

A Counterexample to the Area Inequality for Geometric Means of Exterior Univalent Functions

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Abstract

A question of D. K. Thomas, recorded as Problem 6.50 by Hayman and Lingham, asks whether the geometric mean of two normalized exterior univalent functions satisfies the area-theorem coefficient inequality. We give an explicit negative answer with exponent $1/2$. Both factors are odd, zero-free, and univalent on $|z| > 1$, and their normalized geometric mean is analytic there. Its first four nonzero negative Laurent coefficients already have weighted squared sum $269463/262144 > 1$. A fixed radial dilation yields factors and a mean analytic across the unit circle, with the full coefficient sum finite and greater than $101/100$. The proof uses elementary conformal maps and exact rational coefficient calculations.

1 The question and an explicit pair

Write $\mathbb{D} = \{w : |w| < 1\}$ and $\mathbb{E} = \{z : |z| > 1\}$. We use the normalization

$$\Sigma_0 = \left\{ f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} : f \text{ is analytic and univalent on } \mathbb{E} \right\}.$$

Thus the constant term is zero. This is the class denoted by Σ in [1, Chapter 6, equation (6.2), p. 114]. The example below also belongs to the usual larger class allowing a constant term.

Problem 6.50 of [1, p. 137], attributed to Thomas, asks whether the normalized geometric mean

$$F = f^{1-\alpha} g^\alpha = z + \sum_{n=0}^{\infty} A_n z^{-n}, \quad 0 \leq \alpha \leq 1,$$

must satisfy $\sum_{n \geq 1} n|A_n|^2 \leq 1$. The source refers to [2], which studies the coefficients of such means. Our example has no exterior zeros or branch obstruction: F is a single-valued analytic function throughout $\mathbb{E}$.

Let $s(w) = \sqrt{1 - w^2}$ denote the analytic square root on $\mathbb{D}$ with $s(0) = 1$, and put

$$B(w) = \frac{(1 - i)s(w) + (1 + i)(1 - iw)}{2}. \quad (1)$$

Define

$$f(z) = z - z^{-1}, \quad g(z) = zB(z^{-2}). \quad (2)$$

Theorem 1. *The functions f, g in (2) belong to Σ_0 and have no zeros in $\mathbb{E}$. Their normalized geometric mean*

$$F(z) = z\sqrt{(1 - z^{-2})B(z^{-2})}$$

is analytic on $\mathbb{E}$, where the square root is normalized to 1 at infinity. Its nonzero coefficients through order z^{-7} are

$$A_1 = -\frac{1+i}{4}, \quad A_3 = \frac{-6+5i}{16}, \quad A_5 = \frac{-3-9i}{64}, \quad A_7 = \frac{-15+52i}{512}.$$

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Consequently

$$|A_1|^2 + 3|A_3|^2 + 5|A_5|^2 + 7|A_7|^2 = \frac{269463}{262144} > 1. \quad (3)$$

In particular, the answer to Problem 6.50 is negative.

The constructions used to establish univalence are classical. The contribution is the explicit pair (2), its finite coefficient certificate, and the regularized example below. We make no claim that the mapping tools are new, or that a bounded literature search establishes absolute priority.

2 Univalence and the analytic branches

We first record the elementary root transform needed for g .

Lemma 2. *Suppose h is analytic and univalent on $\mathbb{D}$, with $h(0) = 0$ and $h'(0) = 1$. Let b be the analytic square root of $w/h(w)$ on $\mathbb{D}$, normalized by $b(0) = 1$. Then $G(z) = zb(z^{-2})$ belongs to Σ_0 , is odd, and has no zeros on $\mathbb{E}$.*

Proof. Univalence implies that h has no zero except its simple zero at 0. Thus $w/h(w)$ extends to a nonvanishing analytic function on the simply connected disk, and b exists and never vanishes. The expansion of b at zero gives the required exterior normalization and oddness. Moreover

$$G(z)^2 = \frac{1}{h(z^{-2})}.$$

If $G(z_1) = G(z_2)$, univalence of h gives $z_1^{-2} = z_2^{-2}$, so $z_2 = z_1$ or $z_2 = -z_1$. The latter alternative contradicts oddness and nonvanishing. $\square$

Let

$$k(w) = \frac{w}{(1-w)^2}.$$

The Cayley map $c(w) = (1+w)/(1-w)$ maps $\mathbb{D}$ onto the right half-plane, and $k = (c^2 - 1)/4$. Squaring is univalent on that half-plane and maps it onto $\mathbb{C} \setminus (-\infty, 0]$. Hence

$$k : \mathbb{D} \longrightarrow \mathbb{C} \setminus (-\infty, -1/4]$$

is a conformal bijection. Since $\frac{1}{2}k(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/8] \subset k(\mathbb{D})$, the function

$$\phi(w) = k^{-1}\left(\frac{1}{2}k(w)\right)$$

is an analytic univalent self-map of $\mathbb{D}$ fixing 0, with $\phi'(0) = 1/2$. Set

$$p(w) = -i\phi(iw), \quad h(w) = \frac{2p(w)}{(1+p(w))^2}. \quad (4)$$

The first map is a rotation of ϕ ; the second is twice $k_-(p(w))$, where $k_-(u) = u/(1+u)^2 = -k(-u)$ is univalent on $\mathbb{D}$. Therefore h is analytic and univalent on $\mathbb{D}$ and satisfies $h(0) = 0$, $h'(0) = 1$.

The defining identity for ϕ gives

$$\frac{p(w)}{(1-ip(w))^2} = \frac{w}{2(1-iw)^2}. \quad (5)$$

It follows that

$$\frac{w}{h(w)} = \left((1-iw) \frac{1+p(w)}{1-ip(w)} \right)^2.$$

All denominators are nonzero on $\mathbb{D}$, because $|p(w)| < 1$. The expression inside the square is analytic, nonvanishing, and equal to 1 at 0. It is therefore the normalized square root appearing in Lemma 2.

For clarity, this expression is precisely (1). Indeed, (5) implies

$$(1-iw)^2 \left(\frac{1+ip(w)}{1-ip(w)} \right)^2 = 1-w^2.$$

Normalization at 0 identifies its analytic square root, so

$$s(w) = (1 - iw) \frac{1 + ip(w)}{1 - ip(w)}.$$

Substituting this into (1) and simplifying yields

$$B(w) = (1 - iw) \frac{1 + p(w)}{1 - ip(w)}.$$

Lemma 2 proves all claims about g . The function f is the same root transform applied to k , since $w/k(w) = (1 - w)^2$. Finally $(1 - w)B(w)$ is nonvanishing on $\mathbb{D}$ and equals 1 at 0. Its normalized analytic square root defines F , and $F^2 = fg$ throughout $\mathbb{E}$.

3 A four-coefficient certificate

From $s(w) = 1 - \frac{1}{2}w^2 - \frac{1}{8}w^4 + O(w^6)$, we obtain

$$B(w) = 1 + \frac{1-i}{2}w - \frac{1-i}{4}w^2 - \frac{1-i}{16}w^4 + O(w^6).$$

Write

$$H(w) = \sqrt{(1-w)B(w)} = 1 + \sum_{j \geq 1} c_j w^j.$$

Equating coefficients in $H(w)^2 = (1-w)B(w)$ successively gives

j	$c_j = A_{2j-1}$	$(2j-1) c_j ^2$
1	$-(1+i)/4$	1/8
2	$(-6+5i)/16$	183/256
3	$(-3-9i)/64$	450/4096
4	$(-15+52i)/512$	20503/262144

Their sum is (3). Every omitted term in the area sum is nonnegative, so no tail estimate is needed. This completes the proof of Theorem 1.

The coefficient calculation can also be reproduced using the recurrence

$$c_0 = 1, \quad c_j = \frac{1}{2} \left(d_j - \sum_{\ell=1}^{j-1} c_\ell c_{j-\ell} \right), \quad (1-w)B(w) = \sum_{j \geq 0} d_j w^j.$$

Only arithmetic in $\mathbb{Q}(i)$ is involved.

4 An example analytic across the unit circle

Corollary 3. *Set $R = 1001/1000$ and*

$$f_R(z) = R^{-1}f(Rz), \quad g_R(z) = R^{-1}g(Rz).$$

Both factors are univalent and zero-free on $|z| > R^{-1}$, and their normalized geometric mean $F_R(z) = R^{-1}F(Rz)$ is analytic there. If $F_R(z) = z + \sum_{n \geq 1} A_n^{(R)} z^{-n}$, then

$$\frac{101}{100} < \sum_{n \geq 1} n |A_n^{(R)}|^2 < \infty.$$

Proof. The mapping statements follow immediately by dilation. The Laurent coefficients satisfy $A_n^{(R)} = A_n R^{-(n+1)}$. Since $R > 1$, the first four contributions therefore give

$$\sum_{n \geq 1} n |A_n^{(R)}|^2 \geq R^{-16} \frac{269463}{262144}.$$

The binomial expansion and $\binom{16}{j} \leq 16^j$ give

$$R^{16} < \sum_{j=0}^{\infty} (16/1000)^j = \frac{125}{123}.$$

Thus the preceding lower bound is strictly greater than

$$\frac{123\,269463}{125\,262144} = \frac{33143949}{32768000} > \frac{101}{100}.$$

To see finiteness, choose any r with $R^{-1} < r < 1$ and apply Cauchy's Laurent coefficient estimate on $|z| = r$ to $F_R(z) - z$. It gives $|A_n^{(R)}| \leq Mr^n$ for some finite M , which implies convergence of the weighted squared sum. $\square$

The final argument is a complete counterexample to the stated area inequality, including the case where the mean is analytic across the boundary. It does not assert an optimal upper bound for geometric means. AI assistance was used in exploration, algebraic verification, and manuscript preparation. The written mapping argument and the finite rational calculation supply the proof; the accompanying code is a reproducibility aid. This preprint has not undergone independent peer review or formal proof verification.

References

- [1] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory (New Edition)*, 2018, Chapter 6, equation (6.2) and Problem 6.50. [arXiv:1809.07200v2](https://arxiv.org/abs/1809.07200v2).
- [2] D. K. Thomas, On the coefficients of meromorphic univalent functions, *Proc. Amer. Math. Soc.* **47** (1975), 161–166. [doi:10.1090/S0002-9939-1975-0357765-2](https://doi.org/10.1090/S0002-9939-1975-0357765-2).