $s \rightarrow \infty$, we have $\theta(X) < 1$ and $\theta(X) \rightarrow 1$.

For the last statement, (b) gives $A \geq \frac{1}{2}\sqrt{6}$ on letting $X \rightarrow \infty$, and (c) gives $B \geq \theta(X)$ for every $X > 1$ on letting $\varepsilon \rightarrow 0$, hence $B \geq 1$. $\square$

The proof of Lemma 2.2(iv) gives $(A, B) = (3, 2)$. We do not know the best constants; the values found by computation are reported in Section 9.

8 The ratio c_n/b_n

For $(a_n) \in \mathcal{A}$ and $X \geq 1$ let

$$G'_X = \{n \geq 1 : c_n \leq Xb_n\}, \quad W'(X) = \sum_{n \in G'_X} \frac{b_n}{c_n}.$$

Here all indices are admitted, since $b_n > 0$ for every n .

Lemma 8.1. *Let $(a_n) \in \mathcal{A}$ and $X \geq 1$.*

- (i) $c_n \geq b_n^2/(2n)$ for all $n \geq 1$.
- (ii) If $n \in G'_X$, then $b_n \leq 2nX$ and $c_n \leq 2nX^2$.
- (iii) If $m \in G'_X$ and $1 \leq n \leq m/2$, then $b_n < 4X^2$.
- (iv) G'_X is finite and contains 1, and $W'(X) < 5 \log X + 2 \log(1/a_1) + 2 + 4 \log 2$.

Proof. (i) For $0 \leq j \leq n-1$ we have $b_{n-j} \geq b_n - jn$, because each of the j terms a_ν with $n-j < \nu \leq n$ is at most n . Let $J = \lfloor b_n/n \rfloor$. If $n \geq 2$, then $b_n \leq \frac{1}{2}n(n+1) < n^2$, so $J \leq n-1$; the same holds if $n = 1$ and $a_1 < 1$. Then

$$c_n \geq \sum_{j=0}^J b_{n-j} \geq \sum_{j=0}^J (b_n - jn) = (J+1) \left(b_n - \frac{nJ}{2} \right) \geq (J+1) \frac{b_n}{2} > \frac{b_n^2}{2n},$$

because $nJ \leq b_n < n(J+1)$. If $n = 1$ and $a_1 = 1$, then $c_1 = 1 > \frac{1}{2}$.

(ii) By (i), $b_n^2/(2n) \leq c_n \leq Xb_n$, so $b_n \leq 2nX$ and $c_n \leq Xb_n \leq 2nX^2$.

(iii) By Lemma 2.1(c) and (ii), $\frac{m}{2}b_n < (m-n+1)b_n \leq c_m \leq 2mX^2$.

(iv) $1 \in G'_X$ because $c_1 = b_1$. If $n, m \in G'_X$ and $m \geq 2n$, then $na_1 \leq c_n \leq Xb_n < 4X^3$ by (iii); as in the proof of Lemma 2.2(iii), G'_X is finite. Let $N = \max G'_X$, $E = \{n \in G'_X : 2n \leq N\}$ and $T = \{n \in G'_X : 2n > N\}$. For $n \geq 2$ we have $b_n/c_n = 1 - c_{n-1}/c_n \leq \log(c_n/c_{n-1})$, and every term of $W'(X)$ is at most 1. With $n_0 = \min T$ we obtain, by telescoping from $n_0 + 1$ to N ,

$$\sum_{n \in T} \frac{b_n}{c_n} \leq 1 + \log \frac{c_N}{c_{n_0}} < 1 + \log \frac{2NX^2}{Na_1/2} = 1 + \log \frac{4X^2}{a_1},$$

because $c_N \leq 2NX^2$ by (ii) and $c_{n_0} \geq n_0 a_1 > \frac{1}{2}Na_1$. If $E \neq \emptyset$, then $1 \in E$, with the term 1, and with $n_E = \max E$, by telescoping from 2 to n_E ,

$$\sum_{n \in E} \frac{b_n}{c_n} \leq 1 + \log \frac{c_{n_E}}{a_1} < 1 + \log \frac{4X^3}{a_1},$$

because $c_{n_E} \leq Xb_{n_E} < 4X^3$ by (iii) with $m = N$. Adding the two bounds gives $W'(X) < 2 + \log(16X^5/a_1^2)$. $\square$

Theorem 8.2. *Let $(a_n) \in \mathcal{A}$.*

- (a) *Let $x_0 \geq 1$ and let $g: [x_0, \infty) \rightarrow [0, \infty)$ be non-increasing. With $A' = 5$ and $B' = 2\log(1/a_1) + 2 + 4\log 2$,*

$$\sum_{n: c_n \geq x_0 b_n} \frac{b_n}{c_n} g\left(\frac{c_n}{b_n}\right) \leq (A' \log(2x_0) + B') g(x_0) + A' \int_{x_0}^{\infty} g(u) \frac{du}{u}.$$

- (b) *Let $f: [1, \infty) \rightarrow \mathbb{R}$ and $x_0 \geq 1$, and suppose that on $[x_0, \infty)$ the function f is non-negative and $xf(x)$ is non-increasing. If $\sum_n f(n) < \infty$, then $\sum_n f(c_n/b_n)$ converges; if $x_0 = 1$, then*

$$\sum_{n=1}^{\infty} f\left(\frac{c_n}{b_n}\right) \leq 5 \sum_{n=1}^{\infty} f(n) + \left(2\log \frac{1}{a_1} + 2 + 9\log 2\right) f(1).$$

If $\sum_n f(n) = \infty$, then $\sum_n f(c_n/b_n) = \infty$ for $a_n = 1$.

Proof. (a) This is the proof of Theorem 3.1 with c_n/b_n in the place of r_n , the weights b_n/c_n in the place of $r_n^{-1/2}$, and Lemma 8.1(iv) in the place of Lemma 2.2(iv).

(b) Let $g(x) = xf(x)$ for $x \geq x_0$. Then $f = g(x)/x$ is non-increasing on $[x_0, \infty)$, and $\int_{x_0}^{\infty} g(u) du/u = \int_{x_0}^{\infty} f(u) du \leq f(x_0) + \sum_{n \geq x_0} f(n)$. By (a) the sum of the terms with $c_n \geq x_0 b_n$ is finite, and the other indices lie in the finite set G'_{x_0} . For $x_0 = 1$ the integral is at most $\sum_{n \geq 1} f(n)$, and $A' \log 2 + B' = 2\log(1/a_1) + 2 + 9\log 2$. For $a_n = 1$ we have $c_n/b_n = \frac{1}{2}(n+1) \leq n$, hence $f(c_n/b_n) = g(c_n/b_n) b_n/c_n \geq g(n)/n = f(n)$ as soon as $\frac{1}{2}(n+1) \geq x_0$. $\square$

Remark 8.3. (1) *Relation to Borwein's theorem.* In the notation of Section 1.3, $u_n = b_n$ and $s_n = c_n$. For bounded (k_n) , Borwein's theorem concerns positive non-decreasing sequences with $b_n \leq Kn$. Such a sequence has increments $a_n = b_n - b_{n-1}$ with $0 \leq a_n \leq b_n \leq Kn$ and $a_1 = b_1 > 0$. So, by the scaling $a_n \mapsto a_n/K$, which leaves c_n/b_n unchanged, Theorem 8.2(b) contains the unweighted conclusion $\sum f(c_n/b_n) < \infty$ of [Bor65], and extends it to all sequences with $0 \leq a_n \leq Kn$, for which b_n may be as large as $\frac{1}{2}Kn(n+1)$. (For positive non-increasing f the conditions $\int^{\infty} f < \infty$ and $\sum f(n) < \infty$ are equivalent.) The proofs are different. The weighted conclusion of [Bor65] for unbounded (k_n) is not treated here.

(2) *No reduction to the ratio c_n/b_n .* One might try to derive Theorem 1.1 from Theorem 8.2 and a similar statement for the ratio b_n/a_n , writing $c_n/a_n = (c_n/b_n)(b_n/a_n)$. Under (1) there is no such statement. Let $a_n = n$ if n is a power of 2, and $a_n = 0$ otherwise. For $n = 2^k$ we have $b_n = 2^{k+1} - 1$, so that $b_n/a_n < 2$; hence $\sum h(b_n/a_n) = \infty$ for every positive non-increasing h , although $c_n/a_n = \frac{2}{3} \cdot 2^k + 1 - \frac{2}{3} \cdot 2^{-k} \rightarrow \infty$ along these n .

9 Computations

All theorems, propositions and lemmas of this note are proved without computation. The computations are tests: they evaluate the inequalities and the constructions on explicit sequences. The tested sequences are exact (integers over a common denominator), and membership in G_X and the identities are decided exactly; floating point enters only through square roots and logarithms, and in the searches, whose results are evaluated again exactly. For a given sequence, W is a step function which jumps at the values r_n , and all bounds increase with X ; so testing the thresholds $X = r_n$ tests all $X \geq 1$. The reproducibility package has six parts.

The program of this note (`writing_stage/check_note.py`) tests the statements with the constants as printed. (A) Lemma 2.2 at every threshold of 69,442 sequences (588,035 pairs of a sequence and a threshold): smooth, lacunary and greedy sequences with a_1 down to 10^{-30} , single terms after up to 10^{12} zeros, bursts all of whose indices exceed $2X^2/a_1$, greedy sequences with exact ties $c_n = Xa_n$, random sequences, and all 65,536 sequences of length 8 with $a_1 \in \{1, \frac{1}{2}, \frac{1}{8}, \frac{1}{64}\}$ and $a_n \in \{0, \frac{n}{4}, \frac{n}{2}, n\}$. The largest value of $W(X)$ divided by the bound of (iv) is 0.4978; the sums (5) and (4) reach 0.8502 and 0.8156 of their bound; in (i), $b_n/(2X)$ comes within $5 \cdot 10^{-12}$ of 1. (B) Theorem 3.1 in 1,016 tests with random x_0 and random step functions g (largest quotient of the two sides 0.3890) and (2) in 14,253 tests with powers and with the functions $x^{-1/2}$ cut off at the values r_n (0.4530), on 254 sequences. (C) Lemma 4.1 and Proposition 4.2 in exact rational arithmetic for 403 lists of values Y_l (98,036 indices; the largest a_n with $n \geq 2$ is 0.2388). (D) The numbers of Proposition 6.1: $m_1, \dots, m_4$ as stated; the partial sums of $\sum f_0(c_n/a_n)$ over L complete phases are 0.75, 1.25, 1.6347, 3.0228 and 4.8448 for $L = 1, 2, 3, 10, 40$, while $\sum_n f_0(n^2) = 1.609629$. (E) Proposition 6.2 for three functions ψ . (F) Proposition 7.1. (G) Lemma 8.1 at every threshold of 12,752 sequences, each extended by zeros (347,759 pairs; largest quotient 0.6002), and the inequality of Theorem 8.2(b) for powers. (H) The examples of the introduction and of Remarks 2.3 and 8.3. The run reports no failure.

The programs with which the results were first obtained (`original/`) test the statements in their first form, in which the bound of Lemma 2.2(iv) was $\frac{9}{2} \log X + 3 \log(1/a_1) + 2 + 2 \log 2$, the right-hand side of (2) was $9 \sum f(n^2) + (7 + 3 \log(1/a_1))f(1)$, and the bound of Lemma 8.1(iv) was $7 \log X + 3 \log(1/a_1) + 2 + 5 \log 2$. Recorded: 53,900 pairs of a sequence and one of 14 fixed values of X (largest quotient 0.3346); all 262,144 sequences of length 9 on a grid of four values, at 7 values of X (0.2964), and eight hill climbs (0.3321); 5,096 and 6,000 tests of the two forms of the sufficiency theorem (0.2958 and 0.2879); Proposition 4.2 exactly for 48 lists (53,689 indices); and the examples of Section 6.

An earlier check (`earlier_check/`), AI-assisted and made with the first version, with programs of its own, covers in its recorded run 2,094,005 sequences for Lemma 2.2 (largest quotient 0.3362), 3,280,027 and 922,280 tests of the sufficiency theorem, 10,441 lists for Proposition 4.2 and 358,186 sequences for Lemma 8.1, without a violation. Six of its nine programs run for a fixed time, so these counts are not reproduced exactly by a new run.

The programs of the two independent verification runs of the first version, runs A and B of the next paragraph, are in `verification_run_A/` and `verification_run_B/`. When this note was written, all programs with a recorded output were run again, except the six with a time budget and the search of run B; the outputs agree with the recorded ones up to running times.

The programs of the final verification run (`independent_run_2/`) test the statements as printed, at every threshold and with the zero tail: Lemmas 2.2 and 8.1 on all 3,155,623 sequences of nine grids (27.0 and 43.1 million pairs of a sequence and a threshold; largest quotients 0.3968 and 0.4959), on 6,104 structured and random sequences with a_1 down to 10^{-300} and single terms after up to 10^{15} zeros (for Lemma 8.1 on 4,615 of them; 0.4977 and 0.6157; the sums (5) and (4) reach 0.7318 and 0.8521 of their bound) and in a randomized search (150 configurations of 40,000 steps; 0.4827 and 0.5980); the constructions of Sections 4, 6 and 7, among them Proposition 4.2 for 362 lists and the inequality $a_n \leq \frac{5}{16}$ of Proposition 6.1(a) at $n = 639,572$ in exact integer arithmetic; (2) and the inequalities

of Theorems 3.1 and 8.2 in 607,935 tests on 5,268 sequences (largest quotient 0.4809); and seven functions with point support chosen to separate condition (d) of Theorem 1.2 from the truth, with the example of Remark 5.4(4). No violation was found.

Observed size of W . For $a_1 = 1$ run B searched for sequences with large $W(X)$. The largest values found, each evaluated again exactly, are 2.2306, 4.3010, 7.8577, 11.0299 and 13.8586 for $X = 3, 10, 10^2, 10^3$ and 10^4 , that is, $W(X)/\log X = 2.03, 1.87, 1.71, 1.60$ and 1.50 . For $X = 10^5$ and 10^6 the search did not improve on the greedy sequence (15.8785 and 18.3520); for $X = 10^{10}$ the sequence $a_n = n$ gives 28.7098, that is, $1.25 \log X$. For small a_1 the values found exceed $\log(1/a_1)$ by amounts comparable with those for $a_1 = 1$. The largest quotient of $W(X)$ by the bound of Lemma 2.2(iv) in the tests of its present form is 0.501; for $W'(X)$ and Lemma 8.1(iv) it is 0.6157. These values are lower bounds for the suprema, found by heuristic searches; they are consistent with Proposition 7.1 and do not determine the best constants.

Weak points of the tests. The first of the original programs evaluates each sequence at 14 fixed values of X up to 10^{10} , not at its thresholds. In the first form of the proof of Lemma 2.2(iv) the set G_X was split at $2X^2/a_1$; in that program the upper part is non-empty for only 243 of the 53,900 pairs, so its bound was rarely exercised. Run B closed this gap with 864 configurations in which that part has 150,176 elements in all (its sum reaches 0.81 of its bound). Two of the eight hill climbs of the original programs ($X = 10^4$ and 10^5) use sequences too short to carry information, and those with $a_1 = 1$ stay 0.5 to 6 per cent below the values found by run B. The original test of Lemma 8.1 sums over the stored prefix of each sequence only, at six values of X ; run B, part (G) above and the final run include the zero tail. Proposition 6.1(c) was tested numerically only through the two inequalities used in its proof, by the final run for one function η . A statement tested on examples is not proved by the examples.

Verification. The results were first obtained, with proofs and with the programs in `original/`, with the constants $\frac{9}{2}$ and 3 in Lemma 2.2(iv). The earlier check mentioned above followed. Then two independent verification runs, both AI-assisted, examined this first version, each with programs of its own; we call them run A and run B. After the note had been written, a further independent verification run, also AI-assisted and with programs of its own, examined the final text; we call it the final run.

Run A re-derived every statement and every proof line by line and found no error, no gap and no wrong constant. It compared the statement of the problem with the $\text{\TeX}$ sources of both arXiv versions of [HL18], which agree for this problem, and it read Borwein's note; the description of [Bor65] in Section 1.3, the class $\mathcal{A}_b$ in Theorem 1.2, Remark 5.2, and the presentation of Section 8 and Proposition 6.2 go back to this run. Its programs tested Lemma 2.2 in its first form at 18.6 million pairs of a sequence and a threshold (largest quotient 0.3346) and in 7,680 simulated-annealing searches (0.3335); the sufficiency theorem in 20,041 tests; Proposition 4.2 exactly on 803 lists (271,840 indices; with the numbers m_l of part (c) doubled, $a_n \leq 1$ fails); the examples of Section 6; Lemma 8.1 in its first form; and five functions chosen to separate condition (d) of Theorem 1.2 from the truth, among them the two of Remark 5.4(2), one of variable sign and one that is unbounded. In all five cases the partial sums behave as Theorem 1.2 predicts.

Run B made independent computations and a literature search. It proved the bound of Lemma 2.2(iv) with the constants 3 and 2 by the splitting used here, and Proposition 7.1. It tested both forms of the bound, without a violation: on 865,376,700 sequences in 18 exhaustive enumerations (a shorter enumeration of a grid being contained in the longer one), on 9,194 structured and random sequences with a_1 down to 10^{-300} , on the 864 configurations mentioned above, and in the search described above; the largest quotients are 0.337 for the first form and 0.501 for the present one. It rebuilt the sequences of Proposition 6.1 exactly (in (a) the first three phases index by index and the

last index, $n = 639,572$, of the fourth), ran the six original programs again, with outputs identical to the recorded ones up to running times, and examined them for vacuous tests; none was found, and the weak points are those listed above. All corrections required by the two runs were made.

When the note was written, the following was changed after these two runs. Lemma 2.2(iv) is stated with the constants of run B and proved by its splitting, and Theorem 3.1 and (2) carry the constants that follow from it. Lemma 8.1(iv) is proved by the same splitting, which gives the constants 5, 2 and $2 + 4 \log 2$ instead of 7, 3 and $2 + 5 \log 2$; the explicit inequality and the divergence statement in Theorem 8.2(b) were added. The proofs of Proposition 7.1 were written out.

The final run examined the text as a whole, and these changes first. It re-derived every proof line by line and found no wrong theorem, proposition or lemma, no gap and no wrong constant. It compared the statement of the problem with the source, and every sentence of this note about Borwein's note with the scan, again, and repeated the literature search. It required nine corrections of the text and of the package, which were made. One concerns a fact: the text had named the list [BBH84] of 1984 as the probable first appearance of Problem 7.31, whereas the tables of the collection name the list [ABB77] of 1977. Two concern mathematics: an unproved remark that the constants of (3) are not sharp was replaced by the statement that the best constants are not known, and the sentence on conditional convergence after Theorem 1.2 was made precise, with Remark 5.4(4) added. The programs of this run found no violation; it also ran the quick part of the package again from the archive, and all programs reproduce the recorded outputs. No proof depends on a computation.

Scope and priority. The note answers both questions of Problem 7.31 for the class of all admissible sequences: Theorem 1.1 answers the second question, with the converse for the class of all admissible sequences, and Theorem 1.2 answers the first by a necessary and sufficient condition. Three things are not treated: subclasses such as monotone sequences (Remark 5.4(3)); the analogous question for sums iterated more than twice; and the best constants in Lemma 2.2(iv).

The problem is due to W. K. Hayman [HL18]. The convergence of $\sum (a_n/c_n)^\alpha$ for $\alpha > \frac{1}{2}$ and the result for c_n/b_n are quoted in the problem as known. For the first, the collection gives no proof and no reference, and we could not identify its origin. For the second, it refers to Borwein [Bor65], whose note proves what is described in Section 1.3. Theorem 8.2 extends its unweighted conclusion to increments $0 \leq a_n \leq Kn$; it does not cover the weighted theorem. According to Borwein's note, the Monthly problem of Hayman and Barry is the statement $\sum (u_n/s_n)^\alpha < \infty$ for $u_n \leq n$ and $\alpha > 1$, that is, a statement on b_n/c_n and not the statement on a_n/c_n of Problem 7.31.

What was read. Problem 7.31 and its update in version 2, the latest, of the arXiv text [HL18], in the $\text{\TeX}$ source and in the PDF, and the three tables at the end of this text. Borwein's note [Bor65], completely, in the scan named in Section 1.3: by run A, when the note was written, and by the final run, each time with the same result. Of the Springer edition [HL19], the front matter, the appendix with Tables A.1–A.3 and the list of references, which are freely accessible on the publisher's site.

What was not read. The proposal [BH64] and the solution [BHB65] of Monthly problem 5167 were not accessible to us. Borwein cites the year and the page 99 only; that the problem he cites is problem 5167 of Barry and Hayman, the pages of the proposal and the names attached to the solution are taken from Crossref records. The problem list [ABB77] of 1977 was not accessible to us. According to Table 2 of [HL18] (Table A.2 of [HL19]), Problems 7.23–7.32 were proposed in it (the tables add Hayman as a fourth author). So the original wording of the problem, and what [ABB77] says about the statement for $\alpha > \frac{1}{2}$, are not known to us. The later list [BBH84] was not accessible to us either, and the other later lists were not consulted; the table of comments on the problems (Table 3 of [HL18], Table A.3 of [HL19]) has no entry for Problem 7.31. The Semantic Scholar index lists one work citing [Bor65]; its record carries the identifier of [BBH84] and the authors of [ABB77]. Crossref and OpenAlex list no citing work. The chapter of the Springer edition [HL19] with Update 7.31 was

not accessible to us; the update quoted in the introduction is that of the arXiv text. In the list of references of [HL19], which names the problems in which each reference is cited, only Borwein’s note carries the mark 7.31.

Searches. They were made on 10 and 11 October 2026, when the results were first obtained, by runs A and B, when the note was written, and by the final run: queries to the arXiv interface, to zbMATH Open, to OpenAlex (among them the works citing the collection and the lists of 1977, 1984 and 1989), to Crossref and to Semantic Scholar, and ten web searches. No paper on Problem 7.31, on series $\sum f(c_n/a_n)$ or on estimates of the type of Lemma 2.2 was found. The closest general work found is that of Bennett and Grosse-Erdmann [BGE05] on deducing the convergence of one series of positive terms from that of another; only its review in zbMATH was read, and it does not mention the problem. The literature on series of this kind from the years 1965 to 1990 is poorly covered by these indexes, and the sources named above were not read. A search that finds nothing is not a proof of novelty. We claim no priority; the arguments are elementary and may be known to specialists.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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