

Dual Tests and Spatial Obstructions for Positive-Real-Part Majorants

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Abstract

We give an attained variational test for almost-everywhere domination of a nonnegative measurable function on the circle by the boundary modulus of a holomorphic function with positive real part. For $h = \log \max(1, f)$, the logarithm of the least center modulus for a majorant of $\max(1, f)$ is

$$\sup_{\substack{g \in C^\infty(\mathbb{T}), g \geq 0 \\ \int g \, dm = 1}} \left\{ \int h g \, dm - \frac{\pi}{2} \|\mathcal{H}g\|_1 \right\},$$

where $\mathcal{H}$ is the periodic Hilbert transform; finiteness is the exact existence criterion. The tests can be restricted to a countable family of squared polynomial moduli. We also construct explicit separated spikes in weak L^1 which admit no such majorant, although their symmetric decreasing rearrangement does. For N spikes, the weak L^1 quasinorm stays bounded while the least center modulus lies between $N/8$ and $N + 1$.

1 The boundary majorant problem

Let m be normalized arclength on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Write $\mathcal{P}$ for the holomorphic functions F on $\mathbb{D}$ with $\operatorname{Re} F > 0$, without imposing a normalization on $F(0)$. Their finite nontangential boundary values are denoted by F^* . Korenblum's question, recorded as Problem 5.74 in Hayman–Lingham [1], asks which nonnegative measurable f satisfy $f \leq |F^*|$ almost everywhere for some $F \in \mathcal{P}$.

We treat this question by an explicit dual variational criterion. The logarithm converts the half-plane condition into a bound on a harmonic conjugate, and weak-star compactness gives the converse implication and attainment. This uses classical harmonic analysis and convex separation. The accompanying separated-spike construction shows that distributional integrability alone does not characterize the class. We make no absolute priority claim for these formulations.

Use the Hilbert-transform convention

$$\widehat{\mathcal{H}u}(n) = -i \operatorname{sgn}(n) \widehat{u}(n), \quad \mathcal{H}1 = 0.$$

All norms and integrals on $\mathbb{T}$ use m . For a nonnegative measurable f , put $h = \log \max(1, f)$ and define

$$\mathcal{D}(h) = \sup_{\substack{g \in C^\infty(\mathbb{T}; \mathbb{R}), g \geq 0 \\ \int g \, dm = 1}} \left(\int h g \, dm - \frac{\pi}{2} \|\mathcal{H}g\|_1 \right) \quad (1)$$

when $h \in L^1$, and set $\mathcal{D}(h) = +\infty$ otherwise.

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Theorem 1. *A nonnegative measurable f has an almost-everywhere majorant $|F^*|$ with $F \in \mathcal{P}$ if and only if $\mathcal{D}(h) < \infty$. When finite,*

$$\mathcal{D}(h) = \min_{\substack{F \in \mathcal{P} \\ |F^*| \geq \max(1, f) \text{ a.e.}}} \log |F(0)|. \quad (2)$$

Moreover, the supremum in (1) is unchanged on restricting the tests to

$$g_p = \frac{|p|^2}{\int |p|^2 dm}, \quad 0 \neq p \in (\mathbb{Q} + i\mathbb{Q})[z]. \quad (3)$$

The clamping in (2) is part of the statement: the formula does not assert the same optimum for the unclamped datum. Indeed, $F + 1$ majorizes $\max(1, f)$ whenever F majorizes f , because $|1 + w| \geq \max(1, |w|)$ for $\operatorname{Re} w \geq 0$. This proves equivalence of existence, but does not preserve the center modulus. The criterion consists of prescribed real-variable tests, not a finite decision procedure for arbitrary measurable data.

2 Logarithms and duality

Let $P[u]$ denote Poisson extension. We first record the precise boundary representation needed for the compactness argument.

Lemma 2. *For $F \in \mathcal{P}$ there is a real $\theta \in L^\infty(\mathbb{T})$, $|\theta| \leq \pi/2$, such that*

$$\log |F^*| = C - \mathcal{H}\theta \quad \text{a.e.}, \quad C = \log |F(0)|. \quad (4)$$

Conversely, every real C and such θ give the boundary logarithmic modulus of a function in $\mathcal{P}$.

Proof. Take the branch $L = \log F$ with $|\operatorname{Im} L| < \pi/2$. Its bounded harmonic imaginary part has the form $P[\theta]$. The L^2 Hilbert transform and harmonic conjugacy imply

$$L = C - P[\mathcal{H}\theta] + iP[\theta].$$

There is no extra singular harmonic term: the harmonic conjugate of $P[\theta]$ is unique up to a real constant. Since θ and $\mathcal{H}\theta$ belong to L^2 , this expression belongs to H^2 and has finite nontangential boundary values almost everywhere. Exponentiation gives (4) and the asserted boundary values of F .

Conversely, the displayed expression defines a holomorphic L . Unless θ is identically one of the endpoint constants $\pm\pi/2$, the maximum principle gives $|\operatorname{Im} L| < \pi/2$, so $F = e^L \in \mathcal{P}$. In the endpoint-constant cases, $\mathcal{H}\theta = 0$ and the positive constant e^C has the same boundary modulus. Thus strict positivity survives the limiting case as well. $\square$

Proof of Theorem 1. First suppose F majorizes $\max(1, f)$. Lemma 2 gives $0 \leq h \leq C - \mathcal{H}\theta$. In particular $h \in L^1$. The skew-adjointness of $\mathcal{H}$ gives, for every admissible g ,

$$\int hg \, dm \leq C - \int (\mathcal{H}\theta)g \, dm = C + \int \theta \mathcal{H}g \, dm \leq C + \frac{\pi}{2} \|\mathcal{H}g\|_1.$$

Hence $\mathcal{D}(h) \leq \log |F(0)|$.

Now suppose $C = \mathcal{D}(h) < \infty$. Consider the weak-star compact real ball

$$B = \{\theta \in L^\infty(\mathbb{T}; \mathbb{R}) : |\theta| \leq \pi/2\}.$$

For each nonnegative smooth g , impose the weak-star closed constraint

$$\int \theta \mathcal{H}g \, dm \geq \int hg \, dm - C \int g \, dm. \quad (5)$$

The functional on the left is continuous because $\mathcal{H}g \in L^1$. These constraints have the finite intersection property. Indeed, suppose the constraints for $g_1, \dots, g_k$ were inconsistent. The compact convex image of B under $\theta \mapsto (\int \theta \mathcal{H}g_j dm)_{j=1}^k$ could then be strictly separated from the closed upper orthant with corner $a_j = \int h g_j dm - C \int g_j dm$. A separating normal has coordinates $\lambda_j \geq 0$, and gives

$$\sup_{\theta \in B} \int \theta \mathcal{H} \left(\sum_j \lambda_j g_j \right) dm < \int (h - C) \left(\sum_j \lambda_j g_j \right) dm.$$

The supremum on the left equals $(\pi/2) \|\mathcal{H}(\sum_j \lambda_j g_j)\|_1$. The nonnegative sum is nonzero in a strict separation, so normalization contradicts (1).

Compactness therefore gives one θ satisfying all (5). Since $h, \mathcal{H}\theta \in L^1$, these inequalities mean $h \leq C - \mathcal{H}\theta$ almost everywhere. Lemma 2 supplies an $F \in \mathcal{P}$ with the required majorant and $\log |F(0)| = C$. This proves both sufficiency and attainment. The constant test $g = 1$ also gives $\mathcal{D}(h) \geq \int h dm \geq 0$.

Finally, the Fejér means of a smooth nonnegative g are nonnegative trigonometric polynomials converging uniformly to g , with Hilbert transforms converging in L^2 and hence in L^1 . By the classical Fejér–Riesz factorization, recalled in [3, Section 1], each is $|p|^2$ for an analytic polynomial. Approximate the coefficients of p by rational complex numbers and normalize its squared modulus. Uniform convergence controls pairing with $h \in L^1$, and L^2 convergence controls the Hilbert-transform norm. This proves (3). $\square$

Proposition 3. *The coefficient $\pi/2$ in (1) cannot be decreased. For $h_0(t) = \log(2/|1 - e^{it}|)$, one has $\mathcal{D}(h_0) = \log 2$.*

Proof. The function $2/(1 - z)$ belongs to $\mathcal{P}$ and has boundary modulus at least one. Its center value and the constant test give $\mathcal{D}(h_0) = \log 2$. For the Poisson kernel $g_r = P_r$ centered at 1,

$$\mathcal{H}g_r(t) = \frac{2r \sin t}{1 - 2r \cos t + r^2}, \quad \|\mathcal{H}g_r\|_1 = \frac{2}{\pi} \log \frac{1+r}{1-r}.$$

The norm identity follows by integrating on $(0, \pi)$ and using symmetry. Poisson extension of h_0 gives

$$\int h_0 g_r dm = \log 2 - \log(1 - r).$$

If $\pi/2$ in (1) is replaced by $a < \pi/2$, the expression at g_r diverges to $+\infty$ with leading term $(1 - 2a/\pi) \log(1/(1 - r))$, despite the explicit majorant. $\square$

3 A local mass obstruction

For a finite positive measure ν on $\mathbb{T}$, write

$$\mathcal{G}_\nu(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta).$$

The classical Herglotz representation [2] says that every $F \in \mathcal{P}$ can be written

$$F = ib + \mathcal{G}_\mu, \quad M := \mu(\mathbb{T}) = \operatorname{Re} F(0), \quad b = \operatorname{Im} F(0). \quad (6)$$

For completeness, this follows by taking a weak limit of the positive measures $\operatorname{Re} F(re^{it}) dm(t)$, whose masses equal $\operatorname{Re} F(0)$. The Poisson formula on smaller disks identifies the limit's Poisson integral with $\operatorname{Re} F$, and harmonic conjugacy fixes the remaining imaginary constant.

Lemma 4. *If ν is finite and positive and $\lambda > 0$, then*

$$m\{|\mathcal{G}_\nu^*| \geq \lambda\} \leq \frac{2\nu(\mathbb{T})}{\lambda + \nu(\mathbb{T})} \leq \frac{2\nu(\mathbb{T})}{\lambda}. \quad (7)$$

Proof. The zero measure is immediate. Otherwise $\operatorname{Re} \mathcal{G}_\nu > 0$. The analytic function $Q = \mathcal{G}_\nu/(\lambda + \mathcal{G}_\nu)$ is bounded and has real part between zero and one. For $\operatorname{Re} w \geq 0$,

$$2 \operatorname{Re} \frac{w}{\lambda + w} - 1 = \frac{|w|^2 - \lambda^2}{|\lambda + w|^2}.$$

Thus $\operatorname{Re} Q^* \geq 1/2$ wherever $|\mathcal{G}_\nu^*| \geq \lambda$. Its boundary mean equals $\operatorname{Re} Q(0) = \nu(\mathbb{T})/(\lambda + \nu(\mathbb{T}))$, proving (7). Boundary values of $\mathcal{G}_\nu$ follow from Lemma 2; boundedness of Q justifies passage to its boundary mean. $\square$

Lemma 5. *Let J be an arc of angular radius $0 < \rho \leq 1$ and I a concentric arc of radius at most $\rho/2$. Suppose $F = ib + \mathcal{G}_\mu \in \mathcal{P}$ and $|F^*| \geq A$ a.e. on I . If*

$$A \geq 2(|b| + 4M/\rho), \quad (8)$$

then

$$\mu(J) \geq \frac{A m(I)}{4}. \quad (9)$$

Proof. For $e^{it} \in I$ and $e^{is} \notin J$, circular angular distance is at least $\rho/2$. The boundary kernel has modulus $|\cot((t-s)/2)| \leq 4/\rho$. Consequently $|\mathcal{G}_{\mu|_{\mathbb{T} \setminus J}}^*| \leq 4M/\rho$ on I . This part extends analytically across I , so endpoint atoms cause no issue. By (8), $|\mathcal{G}_{\mu|_J}^*| \geq A/2$ a.e. on I . Apply Lemma 4 with $\lambda = A/2$. $\square$

4 Separated spikes and their rearrangements

Use angular coordinates $(-\pi, \pi]$. For $n \geq 20$, set

$$t_n = \frac{1}{n}, \quad \rho_n = \frac{1}{10n^2}, \quad J_n = (t_n - \rho_n, t_n + \rho_n), \quad I_n = (t_n - \pi 2^{-n}, t_n + \pi 2^{-n}). \quad (10)$$

Define

$$f = \sum_{n=20}^{\infty} 2^n \mathbf{1}_{I_n}, \quad f_N = \sum_{n=20}^{N+19} 2^n \mathbf{1}_{I_n}.$$

The intervals accumulate only at 0; set $f(0) = 0$.

Theorem 6. *The datum f is finite everywhere and belongs to weak $L^1(\mathbb{T})$, with*

$$\sup_{\lambda > 0} \lambda m\{f > \lambda\} \leq 2.$$

It admits no majorant $|F^*|$ with $F \in \mathcal{P}$. Its symmetric decreasing rearrangement does admit such a majorant. For every $N \geq 1$,

$$\frac{N}{8} \leq \mathcal{M}(f_N) := \inf_{\substack{F \in \mathcal{P} \\ |F^*| \geq \max(1, f_N) \text{ a.e.}}} |F(0)| \leq N + 1. \quad (11)$$

Proof. The arcs J_n are pairwise disjoint, because

$$\rho_n + \rho_{n+1} < \frac{1}{5n^2} < \frac{1}{n(n+1)} = t_n - t_{n+1}.$$

Also I_n lies in the middle half of J_n . Indeed, $20\pi n^2 \leq 2^n$ holds at $n = 20$ and its ratio increases thereafter. In particular $m(I_n) = 2^{-n}$. Summing the geometric tail of these lengths gives the weak L^1 bound.

If $F = ib + \mathcal{G}_\mu$ majorized f , then $2^n \rho_n \rightarrow \infty$ would make (8) valid for all sufficiently large n . Lemma 5 would give $\mu(J_n) \geq 1/4$ for all those n , contradicting their disjointness and finiteness of μ .

For the upper bound in (11), use $F = \mathcal{G}_{u_N m}$ with $u_N = \max(1, f_N)$. Its boundary real part is u_N a.e., so it is a majorant and $F(0) = \int u_N dm \leq N + 1$.

For the lower bound, suppose $B = |F(0)| < N/8$. Then $M \leq B$ and $|b| \leq B$. The elementary estimate $2^n \geq 100n^3$ holds for $n \geq 20$: it holds at 20 and follows inductively from $2n^3 > (n+1)^3$. Thus $2^n \rho_n \geq 10n$. For the last $\lceil N/2 \rceil$ indices in the sum defining f_N , one has $n \geq 20 + \lfloor N/2 \rfloor > N/2$ and

$$2^n \rho_n > 5N > 10B \geq 2(|b|\rho_n + 4M),$$

using $\rho_n \leq 1$. Apply Lemma 5 to those disjoint arcs: $M \geq \lceil N/2 \rceil/4 \geq N/8$, contrary to $M \leq B$. This argument allows complex $F(0)$; no rotation of the half-plane has been made.

Finally, let $f^\#$ be the symmetric decreasing rearrangement. The weak L^1 bound gives $f^\#(e^{it}) \leq 2\pi/|t|$ for $0 < |t| \leq \pi$. Since $|1 - e^{it}| \leq |t|$,

$$f^\#(e^{it}) \leq \left| \frac{2\pi}{1 - e^{it}} \right|.$$

The holomorphic function $2\pi/(1 - z)$ has positive real part. Values at $t = 0$ may be changed without affecting the a.e. conclusion. $\square$

Corollary 7. *The class of nonnegative data with positive-real-part modulus majorants is not rearrangement invariant. Its least clamped center modulus cannot be bounded by the weak L^1 quasinorm alone.*

5 Scope and verification

Theorem 1 is an exact dual characterization of almost-everywhere majorizability. It retains spatial information through the Hilbert transform and uses only a countable collection of prescribed tests. It is not reduced to a rearrangement-invariant integrability condition; Theorem 6 excludes that possibility. The attainment statement concerns the center modulus of a majorant of $\max(1, f)$.

The arguments are analytic. Supplementary exact algebra checks the Cayley identity, the Hilbert-transform sign convention, and the spike inequalities; numerical quadrature checks the two Poisson identities used in Proposition 3. These computations are not a substitute for the proofs or a formal verification. AI assistance was used in developing the arguments, preparing the manuscript, and carrying out these checks. This preprint is unrefereed.

References

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