

Finite-Area Stolz Images at Almost Every Plessner Point

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Abstract

Ivrii recently constructed a Gaussian lacunary series for which almost every boundary point is a Plessner point, although every Stolz image has asymptotic area density zero. We strengthen the area conclusion by slowing the variance clock. If $0 < v_n \leq C/n$, $\sum_n v_n = \infty$, and $\sum_n v_n / \log(n+1) < \infty$, then almost surely the series $\sum_n \sqrt{v_n} \xi_n z^{2^n}$ has almost every boundary point Plessner, every Stolz image there has finite two-dimensional Lebesgue measure, and the areas of its truncated Stolz images tend to zero. Area is taken without multiplicity. The hypotheses hold for $v_n = 1/[n(\log(n+1))^\beta]$ with $0 < \beta \leq 1$. The proof combines Ivrii's shadowing argument with a summable covering by unit-time Wiener sausages.

1 Statement and relation to earlier work

Write $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let m_2 denote planar Lebesgue measure. For $\zeta \in \partial\mathbb{D}$, $A > 1$, and $0 < h < 1$, put

$$\Gamma_A(\zeta) = \{z \in \mathbb{D} : |z - \zeta| < A(1 - |z|)\}, \quad \Gamma_A(\zeta, h) = \Gamma_A(\zeta) \cap \{|z| > 1 - h\}.$$

A Plessner point is a boundary point at which the image of every truncated Stolz angle is dense in $\mathbb{C}$. Plessner's theorem gives the almost-everywhere alternative between an angular limit and a Plessner point; see [3].

Collingwood and Baernstein asked whether density in this conclusion could be replaced by omission of only a set of area zero, or of logarithmic capacity zero, respectively. These are Problems 5.20 and 5.57 in [2]. Ivrii [1] has already answered both questions negatively. For independent standard complex Gaussians ξ_n , he proved that $\sum_n n^{-1/2} \xi_n z^{2^n}$ has almost every boundary point Plessner, yet its Stolz images have asymptotic area density zero. He also proved that the exact nontangential range has area zero and, in general, has Hausdorff dimension two at almost every Plessner point.

The present note extends that construction to obtain finite total image area and vanishing area of truncated images. The negative resolution of the historical questions and the lacunary shadowing/Brownian-embedding method are due to Ivrii. Our additional step is the summability criterion for a slower variance clock.

Theorem 1. *Let $(v_n)_{n \geq 1}$ be deterministic positive numbers satisfying*

$$v_n \leq \frac{C}{n}, \quad \sum_{n=1}^{\infty} v_n = \infty, \quad \sum_{n=1}^{\infty} \frac{v_n}{\log(n+1)} < \infty. \quad (1)$$

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Let ξ_n be independent complex Gaussian variables with density $\pi^{-1}e^{-|z|^2}$. Almost surely

$$f(z) = \sum_{n=1}^{\infty} \sqrt{v_n} \xi_n z^{2^n} \quad (2)$$

converges locally uniformly and defines a holomorphic function on $\mathbb{D}$. There is a set $E \subseteq \partial\mathbb{D}$ of full arc measure such that every $\zeta \in E$ is a Plessner point and, for every $A > 1$,

$$m_2(f(\Gamma_A(\zeta))) < \infty, \quad \lim_{h \downarrow 0} m_2(f(\Gamma_A(\zeta, h))) = 0. \quad (3)$$

In particular, the images in the second limit are dense open subsets of $\mathbb{C}$ of arbitrarily small area.

Here and throughout, area is the measure of the image set. It does not count multiplicity, and no finiteness assertion for $\int_{\Gamma_A(\zeta)} |f'|^2 dm_2$ is made. This distinction is essential when comparing with finite-area hypotheses in Dirichlet-space results such as [5].

Corollary 2. *For every fixed $0 < \beta \leq 1$, the conclusions of Theorem 1 hold for*

$$f_\beta(z) = \sum_{n=1}^{\infty} \frac{\xi_n}{\sqrt{n} (\log(n+1))^{\beta/2}} z^{2^n}.$$

Proof. Integral comparison gives divergence of $\sum 1/[n(\log(n+1))^\beta]$ and convergence of $\sum 1/[n(\log(n+1))^{\beta+1}]$. The upper bound in (1) is immediate. $\square$

2 A finite-time sausage estimate

Normalize planar Brownian motion by $\mathbb{E}|B_t|^2 = t$, so that its transition density is

$$p_t(x, y) = \frac{1}{\pi t} \exp\left(-\frac{|y-x|^2}{t}\right).$$

We use its classical recurrence, strong Markov property, and reflection principle; see [4]. The following standard-order upper estimate is included with a proof.

Lemma 3. *For $0 < \epsilon \leq 1/2$, let $K_\epsilon = \bigcup_{0 \leq t \leq 1} \overline{B}(B_t, \epsilon)$. Then*

$$\mathbb{E} m_2(K_\epsilon) \leq \frac{\pi e^9}{\log(1/\epsilon)}. \quad (4)$$

Proof. For a fixed $a \in \mathbb{C}$, let τ_a be the first hitting time of $\overline{B}(a, \epsilon)$. If $|x-a| \leq \epsilon$, $|y-a| < 2\epsilon$, and $\epsilon^2 \leq u \leq 1$, then $|y-x| \leq 3\epsilon$ and $p_u(x, y) \geq e^{-9}/(\pi u)$. Thus Brownian motion started at x has expected occupation time in $B(a, 2\epsilon)$ during its next unit interval at least

$$4e^{-9}\epsilon^2 \int_{\epsilon^2}^1 \frac{du}{u} = 8e^{-9}\epsilon^2 \log(1/\epsilon).$$

On $\{\tau_a \leq 1\}$ the interval $[\tau_a, \tau_a+1]$ is contained in $[0, 2]$. The strong Markov property therefore gives

$$8e^{-9}\epsilon^2 \log(1/\epsilon) \mathbb{P}(\tau_a \leq 1) \leq \mathbb{E} \int_0^2 \mathbf{1}_{B(a, 2\epsilon)}(B_t) dt.$$

Integrate over a . The right side is $8\pi\epsilon^2$ by Tonelli. Continuity of the path on $[0, 1]$ gives $\{\tau_a \leq 1\} = \{a \in K_\epsilon\}$, so another application of Tonelli proves (4). $\square$

3 Summable variance-clock blocks

Put $T_0 = 0$ and $T_n = \sum_{k=1}^n v_k$. The next lemma isolates the extra summability responsible for finite area.

Lemma 4. *Under (1), almost surely*

$$U_N = \bigcup_{n \geq N} B(B_{T_n}, n^{-1/8})$$

has finite area for every $N \geq 1$, and $m_2(U_N) \rightarrow 0$ as $N \rightarrow \infty$.

Proof. For positive integers j define $n_j = \min\{n : T_n \geq j\}$. Since $T_n \rightarrow \infty$, these indices exist and tend to infinity. Minimality gives

$$j \leq T_{n_j} < j + v_{n_j}.$$

For all sufficiently large j , $v_{n_j} \leq 1/2$. Successive indices are then distinct and

$$T_{n_j} - T_{n_{j-1}} > 1 - v_{n_{j-1}} \geq \frac{1}{2}.$$

Consequently,

$$\sum_{n=n_{j-1}+1}^{n_j} \frac{v_n}{\log(n+1)} \geq \frac{T_{n_j} - T_{n_{j-1}}}{\log(n_j+1)} \geq \frac{1}{2 \log(n_j+1)}. \quad (5)$$

The blocks on the left are disjoint. The last hypothesis in (1), and comparability of $\log(n_j+1)$ with $\log n_j$ for large j , imply

$$\sum_{j \text{ large}} \frac{1}{\log n_j} < \infty. \quad (6)$$

For large j let

$$C_j = \bigcup_{j \leq t \leq j+1} \bar{B}(B_t, n_j^{-1/8}).$$

The radius is eventually at most $1/2$. Stationary increments, translation invariance of area, and Lemma 3 give

$$\mathbb{E} m_2(C_j) \leq \frac{8\pi e^9}{\log n_j}.$$

It follows from (6) and Tonelli that $\sum_{j \text{ large}} m_2(C_j) < \infty$ almost surely.

If $T_n \in [j, j+1)$, then $n \geq n_j$, and hence $B(B_{T_n}, n^{-1/8}) \subseteq C_j$. For all sufficiently large N ,

$$U_N \subseteq \bigcup_{j \geq \lfloor T_N \rfloor} C_j, \quad m_2(U_N) \leq \sum_{j \geq \lfloor T_N \rfloor} m_2(C_j) \rightarrow 0.$$

Adding finitely many initial disks proves finiteness for every N . No independence between the sausage areas is needed. $\square$

4 Shadowing and radial oscillation

We give the details of the adaptation of [1, Lemma 3.1 and Section 5]. The Gaussian tail formula and Borel–Cantelli imply that almost surely, for some $M < \infty$,

$$|\sqrt{v_n} \xi_n| \leq M n^{-1/4} \quad (n \geq 1), \quad (7)$$

because $\mathbb{P}(|\xi_n| > n^{1/4}) = e^{-\sqrt{n}}$ and $v_n \leq C/n$. In particular (2) converges absolutely and uniformly on compact subdisks. Set

$$S_n(\theta) = \sum_{k=1}^n \sqrt{v_k} \xi_k e^{i2^k \theta}, \quad S_0(\theta) = 0.$$

For $\zeta = e^{i\theta}$, let

$$V_n = \Gamma_A(\zeta) \cap \{1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\}, \quad n \geq 0.$$

If $z \in V_n$, factoring a difference of powers and using the definition of Γ_A gives

$$|z^{2^k} - \zeta^{2^k}| \leq A2^{k-n} \quad (k \leq n).$$

For $k > n$ we have $|z^{2^k}| \leq e^{-2^{k-n-1}}$. Thus

$$|f(z) - S_n(\theta)| \leq C_{A,\omega}(n+1)^{-1/4}, \quad z \in V_n, \quad (8)$$

uniformly in θ . Indeed, in the sum $\sum_{k \leq n} k^{-1/4} 2^{k-n}$, the terms $k \leq n/2$ contribute $O(n2^{-n/2})$, while the other terms contribute $O(n^{-1/4})$. The remaining tail is bounded by $n^{-1/4} \sum_{\ell \geq 1} e^{-2^{\ell-1}}$. The cases $n = 0, 1$ are absorbed into the constant. On the coefficient event (7), this argument works for every $A > 1$.

Fix θ . Circular symmetry and independence of the Gaussians show that the discrete processes $(S_n(\theta))_{n \geq 0}$ and $(B_{T_n})_{n \geq 0}$ have the same law. We check that recurrence survives sampling at T_n . The reflection principle, applied to both coordinates, gives for all large n

$$\mathbb{P}\left(\sup_{T_n \leq t \leq T_{n+1}} |B_t - B_{T_n}| > n^{-1/3}\right) \leq 4e^{-cn^{1/3}}$$

for a constant $c > 0$, since $T_{n+1} - T_n \leq C/(n+1)$. The probabilities are summable. Hence the oscillations between successive sampling times tend to zero almost surely. The intervals exhaust $[T_1, \infty)$, so recurrence and unboundedness of planar Brownian motion imply

$$\liminf_{n \rightarrow \infty} |B_{T_n}| = 0, \quad \limsup_{n \rightarrow \infty} |B_{T_n}| = \infty \quad \text{almost surely.}$$

Transfer this event in law to $S_n(\theta)$ and apply Fubini in θ . Almost surely, the same oscillations hold for almost every θ . Taking $r_n = 1 - 2^{-n}$ in (8) then yields

$$\liminf_{r \uparrow 1} |f(r\zeta)| = 0, \quad \limsup_{r \uparrow 1} |f(r\zeta)| = \infty \quad (9)$$

at almost every boundary point. These exclude an angular limit, finite or infinite. Plessner's theorem now makes almost every boundary point a Plessner point.

5 Proof of the area conclusions

For each fixed θ , Lemma 4 and equality in law give almost surely

$$U_{\theta,N} = \bigcup_{n \geq N} B(S_n(\theta), n^{-1/8}), \quad m_2(U_{\theta,1}) < \infty, \quad m_2(U_{\theta,N}) \longrightarrow 0. \quad (10)$$

The area in this formula is jointly measurable in the Gaussian sample and θ : it is the integral over $\mathbb{C}$ of the indicator of a countable union of open disks with measurable centers. Finiteness

and the limiting condition are measurable events. Fubini therefore gives a probability-one event on which (10) holds for almost every θ .

Intersect with the coefficient event and the full-measure conclusions following (9). Fix one such sample and one of its remaining boundary points $\zeta = e^{i\theta}$. For any $A > 1$, (8) implies

$$f(V_n) \subseteq B(S_n(\theta), n^{-1/8}) \quad \text{for all sufficiently large } n,$$

because $C_{A,\omega}(n+1)^{-1/4} < n^{-1/8}$ eventually. The union of the omitted initial shells lies in a compact subdisk, and its image is bounded. The remaining image is contained in $U_{\theta,N}$ for an appropriate N , so the full Stolz image has finite area.

For $h \leq 2^{-N}$ the region $\Gamma_A(\zeta, h)$ is contained in $\bigcup_{n \geq N} V_n$. For sufficiently large N , its image therefore has area at most $m_2(U_{\theta,N})$, which tends to zero. The same disk-union event works for every aperture; only the finite threshold depends on A . Thus no uncountable intersection of probability-one events is involved. The full-measure set can be chosen independently of A . Density follows from the Plessner property and openness from the open mapping theorem. This completes the proof of Theorem 1.

6 Consequences and scope

The exact range along a fixed aperture is

$$R_A(f, \zeta) = \bigcap_{m \geq 1} f(\Gamma_A(\zeta, 2^{-m})).$$

It consists of values attained exactly arbitrarily close to ζ , not all subsequential limit values. Equation (3) gives $m_2(R_A(f, \zeta)) = 0$. A countable cofinal sequence of apertures shows that $m_2(\bigcup_{A > 1} R_A(f, \zeta)) = 0$ at the same boundary points. This recovers an area-zero range conclusion already obtained by Ivrii for his original series.

The complement of each full Stolz image in our construction has infinite area. It therefore contains a compact set of positive area, and thus of positive logarithmic capacity. This recovers Ivrii's negative answer to the historical questions; the present contribution is the stronger area conclusion. No necessity of the summability condition in (1) is asserted. In particular, we do not determine the total image area for the original choice $v_n = 1/n$. Nor do we assert finite area of $f(\mathbb{D})$, a uniform bound over all apertures, or the conclusions at every boundary point.

This is an almost-sure existence construction, not a prescribed deterministic realization of the coefficients. AI tools assisted the mathematical development, literature search, and exposition. The written argument was self-audited; independent peer review, formal verification, and exhaustive priority certification are not claimed.

References

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