

On Krzyż’s Problem 7.23 in Hayman and Lingham: Kühnau’s Two-Point Condition Implies Quasiconformal Extendability If and Only If $q < 1/3$

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October 20, 2026

Abstract

For f analytic in the unit disc $\mathbb{D}$ let $u(z, \zeta) = f'(z)f'(\zeta)/(f(z) - f(\zeta))^2 - (z - \zeta)^{-2}$. Kühnau showed that a quasiconformal extension of f to the whole plane with $|\mu| \leq q < 1$ implies $|u(z, \zeta)| \leq q(1 - |z|^2)^{-1}(1 - |\zeta|^2)^{-1}$. J. G. Krzyż asked (Hayman–Lingham, *Research Problems in Function Theory*, Problem 7.23) to show that this condition is also sufficient for a quasiconformal extension, “possibly with $1/3 \leq q < 1$ ”. We show that it is sufficient if and only if $q < 1/3$; in the range $1/3 \leq q < 1$ named in the problem it is not sufficient, for any q . Here u carries the square on $f(z) - f(\zeta)$ (missing in the printed Problem 6.57, but forced by the two properties of u stated in Problem 7.23), the extension is a quasiconformal self-map of the Riemann sphere that agrees with f on $\mathbb{D}$, and “sufficient” means that some quasiconformal extension exists; the answer is the same for the exterior normalisation of Problem 6.57 and under both readings of the parenthesis, and for extensions to the plane it is the same for bounded f (unbounded f have none). The counterexample is the bounded Möbius image $1/(\operatorname{artanh} z - i)$ of the strip map: the supremum of $(1 - |z|^2)(1 - |\zeta|^2)|u(z, \zeta)|$ is exactly $1/3$, and the map has no homeomorphic extension to the closed disc. For $q < 1/3$ the Ahlfors–Weill theorem gives an extension to the sphere with $|\mu| \leq 3q$, and the best dilatation that the condition guarantees tends to 1 as $q \rightarrow 1/3^-$. For extensions with the same constant q the answer is negative for every $q \in (0, 1)$; this is due to Kühnau (1981), and we give a second proof. What is new is only the computation for the strip map and the resulting sharp threshold. No printed answer to the question in the main reading (some quasiconformal extension) was found in the sources searched; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

We show that Kühnau’s two-point condition (7.4) below is sufficient for a quasiconformal extension if and only if $q < 1/3$; in the range $1/3 \leq q < 1$ named in Problem 7.23 of Hayman and Lingham it is not sufficient, for any q (Corollary 1). Readings (Section 1.2): u carries the square on $f(z) - f(\zeta)$; the disc normalisation of Problem 7.23 is equivalent to that of Problem 6.57; an extension is a quasiconformal self-map of the sphere agreeing with f on $\mathbb{D}$; “sufficient” means that some extension exists (R1), and the same-constant reading (R2) and both readings of the parenthesis are answered as well.

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1.1 The problem

Let $\mathbb{D} = \{|z| < 1\}$ and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For f analytic and non-constant in $\mathbb{D}$ put

$$u_f(z, \zeta) = \frac{f'(z)f'(\zeta)}{(f(z) - f(\zeta))^2} - \frac{1}{(z - \zeta)^2} \quad (z \neq \zeta, f(z) \neq f(\zeta)), \quad (1.1)$$

$$Q_f(z, \zeta) = (1 - |z|^2)(1 - |\zeta|^2) |u_f(z, \zeta)|, \quad q(f) = \sup_{z \neq \zeta} Q_f(z, \zeta) \in [0, \infty],$$

with $q(f) = \infty$ if $f(z) = f(\zeta)$ for some $z \neq \zeta$. Problem 7.23 of Hayman and Lingham [HL19, Problem 7.23; arXiv version pp. 166–167], proposed by J. G. Krzyż, recalls a theorem of Kühnau, given there without citation: if f is analytic in $\mathbb{D}$ and has a quasiconformal extension to the whole plane with $|f_{\bar{z}}/f_z| \leq q < 1$ almost everywhere, then

$$|u_f(z, \zeta)| \leq \frac{q}{(1 - |z|^2)(1 - |\zeta|^2)} \quad (z, \zeta \in \mathbb{D}), \quad (7.4)$$

that is, $q(f) \leq q$. The problem asks to show that (7.4) is also sufficient for a quasiconformal extension to the whole plane, “possibly with $1/3 \leq q < 1$ ”. The kernel u is taken from Problem 6.57, and Problem 7.23 states two of its properties: u is invariant under Möbius transformations of f , and $u(z, \zeta) \rightarrow \frac{1}{6}S_f(z)$ as $\zeta \rightarrow z$, where $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$ is the Schwarzian derivative. The 2018 update in [HL19] reports no progress on Problem 7.23. The problem is older: Kühnau [Kü81, p. 126] refers to Problems 6.57, 6.58 and 7.23 of the list of Anderson, Barth and Brannan [ABB77] (which we have not read). It is record AMR-022-7023 of the UnsolvedMath collection [ula26], where it is listed as open.

1.2 Readings

(a) *The square.* In Problem 6.57 the kernel is printed without the square on $f(z) - f(\zeta)$, and this is so in the TeX source of arXiv:1809.07200v2 itself, so the misprint is in the source and not in the extraction of the record. The two properties stated in Problem 7.23 hold only with the square (Lemmas 1 and 2(a)); without it, the expansion at the diagonal begins with $-h^{-2} + f'(z)h^{-1}$, $h = \zeta - z$, and has no limit. Problem 6.58 prints the square, and with the square u_f is the classical kernel $\partial_{z\zeta}^2 \log[(f(z) - f(\zeta))/(z - \zeta)]$ (cf. [Kü81, (3)]). We read u as in (1.1). (b) *Normalisation.* We use the disc normalisation printed in Problem 7.23. Problem 6.57 is set for functions near ∞ (its TeX also has a normalisation slip); by Remark 1 that form of the condition is equivalent, and the counterexample answers it as well. (c) *The whole plane.* Both (7.4) and quasiconformal extendability to $\widehat{\mathbb{C}}$ are invariant under $f \mapsto T \circ f$ for Möbius maps T (Lemma 2), so we read an extension to the whole plane as a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ that agrees with f on $\mathbb{D}$; Kühnau’s own note [Kü74, p. 195] speaks of extensions to the whole sphere. Read literally, as a self-map of $\mathbb{C}$, every unbounded f fails trivially: the Möbius map $z/(1 - z)$ has $u \equiv 0$ and maps $\mathbb{D}$ onto a half-plane. This trivial defect is not our answer: the counterexample below is bounded and has no homeomorphic extension to $\widehat{\mathbb{C}}$ or to $\mathbb{C}$, so the negative answer for $1/3 \leq q < 1$ does not depend on this choice; for $q < 1/3$ a bounded f extends to $\mathbb{C}$ as well (Remark 4). (d) *Sufficient.* In reading R1, the main one by analogy with Problem 6.57 (“possibly for q sufficiently small”), (7.4) should imply that some quasiconformal extension exists; in reading R2, one with $|\mu| \leq q$. (e) *The parenthesis.* Read as naming the range in which sufficiency is asked (the main reading, since $q < 1/3$ is covered by the Ahlfors–Weill theorem), the answer is negative for every q in that range (Theorem A); read as “possibly only for $q < 1/3$ ”, the answer is the positive Theorem B. Corollary 1 settles both readings. (f) *Univalence* is not assumed in Problem 7.23; (7.4) forces it (Lemma 3).

1.3 Results

Theorem A. Let $f(z) = \operatorname{artanh} z = \frac{1}{2} \log \frac{1+z}{1-z}$ and $g = 1/(f - i)$.

- (i) g is analytic, univalent and bounded in $\mathbb{D}$: $|g| < 1/(1 - \pi/4) \approx 4.6598$.
- (ii) $Q_g(z, \zeta) < 1/3$ for all $z \neq \zeta$ in $\mathbb{D}$, and $Q_g(x, x') \rightarrow 1/3$ as $x' \rightarrow x$, for every real $x \in (-1, 1)$. Hence $q(g) = 1/3$; the supremum is not attained, and at non-real z the limit of $Q_g(z, \zeta)$ as $\zeta \rightarrow z$ is $(1 - |z|^2)^2/(3|1 - z^2|^2) < 1/3$. So g satisfies (7.4) for every $q \in [1/3, 1)$.
- (iii) $g(x) \rightarrow 0$ as $x \rightarrow 1^-$ and as $x \rightarrow -1^+$. So g has no continuous injective extension to $\overline{\mathbb{D}}$, and hence no homeomorphic and no quasiconformal extension to $\widehat{\mathbb{C}}$ or to $\mathbb{C}$.

The same holds for f and for every Möbius image $T \circ f$ analytic in $\mathbb{D}$ (with the extension taken in $\widehat{\mathbb{C}}$).

Theorem B. Let f be analytic and non-constant in $\mathbb{D}$ and satisfy (7.4) with some $q < 1/3$. Then f is univalent and has a quasiconformal extension to $\widehat{\mathbb{C}}$ with $|\mu| \leq 3q$ almost everywhere, that is, a K -quasiconformal extension with $K = (1 + 3q)/(1 - 3q)$.

Corollary 1. Let $0 \leq q < 1$. Every non-constant f analytic in $\mathbb{D}$ that satisfies (7.4) has a quasiconformal extension to $\widehat{\mathbb{C}}$ if and only if $q < 1/3$.

For f analytic in $\mathbb{D}$ let $k(f)$ be the infimum of $\|\mu_F\|_\infty$ over the quasiconformal maps F of $\widehat{\mathbb{C}}$ with $F = f$ on $\mathbb{D}$ ($k(f) = 1$ if there is none), and let $k^*(q) = \sup\{k(f) : q(f) \leq q\}$.

Proposition 1. For $0 < q < 1/3$,

$$q\sqrt{1 + 48q^2/25} \leq k^*(q) \leq 3q, \quad \text{and} \quad k^*(q) \rightarrow 1 \quad (q \rightarrow 1/3^-).$$

Theorem C. For every $q \in (0, 1)$ there is an f analytic in $\mathbb{D}$ with $q(f) \leq q$ that has no quasiconformal extension to $\widehat{\mathbb{C}}$ with $|\mu| \leq q$ almost everywhere.

Theorem C answers reading R2 negatively. This answer is due to Kühnau [Kü81] (Section 1.4); Theorem C is a second proof, with the family $\operatorname{artanh}(rz)$, and is not new.

1.4 Prior work and what is new

The necessary condition (7.4) is Kühnau's. It follows from his Grunsky inequality [Kü71] (Section 2.1) and is very likely inequality (6) of [Kü74], which is unchecked: only the first page of that note was read. That page recalls Kraus's bound $|S_f| \leq 6(1 - |z|^2)^{-2}$ for univalent f , the bound $6q(1 - |z|^2)^{-2}$ for maps with a Q -quasiconformal extension to the whole sphere, $q = (Q - 1)/(Q + 1)$, the converses with the constants 2 (Nehari [Neh49]) and $2q$, and announces further contributions.

Kühnau's paper [Kü81], which we read in full, concerns the Grunsky conditions (4)–(7) there for $w(z) = z + a_1 z^{-1} + \dots$ univalent in $|z| > 1$ (class Σ_0); all of them follow from (4) with the same constant, and (7) bounds the kernel U of (3) there (our u in the exterior normalisation) at finitely many points. By Satz 1, (4)–(7) hold with $\varkappa = 1/\lambda$, λ the Fredholm eigenvalue of the image of the unit circle, and not with a smaller constant. Satz 2: there are maps in Σ_0 that satisfy (4)–(7) with $\varkappa = q$ but have no Q -quasiconformal extension, $q = (Q - 1)/(Q + 1)$; they form a one-parameter family of Teichmüller maps built from an ellipse by the chain (11) there ($z^{3/2}$, the affine map $\mathfrak{z} + r^2 \bar{\mathfrak{z}}$ inside the disc, $\mathfrak{w}^{2/3}$), with extremal dilatation $q^* = r^2$, and the Fredholm eigenvalue argument of Section 3 there gives $1/\lambda < q^*$. On p. 126 Kühnau adds that the same holds for other inequalities that follow from (4), for example (6) of [Kü74], and refers to Problems 6.57, 6.58 and 7.23 of [ABB77]. This answers reading R2 negatively, and the deduction does not depend on identifying (6) of [Kü74]:

the maps of Satz 2 satisfy (4) with $\varkappa = q$, hence (7.4) with q by Section 2.1 (with Remark 1 and a Möbius normalisation such as $1/(w(1/z) - c)$, c a value omitted by w), and have no Q -quasiconformal extension. Reading R1, the existence of some quasiconformal extension for $1/3 \leq q < 1$, is not treated in [Kü81]: Kühnau's examples do have quasiconformal extensions, with dilatation $q^* > q$. We note neutrally that the 2018 update of [HL19] reports no progress on Problem 7.23, although [Kü81] refers to it.

As reported in [Kü81, p. 126] (we have not read these papers), Pommerenke [Pom75] showed that the Grunsky condition with some $\varkappa < 1$ implies a quasiconformal extension, Kühnau [Kü70] obtained a $(1+3\varkappa)/(1-3\varkappa)$ -quasiconformal extension from (4) with $\varkappa < 1/3$, and Zhuravlev [Zhu78] obtained qualitative analogues; Kühnau also points to Becker and Pommerenke [BP78]. Theorem B is the analogue of [Kü70] for (7.4), with the same constant; it is the Ahlfors–Weill theorem [AW62] applied to the diagonal limit.

What is new, as far as our searches found, is only the computation $q(\operatorname{artanh}) = 1/3$ (Theorem A(ii)) and the resulting sharp threshold in reading R1 (Corollary 1), with Proposition 1 as a consequence. The counterexample is the classical extremal function of the Nehari class [GP84], and the computation is short, so a remark in one of the unread sources listed in the paragraph “Scope and priority” is possible.

2 Preliminaries

Lemma 1 (diagonal limit). *If f is analytic near z and $f'(z) \neq 0$, then $u_f(z, \zeta) \rightarrow S_f(z)/6$ as $\zeta \rightarrow z$.*

Proof. Put $h = \zeta - z$ and $f(z+h) = c_0 + c_1h + c_2h^2 + c_3h^3 + O(h^4)$, $c_1 = f'(z)$, $\alpha = c_2/c_1$, $\beta = c_3/c_1$. Then

$$u_f(z, z+h) = \frac{c_1 f'(z+h)}{(f(z+h) - c_0)^2} - \frac{1}{h^2} = \frac{(1+2\alpha h+3\beta h^2)(1+\alpha h+\beta h^2)^{-2} - 1}{h^2} + O(h) = \beta - \alpha^2 + O(h),$$

since $(1+\alpha h+\beta h^2)^{-2} = 1-2\alpha h+(3\alpha^2-2\beta)h^2+O(h^3)$. Finally $S_f(z) = 6c_3/c_1 - \frac{3}{2}(2c_2/c_1)^2 = 6(\beta - \alpha^2)$. (Without the square, $c_1 f'(z+h)/(f(z+h) - c_0) - h^{-2} = -h^{-2} + c_1 h^{-1} + c_2 + O(h)$.) $\square$

Lemma 2 (invariances). *Let f be analytic and non-constant in $\mathbb{D}$.*

- (a) *If $T(w) = (aw + b)/(cw + d)$, $ad - bc = 1$, and $T \circ f$ is analytic in $\mathbb{D}$, then $u_{T \circ f} = u_f$; hence $q(T \circ f) = q(f)$.*
- (b) *If s is a conformal automorphism of $\mathbb{D}$, then $Q_{f \circ s}(z, \zeta) = Q_f(s(z), s(\zeta))$; hence $q(f \circ s) = q(f)$.*
- (c) *If $0 < r < 1$ and $f_r(z) = f(rz)$, then $u_{f_r}(z, \zeta) = r^2 u_f(rz, r\zeta)$.*

Proof. (a) $T(w_1) - T(w_2) = (w_1 - w_2)/((cw_1 + d)(cw_2 + d))$ and $T'(w) = (cw + d)^{-2}$, so the factors $(cf(z) + d)^{-2}(cf(\zeta) + d)^{-2}$ cancel in the first term of $u_{T \circ f}$. (b) By (a) applied to the identity map, $s'(z)s'(\zeta)/(s(z) - s(\zeta))^2 = (z - \zeta)^{-2}$; hence $u_{f \circ s}(z, \zeta) = s'(z)s'(\zeta) u_f(s(z), s(\zeta))$, and $1 - |s(z)|^2 = |s'(z)|(1 - |z|^2)$. (c) Both terms of u acquire the factor r^2 . $\square$

Lemma 3 (univalence). *If f is analytic and non-constant in $\mathbb{D}$ and Q_f is bounded on the set of pairs $z \neq \zeta$ with $f(z) \neq f(\zeta)$, then f is injective; in particular $f' \neq 0$ and S_f is analytic in $\mathbb{D}$.*

Proof. Let $f(z_1) = f(z_2)$ with $z_1 \neq z_2$. If $f'(z_1)f'(z_2) \neq 0$, then near (z_1, z_2) the numerator $f'(z)f'(\zeta)$ stays away from 0, the denominator $(f(z) - f(\zeta))^2$ tends to 0, and $(z - \zeta)^{-2}$ and the weight $(1 - |z|^2)(1 - |\zeta|^2)$ stay bounded and bounded away from 0; since pairs with $f(z) \neq f(\zeta)$ accumulate at (z_1, z_2) (the zeros of $f - f(z_2)$ are isolated), Q_f is unbounded. If $f'(z_1) = 0$, then f takes every value close to $f(z_1)$ at least twice near z_1 , and since the zeros of f' are isolated there are $z \neq \zeta$ near z_1 with $f(z) = f(\zeta)$ and $f'(z)f'(\zeta) \neq 0$, which is the first case. $\square$

Lemma 4 (symmetric operators). *Let A be a bounded operator on ℓ^2 with $A^T = A$, where $x^T A y = \sum_{m,n} a_{mn} x_m y_n$. Then $\sup_{\|x\|=1} |x^T A x| = \|A\|$.*

Proof. “ $\leq$ ” is clear. Let N be the left side. For unit vectors x, y , $y^T A x = \frac{1}{4}[(x+y)^T A(x+y) - (x-y)^T A(x-y)]$, so $|y^T A x| \leq \frac{N}{4}(\|x+y\|^2 + \|x-y\|^2) = N$. Taking $y = \overline{Ax}/\|Ax\|$ gives $\|Ax\| \leq N$. $\square$

2.1 The Grunsky operator and Kühnau’s inequality

Let f be univalent in $\mathbb{D}$. Then $(f(z) - f(\zeta))/(z - \zeta)$ does not vanish in $\mathbb{D} \times \mathbb{D}$, and

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n \geq 0} d_{mn} z^m \zeta^n, \quad u_f(z, \zeta) = \frac{\partial^2}{\partial z \partial \zeta} \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n \geq 1} mn d_{mn} z^{m-1} \zeta^{n-1},$$

the series converging locally uniformly in $\mathbb{D} \times \mathbb{D}$. Put $a_{mn} = \sqrt{mn} d_{mn}$, $A = (a_{mn})_{m,n \geq 1}$ and $x(z) = (\sqrt{m} z^{m-1})_{m \geq 1}$, so that

$$u_f(z, \zeta) = x(z)^T A x(\zeta), \quad \|x(z)\|^2 = \sum_{m \geq 1} m |z|^{2m-2} = (1 - |z|^2)^{-2}. \quad (2.1)$$

The coefficients d_{mn} , $m, n \geq 1$, do not change if f is replaced by a Möbius image $T \circ f$, since the logarithm changes only by a function of z plus a function of ζ . If $f(0) = 0$ and $F(t) = 1/f(1/t)$, then $(F(t) - F(\tau))/(t - \tau) = \frac{f(z) - f(\zeta)}{z - \zeta} \cdot \frac{z\zeta}{f(z)f(\zeta)}$ with $z = 1/t$, $\zeta = 1/\tau$; so $d_{mn} = -b_{mn}$, where $\log[(F(t) - F(\tau))/(t - \tau)] = -\sum b_{mn} t^{-m} \tau^{-n} + \dots$ defines the Grunsky coefficients b_{mn} of F (the a_{kl} of [Kü81, (2)]). With $x_k = \sqrt{k} y_k$, the condition $|\sum b_{kl} x_k x_l| \leq \varkappa \sum |x_k|^2/k$ of [Kü81, (4)] for all finite systems reads $|y^T A y| \leq \varkappa \|y\|^2$, that is, $\kappa(f) := \|A\| \leq \varkappa$ by Lemma 4 (Grunsky’s inequalities give $\kappa(f) \leq 1$ for univalent f).

Kühnau’s inequality [Kü71], as stated in [Kü81, p. 126] (cited, not re-proved): if $F \in \Sigma_0$ has a Q -quasiconformal extension to $\widehat{\mathbb{C}}$, then (4) holds with $\varkappa = q = (Q - 1)/(Q + 1)$. Kühnau also cites [Pom75] for it, without pages (he cites pp. 290–291 of that book for the argument of his proof of Satz 1 and p. 292 for the question of sufficiency); we could not consult the book and give no theorem numbers. A quasiconformal extension of f with $|\mu| \leq k$ gives one of $F(t) = 1/(f(1/t) - f(0))$, which lies in Σ_0 after an affine normalisation and has the same coefficients d_{mn} ; so the inequality reads, in our notation:

$$f \text{ has a quasiconformal extension to } \widehat{\mathbb{C}} \text{ with } |\mu| \leq k \implies \kappa(f) \leq k. \quad (2.2)$$

By (2.1) and the Cauchy–Schwarz inequality, $|u_f(z, \zeta)| \leq \|A\| \|x(z)\| \|x(\zeta)\|$, so $q(f) \leq \kappa(f)$ for every univalent f , and (2.2) gives Kühnau’s condition (7.4) with $q = k$. Thus (7.4) is the Grunsky condition tested only on the point-evaluation vectors $x(z)$, $x(\zeta)$, a weaker hypothesis than $\kappa(f) \leq q$.

Remark 1 (the normalisation of Problem 6.57). For $F(t) = f(1/t)$, $|t| > 1$, one has $F'(t) = -f'(1/t)/t^2$ and $t - \tau = (\zeta - z)/(z\zeta)$ with $z = 1/t$, $\zeta = 1/\tau$, hence

$$u_F(t, \tau) = \frac{u_f(1/t, 1/\tau)}{t^2 \tau^2}, \quad (|t|^2 - 1)(|\tau|^2 - 1) = |t|^2 |\tau|^2 (1 - |1/t|^2)(1 - |1/\tau|^2).$$

So the condition $|u_F(t, \tau)| \leq q(|t|^2 - 1)^{-1}(|\tau|^2 - 1)^{-1}$ in $|t|, |\tau| > 1$, the form of (7.4) in the setting of Problem 6.57, is equivalent to (7.4) for f , and F has a quasiconformal extension with $|\mu| \leq k$ if and only if f has one. With Lemma 2(a) this covers the normalisation $F(t) = t + b_0 + b_1 t^{-1} + \dots$: for instance $G(t) = 1/\operatorname{artanh}(1/t) = t - \frac{1}{3t} + \dots$ is univalent in $|t| > 1$, satisfies the exterior condition with $q = 1/3$ by Theorem A, and has no homeomorphic extension to $\widehat{\mathbb{C}}$, since $G(t) \rightarrow 0$ as $t \rightarrow 1^+$ and as $t \rightarrow -1^-$.

3 The counterexample

Proof of Theorem A. (i) The map $z \mapsto (1+z)/(1-z)$ sends $\mathbb{D}$ onto the right half-plane, so $f = \operatorname{artanh}$ maps $\mathbb{D}$ univalently onto the strip $\{W : |\operatorname{Im} W| < \pi/4\}$, with inverse $\tanh$. Hence $|f(z) - i| \geq 1 - \operatorname{Im} f(z) > 1 - \pi/4 > 0$, and $g = 1/(f - i)$ is analytic, univalent and bounded by $1/(1 - \pi/4)$.

(ii) By Lemma 2(a), $u_g = u_f$ and $Q_g = Q_f$. Let $z_j = \tanh W_j$ with $W_j = s_j + iy_j$, $|y_j| < \pi/4$ ($j = 1, 2$), $z_1 \neq z_2$, and $\Delta = W_1 - W_2 = a + ib \neq 0$. Since $f'(z) = 1/(1 - z^2) = \cosh^2 W$ for $z = \tanh W$, and $\tanh W_1 - \tanh W_2 = \sinh \Delta / (\cosh W_1 \cosh W_2)$,

$$u_f(z_1, z_2) = \cosh^2 W_1 \cosh^2 W_2 \left(\frac{1}{\Delta^2} - \frac{1}{\sinh^2 \Delta} \right) = f'(z_1) f'(z_2) \varphi(\Delta), \quad \varphi(\Delta) = \frac{1}{\Delta^2} - \frac{1}{\sinh^2 \Delta}.$$

Moreover $|\cosh W|^2 - |\sinh W|^2 = \cos 2y$ for $W = s + iy$, so $1 - |\tanh W|^2 = \cos 2y / |\cosh W|^2$ and $(1 - |z|^2)|f'(z)| = \cos 2y$. Therefore

$$Q_f(z_1, z_2) = \cos 2y_1 \cos 2y_2 |\varphi(\Delta)|. \quad (3.1)$$

From $1/\sin^2 w = \sum_{k \in \mathbb{Z}} (w - \pi k)^{-2}$ and $\sinh \Delta = -i \sin(i\Delta)$ we get $1/\sinh^2 \Delta = \sum_{k \in \mathbb{Z}} (\Delta - i\pi k)^{-2}$, hence

$$\varphi(\Delta) = - \sum_{k \neq 0} \frac{1}{(\Delta - i\pi k)^2}, \quad |\varphi(\Delta)| \leq \sum_{k \neq 0} \frac{1}{a^2 + (b - \pi k)^2}. \quad (3.2)$$

Since $|y_j| < \pi/4$, we have $|b| = |y_1 - y_2| < \pi/2$, both cosines in (3.1) are positive, and

$$\cos 2y_1 \cos 2y_2 = \frac{1}{2} (\cos 2b + \cos 2(y_1 + y_2)) \leq \frac{1}{2} (1 + \cos 2b) = \cos^2 b. \quad (3.3)$$

Case $b = 0$. Then $a \neq 0$, and (3.2) gives $|\varphi(\Delta)| < \sum_{k \neq 0} (\pi k)^{-2} = 1/3$; so $Q_f < 1/3$. *Case $b \neq 0$.* The right side of (3.2) is even in b , and we may take $0 < b < \pi/2$. Dropping a^2 and using $\sum_{k \in \mathbb{Z}} (b - \pi k)^{-2} = 1/\sin^2 b$, (3.1)–(3.3) give

$$Q_f(z_1, z_2) \leq H(b) := \cos^2 b \left(\frac{1}{\sin^2 b} - \frac{1}{b^2} \right) = \cot^2 b - \frac{\cos^2 b}{b^2}.$$

We claim that $H(b) \leq \frac{1}{3} - \frac{2}{9}b^2 + \frac{2}{45}b^4 < \frac{1}{3}$ for $0 < b < \pi/2$. (a) For $0 < b < \pi$,

$$\cot b = \frac{1}{b} - \sum_{k \geq 1} \frac{2b}{\pi^2 k^2 - b^2} \leq \frac{1}{b} - 2b \sum_{k \geq 1} \frac{1}{\pi^2 k^2} = \frac{1}{b} - \frac{b}{3}.$$

(b) For $0 < b < \pi/2$ both $\cot b$ and $\frac{1}{b} - \frac{b}{3}$ are positive (as $b^2 < \pi^2/4 < 3$), so $\cot^2 b \leq (\frac{1}{b} - \frac{b}{3})^2$. (c) Integrating $\cos t \leq 1$ six times from 0 and using evenness, $\cos t \geq 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}$ for all real t ; with $t = 2b$ and $\cos^2 b = \frac{1}{2}(1 + \cos 2b)$ this gives $\cos^2 b \geq 1 - b^2 + \frac{b^4}{3} - \frac{2b^6}{45}$. (d) Hence

$$H(b) \leq \left(\frac{1}{b} - \frac{b}{3} \right)^2 - \frac{1 - b^2 + b^4/3 - 2b^6/45}{b^2} = \frac{1}{3} - \frac{2}{9}b^2 + \frac{2}{45}b^4 = \frac{1}{3} - \frac{2b^2}{45}(5 - b^2),$$

which is less than $1/3$ because $b^2 < \pi^2/4 < 5$. So $Q_g = Q_f < 1/3$ at all pairs $z_1 \neq z_2$.

The diagonal. For $f = \operatorname{artanh}$ we have $f''/f' = 2z/(1 - z^2)$ and $S_f(z) = \frac{2+2z^2}{(1-z^2)^2} - \frac{2z^2}{(1-z^2)^2} = \frac{2}{(1-z^2)^2}$. By Lemma 1, $Q_f(z, \zeta) \rightarrow (1 - |z|^2)^2 |S_f(z)|/6 = (1 - |z|^2)^2 / (3|1 - z^2|^2)$ as $\zeta \rightarrow z$. Since $|1 - z^2| \geq 1 - |z|^2$, with equality if and only if z is real, the limit is $1/3$ at real z and less than $1/3$ elsewhere. Hence $q(g) = q(f) = 1/3$, and the supremum is not attained.

(iii) As $x \rightarrow 1^-$ along the real axis, $f(x) \rightarrow +\infty$, and as $x \rightarrow -1^+$, $f(x) \rightarrow -\infty$; in both cases $g(x) \rightarrow 0$. A continuous extension G of g to $\overline{\mathbb{D}}$ therefore has $G(1) = G(-1) = 0$ and is not injective. A homeomorphism of $\widehat{\mathbb{C}}$ or of $\mathbb{C}$ that agrees with g on $\mathbb{D}$ would restrict to such a G , and quasiconformal maps are homeomorphisms. For f and $T \circ f$ the same argument applies in $\widehat{\mathbb{C}}$, with the common limit ∞ , respectively $T(\infty)$, at ± 1 . The statements (ii) for $T \circ f$ follow from Lemma 2(a). $\square$

Remark 2 (the image domain). The lines $\text{Im } W = \pm\pi/4$ are mapped by $W \mapsto 1/(W - i)$ onto the circles through 0 with centres on the positive imaginary axis and radii $1/(2 - \pi/2)$ and $1/(2 + \pi/2)$. So $g(\mathbb{D})$ is the crescent between two circles that are internally tangent at 0; its boundary is not a Jordan curve. The unboundedness of artanh plays no role: g is bounded.

Remark 3 (where the supremum is approached). Let $Q_g(z_j, \zeta_j) \rightarrow 1/3$, with $z_j = \tanh W_j$, $\zeta_j = \tanh W'_j$ and $\Delta_j = W_j - W'_j = a_j + ib_j$. Then $\Delta_j \rightarrow 0$ and $\text{Im } W_j, \text{Im } W'_j \rightarrow 0$. Equivalently, the automorphisms $s_j(z) = \tanh(\text{artanh } z - \text{Re } W_j)$ of $\mathbb{D}$, which preserve $(-1, 1)$ and Q_g (by Lemma 2(a),(b), since $f \circ s_j = f - \text{Re } W_j$), satisfy $s_j(z_j) \rightarrow 0$ and $s_j(\zeta_j) \rightarrow 0$: up to these automorphisms the supremum is approached only at the real diagonal. Indeed, let $c_j = \cos 2y_j \cos 2y'_j \leq \cos^2 b_j$ and $P(b) = \sum_{k \neq 0} (b - \pi k)^{-2}$, so that $\cos^2 b P(b) = H(b)$ for $b \neq 0$ and $P(0) = 1/3$. By the claim above, $Q_g \leq H(b_j) \leq \frac{1}{3} - \frac{2b_j^2}{45}(5 - b_j^2)$, so $b_j \rightarrow 0$ and $P(b_j) \rightarrow 1/3$; then $Q_g \leq c_j P(b_j)$ forces $c_j \rightarrow 1$, that is, $y_j, y'_j \rightarrow 0$. Finally, for $|b| \leq \pi/2$ and $k = \pm 1$ we have $(b - \pi k)^2 \leq 9\pi^2/4$, so each of these two terms in (3.2) is smaller than $(b - \pi k)^{-2}$ by at least $\varepsilon(a) = 16a^2/(9\pi^2(4a^2 + 9\pi^2))$; hence $Q_g \leq c_j(P(b_j) - 2\varepsilon(a_j))$, and $a_j \rightarrow 0$.

4 The range $q < 1/3$ and the dilatation

Proof of Theorem B. By Lemma 3, f is injective and S_f is analytic. Letting $\zeta \rightarrow z$ in (7.4) and using Lemma 1, $(1 - |z|^2)^2 |S_f(z)| \leq 6q = 2k$ with $k = 3q < 1$. By the theorem of Ahlfors and Weill [AW62], which we cite, f then has a quasiconformal extension F to $\widehat{\mathbb{C}}$ with $|\mu_F| \leq k$ almost everywhere, given explicitly by $F(1/\bar{z}) = f(z) + (1 - |z|^2)f'(z)/(\bar{z} - \frac{1}{2}(1 - |z|^2)f''(z)/f'(z))$ for $z \in \mathbb{D}$. $\square$

Proof of Corollary 1. For $q < 1/3$ this is Theorem B. For $1/3 \leq q < 1$ the function g of Theorem A satisfies (7.4) and has no quasiconformal extension. $\square$

Remark 4 (extensions to $\mathbb{C}$). For the Ahlfors–Weill extension, in general $F(\infty) = f(0) - 2f'(0)^2/f''(0) \neq \infty$. For bounded f , composing F on $\{|z| > 1\}$ with a quasiconformal self-map of $\{|z| > 1\} \cup \{\infty\}$ that fixes the unit circle and moves $F^{-1}(\infty)$ to ∞ extends f to $\mathbb{C}$; we claim no dilatation constant for this. Unbounded f have no extension to $\mathbb{C}$.

Proof of Proposition 1. The upper bound is Theorem B. For the lower bound and the limit we use $f_r(z) = \text{artanh}(rz)$, $0 < r < 1$. By Lemma 2(c),

$$Q_{f_r}(z, \zeta) = r^2 \frac{1 - |z|^2}{1 - r^2|z|^2} \cdot \frac{1 - |\zeta|^2}{1 - r^2|\zeta|^2} Q_{\text{artanh}}(rz, r\zeta) \leq \frac{r^2}{3}$$

by Theorem A(ii), and $Q_{f_r}(z, \zeta) \rightarrow r^2 S_{\text{artanh}}(0)/6 = r^2/3$ as $z, \zeta \rightarrow 0$. So $q(f_r) = r^2/3$, and we take $r = \sqrt{3q}$, so that $q(f_r) = q$.

Lower bound. For $m, n \geq 1$ the coefficients of $\log[(f_r(z) - f_r(\zeta))/(z - \zeta)]$ are $d_{mn}r^{m+n}$, where d_{mn} are those of artanh . From

$$u_{\text{artanh}}(z, 0) = \frac{1}{(1 - z^2) \text{artanh}^2 z} - \frac{1}{z^2} = \frac{1}{3} + \frac{4}{15}z^2 + \frac{44}{189}z^4 + \dots = \sum_{m \geq 1} m d_{m1} z^{m-1}$$

we get $d_{11} = 1/3$, $d_{21} = 0$, $d_{31} = d_{13} = 4/45$. So $a_{11}(r) = r^2/3$ and $a_{31}(r) = \frac{4\sqrt{3}}{45}r^4$, and with $e_1 = (1, 0, 0, \dots)$,

$$\kappa(f_r) \geq \|A_r e_1\| \geq \left(\frac{r^4}{9} + \frac{48r^8}{2025} \right)^{1/2} = q \left(1 + \frac{48q^2}{25} \right)^{1/2} \quad (r^2 = 3q),$$

since $81/2025 = 1/25$. By Kühnau's inequality (2.2) (cited, not re-proved), every quasiconformal extension of f_r has $\|\mu\|_\infty \geq \kappa(f_r)$; so $k^*(q) \geq k(f_r) \geq q\sqrt{1 + 48q^2/25}$.

The limit. It suffices to show $k(f_r) \rightarrow 1$ as $r \rightarrow 1^-$. Suppose instead that $k(f_{r_j}) \leq k_0 < 1$ along a sequence $r_j \rightarrow 1$. Put $k_1 = (1 + k_0)/2$, $g_j = T \circ f_{r_j}$ with $T(W) = 1/(W - i)$, and let G_j be quasiconformal maps of $\widehat{\mathbb{C}}$ with $G_j = g_j$ on $\mathbb{D}$ and $|\mu_{G_j}| \leq k_1$. The points $g_j(0) = i$, $g_j(\frac{1}{2})$, $g_j(-\frac{1}{2})$ converge to the three distinct points $g(0)$, $g(\frac{1}{2})$, $g(-\frac{1}{2})$, with g as in Theorem A. Let M_j be the Möbius maps sending $g_j(0), g_j(\frac{1}{2}), g_j(-\frac{1}{2})$ to $0, 1, \infty$, so that $M_j \rightarrow M$ uniformly on $\widehat{\mathbb{C}}$ in the chordal metric, and let N be the Möbius map sending $0, 1, \infty$ to $0, \frac{1}{2}, -\frac{1}{2}$. Then $H_j = M_j \circ G_j \circ N$ are K_1 -quasiconformal self-maps of $\widehat{\mathbb{C}}$ fixing $0, 1, \infty$, $K_1 = (1 + k_1)/(1 - k_1)$. Such a family is sequentially compact under uniform convergence in the chordal metric, and its limits are K_1 -quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ (Lehto and Virtanen [LV73, Chapter II, Section 5]; cited, not re-proved). So a subsequence converges to a homeomorphism H . On $\mathbb{D}$, $f_{r_j} \rightarrow f$ locally uniformly, so $M_j \circ g_j \rightarrow M \circ g$ locally uniformly in the chordal metric, and $H \circ N^{-1} = M \circ g$ on $\mathbb{D}$. Thus $M^{-1} \circ H \circ N^{-1}$ is a homeomorphism of $\widehat{\mathbb{C}}$ extending g , contrary to Theorem A(iii). Hence $k(f_r) \rightarrow 1$, and $k(f_{\sqrt{3q}}) \leq k^*(q) \leq 3q < 1$ gives $k^*(q) \rightarrow 1$ as $q \rightarrow 1/3^-$. $\square$

Illustration only: for $r = 0.5, 0.8, 0.9, 0.95, 0.99$ ($q(f_r) = 0.083, 0.213, 0.270, 0.301, 0.327$) the 80×80 sections of A_r have norms $0.084, 0.237, 0.331, 0.407, 0.530$, the 160×160 sections $0.0845, 0.2373, 0.3311, 0.4065, 0.5318$: numerical lower bounds for $\kappa(f_r)$.

5 Extensions with the same constant

Proof of Theorem C. For $1/3 \leq q < 1$ take g of Theorem A, which has no quasiconformal extension at all. For $0 < q < 1/3$ take f_r with $r = \sqrt{3q}$; then $q(f_r) = q$, and by the proof of Proposition 1, $\kappa(f_r) \geq q\sqrt{1 + 48q^2/25} > q$. An extension with $|\mu| \leq q$ would give $\kappa(f_r) \leq q$ by Kühnau's inequality (2.2) (cited, not re-proved); so there is none. (For $q = 0$, (7.4) means $u \equiv 0$, so $S_f \equiv 0$ and f is Möbius, which extends conformally.) $\square$

As explained in Section 1.4, this negative answer is due to Kühnau [Kü81, Satz 2 and p. 126], whose Teichmüller maps satisfy (7.4) with q by Section 2.1 and Remark 1.

6 Consistency with known results

By Lemma 1, $q(f) \leq 1/3$ implies $\|S_f\| := \sup_{\mathbb{D}} (1 - |z|^2)|S_f(z)| \leq 2$, and $\|S_g\| = \sup 2(1 - |z|^2)/|1 - z^2|^2 = 2$. *Nehari's theorem* [Neh49] ($\|S_f\| \leq 2$ implies univalence) agrees with the univalence of g , and *the Ahlfors–Weill theorem* [AW62] needs $\|S_f\| \leq 2k < 2$. *Gehring and Pommerenke* [GP84] showed that if $\|S_f\| \leq 2$, then $f(\mathbb{D})$ is a Jordan domain unless it is the image of a parallel strip under a Möbius map (cited as recalled; the paper was not re-read); g is exactly the exceptional case, in agreement with Theorem A(iii). *Pommerenke's converse* [Pom75] (as reported in [Kü81, p. 126]): $\kappa(f) < 1$ implies a quasiconformal extension. So necessarily $\kappa(g) = 1$, while $q(g) = 1/3$; already $\|Ae_1\| \geq (1/9 + 48/2025)^{1/2} = \sqrt{273}/45 = 0.3672 > 1/3$ for artanh , and the 80×80 and 160×160 sections of A have norms 0.61 and 0.635 , numerical lower bounds for $\kappa(\text{artanh}) = 1$. There is no conflict, because (7.4) tests A only on the vectors $x(z)$.

Zhuravlev's integral condition. By Remark 1, (6.21) of Problem 6.57 reads $\iint_{\mathbb{D}} |u_f(z, \zeta)|^2 d\sigma_\zeta \leq \pi q^2 (1 - |z|^2)^{-2}$ in the disc. The functions $\sqrt{n} \zeta^{n-1}$ are orthogonal in $L^2(\mathbb{D})$ with squared norm π , so by (2.1) the left side is $\pi \|Ax(z)\|^2$, and the condition says $\|Ax(z)\| \leq q \|x(z)\|$; it is stronger than (7.4). For artanh it fails at $z = 0$ for every $q < \sqrt{273}/45$, since $x(0) = e_1$; numerically $\|Ae_1\| \geq 0.425$ (from the 160×160 section; the 80×80 section gives 0.4215 ; the partial sums of the column norm

converge slowly), so it fails for every q below this finite-section lower bound. (By $W = \operatorname{artanh} \zeta$ one has $\pi \|Ae_1\|^2 = \iint_{\mathbb{D}} |u_{\operatorname{artanh}}(0, \zeta)|^2 d\sigma_\zeta = \iint_{|\operatorname{Im} W| < \pi/4} |\varphi(W)|^2 d\sigma_W$, and a numerical quadrature of the last integral gives $\|Ae_1\| \approx 0.4352$.) This is consistent with the affirmative answer by Zhuravlev [Zhu78] reported in the update of Problem 6.57 (we have not read [Zhu78]).

Remark 5 (the two conditions). For the full Grunsky condition, $\kappa(f) < 1$ already suffices for some quasiconformal extension [Pom75, Kü81]. For the weaker two-point condition (7.4) the threshold is exactly $1/3$ (Corollary 1): the strip map has $q = 1/3$ but $\kappa = 1$.

Remark 6 (the endpoint $q = 1/3$; not claimed). If $q(f) \leq 1/3$, then $\|S_f\| \leq 2$, so by [GP84] $f(\mathbb{D})$ is a Jordan domain unless $f = T \circ \operatorname{artanh} \circ s$ with T Möbius and s an automorphism of $\mathbb{D}$; such f have $q(f) = 1/3$ and no quasiconformal extension (Lemma 2, Theorem A). Whether every other f with $q(f) \leq 1/3$ has a quasiconformal extension is left open.

7 Computations and reproducibility

All proofs in this note are analytic and do not depend on computation. The reproducibility package has five parts; its README gives command lines, running times and what each program checks. *The programs with which the results were first obtained* (`original/`; `sympy`, `mpmath`, `numpy`) check the identities of Sections 2 and 3 symbolically, evaluate Q_g from the raw definition of u with 100 digits at 19,995 random and adversarial pairs (maximum $0.3333328 < 1/3$; (3.1) agrees with the raw u to about 10^{-59}), evaluate $H(b)$ on a grid, and compute Grunsky sections by a fast Fourier transform. *The program of this note*, `check_note.py` (standard library only, about one second), checks with rational arithmetic Lemma 1 and the expansion without the square on random rational germs, Lemma 2(a),(c) and Remark 1 at rational points, $S_{\operatorname{artanh}}$, the coefficients d_{mn} of artanh from its series ($d_{11} = 1/3$, $d_{31} = 4/45$, $d_{22} = 1/30$, $d_{51} = 44/945$, $d_{33} = 83/2835$, symmetry, parity), $81/2025 = 1/25$, the identity at $r^2 = 3q$, and the polynomial identities of steps (c) and (d); in floating point it checks (3.1) at 300 well-separated pairs, the sign in (3.2), $H(b)$ and step (a) on a grid, and Remark 3.

An interval certificate (`interval_certificate/`, from an independent AI-assisted verification run, included unmodified except for its header comments). By (3.1) and (3.3), with equality in (3.3) at $y_1 = -y_2 = b/2$, $q(g)$ is the supremum of $G(a, b) = \cos^2 b |\varphi(a + ib)|$ over $(a, b) \neq (0, 0)$, $|b| < \pi/2$. The program `b2_interval.py` certifies $G < 1/3$: for $|a| \geq 2$ by the analytic bound $G \leq \frac{1}{4} + \sinh^{-2} 2 = 0.32602$; for $|\Delta| \leq 1$ by a Taylor majorant of φ , certified on 17 polar boxes; and on the rest of $[0, 2] \times [0, \pi/2]$ by adaptive subdivision into 3,506 boxes, in outward-rounded interval arithmetic (`mpmath.iv`, 30 digits). Detection tests reject false targets ($G < 0.25$ on $[1, 2] \times [0, \frac{1}{2}]$, the bound without the factor $\cos^2 b$, and the target 0.333 near the origin). This is a second, computer-assisted route to Theorem A(ii) without the one-variable inequality for $H(b)$; run again from the package copy, it took 6 seconds and reproduced the recorded output byte for byte. *Further programs of that run* (`verification_run_B/`, likewise): symbolic checks; 1,200 local searches for large Q_g (maximum 0.33333325 at 60 digits); Grunsky coefficients by a power-series recursion, up to 160×160 .

The programs of the third verification run (`independent_run_2/`; `sympy`, `mpmath`, `numpy`; about four minutes) check the identities and constants of Sections 2 to 4 symbolically or with 60 digits; evaluate Q_g from the raw definition of u at 20,000 adversarial pairs and in 160 local searches, with the precision raised with the proximity of the two points (up to 170 digits), so that $1/3 - Q_g$ is resolved at every pair (all positive; the smallest, $2.5 \cdot 10^{-52}$, at a real pair at distance 10^{-28}); certify $G < 1/3$ by a route different from `b2_interval.py`: the sum in (3.2) over $0 < |k| \leq 20$ with a rigorous bound for the rest, in their own outward-rounded interval arithmetic on $[0, 4] \times [0, \pi/2]$ outside $|\Delta| < \frac{1}{2}$, an exact rational majorant for $|\Delta| \leq \frac{1}{2}$, and an analytic bound for $a \geq 4$, with detection tests; and compute the Grunsky numbers of Sections 4 and 6 by a two-dimensional fast Fourier transform of

the raw u at 40 digits and, for the first column, by one-variable series, with the integral for $\|Ae_1\|$ in Section 6.

A numerical caution. Evaluating u from its definition in double precision near the diagonal is unreliable because of cancellation: a verification run saw spurious values up to 842 at a pair where 60-digit arithmetic gives 0.333333331. More generally, at two points at distance h the raw formula loses about $3\log_{10}(1/h)$ digits. The checks above from the raw definition use 100 digits at distances at least about 10^{-14} , 60 digits at distances at least 10^{-8} , or a precision raised with $1/h$; `check_note.py` uses the raw u only at well-separated pairs; and the two interval certificates work with (3.1) or (3.2), with a majorant near $\Delta = 0$; none of them is affected.

Verification. The results were first obtained, with proofs, together with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first compared the statement with the source and with the TeX source of arXiv:1809.07200v2 (finding there the missing square in Problem 6.57), re-derived all proofs line by line, checked the cited Kühnau papers in zbMATH, and tested the statements with its own programs (maximum of Q_g at 11,976 adversarial pairs: 0.3333332). The second re-derived the proofs, wrote the programs described above, re-ran the original programs from a copy (identical outputs), and repeated the literature search, finding the survey [AS02]. Neither run found a mathematical error; their required corrections (bibliographic data, citations and wording; see the verification report) were made. When this note was written, [Kü81] was read in full, and Theorem C is accordingly presented as a second proof of a result of that paper.

A third independent verification run, also AI-assisted, then examined the text of this note. It compared the statement again with the TeX source of arXiv:1809.07200v2, read [Kü81] again in full and p. 195 of [Kü74] from page images, checked every printed DOI in Crossref and the cited zbMATH records, re-derived all proofs line by line, wrote the programs in `independent_run_2/` described above, re-ran every program of the package from the archive (all outputs identical to the recorded ones), and repeated part of the literature search. It found no mathematical error. Its required corrections were made: the page numbers of the source, the scope of two sentences of the abstract (extensions to the plane, and the reading in which no printed answer was found), the statement on the convergence of the column norm in Section 6 (where the integral for $\|Ae_1\|$ was added), the pages attributed to [Pom75], the dates of [Kü75] and [Kü70], two related works, and the description of the computations.

Scope and priority. The note answers Problem 7.23 of [HL19] in reading R1 (Corollary 1: (7.4) is sufficient if and only if $q < 1/3$) and in reading R2 (Theorem C, already known from [Kü81, Satz 2 and p. 126]). The question is J. G. Krzyż’s and was already in the list [ABB77], as referred to in [Kü81]. The necessary condition (7.4) is Kühnau’s (probably (6) of [Kü74]; unchecked) and follows from his Grunsky inequality [Kü71]. The strip map, the theorems of Nehari, Ahlfors–Weill and Gehring–Pommerenke, Pommerenke’s converse, the compactness of normalised quasiconformal maps, and the results of [Kü70, Zhu78] next to Theorem B are known and cited with their sources. What is new, as far as the searches found, is only the computation $q(\operatorname{artanh}) = 1/3$ and the resulting sharp threshold in reading R1, with Proposition 1 as a consequence. We make no claim of priority. What was read: Problems 6.57, 6.58 and 7.23 of [HL19] with their updates (arXiv version v2; its TeX source by a verification run); [Kü81] in full; of [Kü74] only p. 195; the zbMATH records of the cited Kühnau papers. Cited as reported in [Kü81] and not read: [Kü70, Zhu78, ABB77, BP78], and [Kü71, Pom75] for the statements used. Cited as standard and not re-read: [AW62, Neh49, GP84, LV73]. The searches, made on 20 October 2026, used the 2018 update of [HL19]; the zbMATH Open API (all records of Kühnau, the records of Krzyż, keyword searches, and the documents citing the two notes of 1974 and, in the third verification run, those citing [Kü81]); the arXiv API; Crossref; OpenAlex and Semantic Scholar (partly refused, with HTTP status 429); and a few web searches. Unread candidates for prior

art are [Kü74] (only p. 195 read), [Kü75] (vol. 24 (1974), no. 6, published 1975; Zbl 0315.30029), and the survey [AS02] of sufficient conditions, which cites [Kü74] (full text not accessible). No printed answer to the question in reading R1 was found in the sources searched; a search that finds nothing is not a proof of novelty. Related work: Shen [She01] uses the same kernel with an integral norm only, and the two-point distortion theorems of Chuaqui, Duren, Ma, Mejía, Minda and Osgood [CDM⁺11] and of Ma, Mejía and Minda concern a different quantity. Later work on the same-constant question for the full Grunsky condition, when it is sufficient for a Q -quasiconformal extension, is due to Kühnau [Kü86] and Krushkal [Kru87] (known to us from their zbMATH reviews); it does not concern (7.4).

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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