

Root-Transform Bounds Do Not Force Logarithmic Coefficient Bounds

Alper Ferudun*

26 October 2026

Abstract

We construct a bounded normalized univalent function whose logarithmic coefficients are not $O(1/n)$, although every root transform satisfies the coefficient estimate $O(n^{2/p-1})$. This gives a negative answer to the converse question recorded as Problem 6.84 by Hayman and Lingham. The normalized logarithm of the constructed function is Hölder continuous of every order below one on the closed disk. Starting from the classical existence of arbitrarily large weighted logarithmic coefficients, we insert dilated seeds by high-order root transforms and normalized composition. A congruence filter transfers a selected coefficient, while a diagonal choice preserves all earlier coefficients and gives the required boundary regularity.

1 The question and the result

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ consist of the injective holomorphic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Write

$$L_f(z) = \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad L_f(0) = 0. \quad (1)$$

The normalized p th root transform is

$$(f(z^p))^{1/p} = z \exp\left(\frac{L_f(z^p)}{p}\right) = z + \sum_{n=1}^{\infty} c_n^{(p)}(f) z^{pn+1}. \quad (2)$$

In particular,

$$c_n^{(p)}(f) = [z^n] \exp(L_f/p). \quad (3)$$

Milin proved that $\gamma_n(f) = O(1/n)$ implies

$$c_n^{(p)}(f) = O(n^{2/p-1}) \quad \text{for every integer } p \geq 1; \quad (4)$$

see [4, Theorem 1]. Duren's question, as recorded in [3, Problem 6.84], asks whether the converse holds. The constants in (4) may depend on f and p .

For $0 < \alpha < 1$, let $\mathcal{A}^\alpha$ denote the functions holomorphic on $\mathbb{D}$ with a $C^\alpha(\overline{\mathbb{D}})$ extension. We use

$$\|U\|_\alpha = \|U\|_\infty + [U]_\alpha, \quad [U]_\alpha = \sup_{\substack{z, w \in \overline{\mathbb{D}} \\ z \neq w}} \frac{|U(z) - U(w)|}{|z - w|^\alpha}.$$

Theorem 1. *There is a bounded function $f \in \mathcal{S}$ such that*

$$L_f \in \bigcap_{0 < \alpha < 1} \mathcal{A}^\alpha, \quad \sup_{n \geq 1} n |\gamma_n(f)| = \infty.$$

For this same function and every integer $p \geq 1$,

$$c_n^{(p)}(f) = O(n^{-1+\eta}) \quad \text{for every } \eta > 0.$$

Consequently (4) holds for every p , while $\gamma_n(f) = O(1/n)$ fails.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

The large-coefficient input is classical. The construction below transfers it into a function with all the stated Hölder regularities. It does not assert a new construction of Pommerenke's original large-coefficient examples. Nor does it address the separate universal Szegő conjecture for any particular unresolved value of p .

2 Smooth seeds and root insertion

We first isolate precisely the known input that is needed.

Lemma 2 (Classical input). *Given $T > 0$, there are $g \in \mathcal{S}$, holomorphic and injective on a neighborhood of $\overline{\mathbb{D}}$, and an integer $m \geq 1$ such that*

$$|m[z^m]L_g| > T.$$

Proof. The classical large-logarithmic-coefficient examples give a bounded $g_* \in \mathcal{S}$ for which $\gamma_n(g_*)$ is not $O(1/n)$. For the stronger failure of $O(n^{-0.83})$, see the explicit statement in [2, p. 338], referring to [1, Theorem 8.4]; see also the attribution to Pommerenke in [3, Problem 6.84]. Choose m with $2m|\gamma_m(g_*)| > T$ and then choose $r < 1$ sufficiently close to one that $2mr^m|\gamma_m(g_*)| > T$. The function $g(z) = g_*(rz)/r$ is holomorphic and injective on $|z| < 1/r$ and satisfies $L_g(z) = L_{g_*}(rz)$. $\square$

Lemma 3 (Root insertion). *Suppose $f, g \in \mathcal{S}$ are holomorphic on neighborhoods of $\overline{\mathbb{D}}$. Put $F = L_f$, $H = L_g$, and choose a real number*

$$A \geq \max\{0, \max_{\overline{\mathbb{D}}} \operatorname{Re} H\}.$$

For each integer $N \geq 1$ define

$$\phi_N(z) = z \exp\left(\frac{H(z^N) - A}{N}\right), \quad f_N(z) = e^{A/N} f(\phi_N(z)). \quad (5)$$

Then $f_N \in \mathcal{S}$, it is holomorphic past $\overline{\mathbb{D}}$, and

$$F_N := L_{f_N} = \frac{H(z^N)}{N} + F(\phi_N(z)). \quad (6)$$

For fixed $0 < \alpha < 1$,

$$\|F_N - F\|_\alpha \longrightarrow 0 \quad (N \rightarrow \infty). \quad (7)$$

If $m \geq 1$ is fixed, then

$$Nm[z^{Nm}]F_N \longrightarrow m[z^m]H. \quad (8)$$

Proof. The logarithms F, H extend to neighborhoods of the closed disk. For example, the growth theorem gives $|f(z)/z| \geq (1 + |z|)^{-2}$ in $\mathbb{D}$, so continuity excludes zeros on $\overline{\mathbb{D}}$, and a slightly larger disk is zero-free. The same argument applies to g .

Let $u_N(z) = z \exp(H(z^N)/N)$. Then $u_N(z)^N = g(z^N)$. If $u_N(z) = u_N(w)$, injectivity of g implies $w^N = z^N$. Apart from the zero case, write $w = \omega z$ with $\omega^N = 1$. The identity $u_N(\omega z) = \omega u_N(z)$ now implies $\omega = 1$. Thus u_N , and hence ϕ_N , is injective. Also

$$|\phi_N(z)| \leq |z| \quad (z \in \overline{\mathbb{D}}), \quad \phi'_N(0) = e^{-A/N}.$$

Therefore (5) is an injective normalized composition. Its extension past the closed disk follows from compactness and the extensions of its factors. Direct substitution gives (6), including its normalization at zero.

All constants in the next estimates may depend on the fixed functions f, g but not on N . Uniformly on $\overline{\mathbb{D}}$,

$$\|\phi_N - \operatorname{id}\|_\infty = O(N^{-1}), \quad \|H(z^N)/N\|_\infty = O(N^{-1}).$$

Since F' is bounded, $\Delta_N = F_N - F$ satisfies $\|\Delta_N\|_\infty = O(N^{-1})$. Furthermore,

$$\phi'_N(z) = \exp\left(\frac{H(z^N) - A}{N}\right) (1 + z^N H'(z^N)), \quad \left(\frac{H(z^N)}{N}\right)' = z^{N-1} H'(z^N).$$

It follows that $\|\Delta'_N\|_\infty = O(1)$. Convexity of $\overline{\mathbb{D}}$ and the fundamental theorem of calculus give, for any such analytic Q ,

$$|Q(z) - Q(w)| \leq \min\{2\|Q\|_\infty, \|Q'\|_\infty |z - w|\},$$

and hence

$$[Q]_\alpha \leq (2\|Q\|_\infty)^{1-\alpha} \|Q'\|_\infty^\alpha.$$

Consequently $\|\Delta_N\|_\alpha = O(N^{\alpha-1})$, proving (7).

To prove the transfer, write $F(z) = \sum_{j \geq 1} b_j z^j$. The Taylor series of $\phi_N(z)^j$ contains only degrees congruent to j modulo N , and has no degree below j . Thus only $j = \ell N$, $1 \leq \ell \leq m$, can contribute to degree Nm in $F(\phi_N)$. Since $\|\phi_N\|_\infty \leq 1$, the Cauchy coefficient bound gives

$$|[z^{Nm}]F(\phi_N)| \leq \sum_{\ell=1}^m |b_{\ell N}|. \quad (9)$$

As F is holomorphic past the closed disk, there are $R > 1$ and $C < \infty$ with $|b_j| \leq CR^{-j}$. Therefore

$$Nm |[z^{Nm}]F(\phi_N)| \leq \frac{NmCR^{-N}}{1 - R^{-N}} \rightarrow 0.$$

Together with (6), this is (8). $\square$

3 A limit retaining infinitely many large coefficients

Proof of Theorem 1. Start with $f_0(z) = z$, $F_0 = 0$, and put $\alpha_k = 1 - 1/(k+1)$. Inductively suppose that $f_0, \dots, f_{k-1}$ and increasing indices $n_1, \dots, n_{k-1}$ have been selected, with each f_j holomorphic past $\overline{\mathbb{D}}$. Lemma 2 supplies a seed g_k and an integer m_k with

$$|m_k[z^{m_k}]L_{g_k}| > 4k + 2. \quad (10)$$

Set

$$\varepsilon_k = 2^{-k} \min(\{1\} \cup \{n_i^{-1} : 1 \leq i < k\}).$$

Apply Lemma 3 to f_{k-1}, g_k . Choose N_k so large that $n_k = N_k m_k$ exceeds every previously selected index and that the resulting $f_k, F_k = L_{f_k}$ satisfy

$$\|F_k - F_{k-1}\|_{\alpha_k} < \varepsilon_k, \quad (11)$$

$$|n_k[z^{n_k}]F_k - m_k[z^{m_k}]L_{g_k}| < 1. \quad (12)$$

This is possible because $m_k, f_{k-1}, g_k, \alpha_k$ are fixed before N_k is chosen. In particular, (10) and (12) imply

$$|n_k[z^{n_k}]F_k| > 4k + 1. \quad (13)$$

Fix $0 < \alpha < 1$. For all sufficiently large k , $\alpha_k \geq \alpha$. Since the diameter of $\overline{\mathbb{D}}$ is 2, we have $\|Q\|_\alpha \leq 2\|Q\|_{\alpha_k}$ for these k . The summable bound (11) therefore makes F_k Cauchy in $C^\alpha(\overline{\mathbb{D}})$. The limits for different α coincide by uniform convergence. Denote the common limit by F . It is holomorphic on $\mathbb{D}$, has $F(0) = 0$, and belongs to every $\mathcal{A}^\alpha$, $0 < \alpha < 1$.

The functions $f_k = ze^{F_k}$ converge locally uniformly to $f = ze^F$. Each f_k is univalent and $f'(0) = 1$, so the univalent limit theorem implies $f \in \mathcal{S}$. The function F is bounded; indeed (11) gives $\|F\|_\infty \leq 1$. Thus f is bounded as well.

It remains essential to check that later insertions do not erase earlier coefficients. For fixed i and every $k > i$, the Cauchy bound and (11) give

$$n_i |[z^{n_i}](F_k - F_{k-1})| \leq n_i \varepsilon_k \leq 2^{-k}.$$

Summing over $k > i$ and using (13), we obtain

$$|n_i[z^{n_i}]F| \geq |n_i[z^{n_i}]F_i| - 2^{-i} > 4i + 1 - 2^{-i} > 4i.$$

Equation (1) now shows $n_i |\gamma_{n_i}(f)| > 2i$, proving failure of $O(1/n)$.

For completeness, the coefficient consequence of Hölder continuity can be seen without a boundary differentiation theorem. If $U \in \mathcal{A}^\alpha$, let $u(t) = U(e^{it})$ and put $h = \pi/n$. Shifting the Fourier integral by h yields

$$2|[z^n]U| \leq \frac{1}{2\pi} \int_0^{2\pi} |u(t+h) - u(t)| dt \leq [U]_\alpha (\pi/n)^\alpha.$$

For every fixed p , the function $\exp(F/p)$ belongs to every $\mathcal{A}^\alpha$: on the bounded range of F the exponential is Lipschitz, as follows by integrating its derivative along the segment between two values. By (3), $c_n^{(p)}(f) = O(n^{-\alpha})$ for each $\alpha < 1$. For $0 < \eta < 1$ choose $\alpha = 1 - \eta$; for $\eta \geq 1$ bounded coefficients suffice. This proves the asserted stronger estimate. For $p \geq 3$ one may take $\alpha = 1 - 1/p$, and for $p = 1, 2$ bounded coefficients already imply (4). $\square$

4 The role of uniformity

The construction uses one function f for all values of p . It does not produce constants uniform in p with the normalization needed for a converse. Milin already proved a converse under a bound on $pA(p)$ when $|c_n^{(p)}| \leq A(p)n^{2/p-1}$ along an unbounded sequence of integers p ; see [4, Theorem 2]. The distinction can be checked directly: for each fixed n ,

$$p[z^n]e^{F/p} \longrightarrow [z^n]F = 2\gamma_n(f).$$

If one had $p|c_n^{(p)}| \leq Cn^{2/p-1}$ with a single C for all n and an unbounded sequence of p , taking $p \rightarrow \infty$ would give $2|\gamma_n(f)| \leq C/n$. Theorem 1 shows why the dependence of the constants cannot simply be discarded.

The construction is an analytic existence argument, with smooth seeds chosen from a classical theorem. It is not a numerical list of the coefficients of the limit function. The accompanying standard-library Python script checks the logarithmic composition identity against direct rational power-series expansion in 15 finite cases, including the congruence filter and its error bound. Those checks supplement, rather than establish, the infinite proof.

This preprint was developed with AI assistance and self-audited. The classical inputs are credited explicitly. Independent peer review, proof-assistant verification, and exhaustive priority certification are not claimed.

References

- [1] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften 259, Springer-Verlag, 1983.
- [2] D. Girela, Logarithmic coefficients of univalent functions, *Ann. Acad. Sci. Fenn. Math.* **25** (2000), 337–350. <https://www.acadsci.fi/mathematica/Vol25/girela.pdf>.
- [3] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, arXiv:1809.07200v2, 2018, Problem 6.84, pp. 146–147. <https://arxiv.org/abs/1809.07200>.
- [4] I. M. Milin, A problem for the coefficients of p -fold symmetric univalent functions, *Mat. Zametki* **38** (1985), 66–73; English translation, *Math. Notes* **38** (1985), 549–553. <https://doi.org/10.1007/BF01137466>.