

Duren's Problem on Integral Means of the Derivative of a Subordinate Function: the Radius $1/2$ for $p \leq 2$ and Bounds for $p > 2$

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Abstract

Let g be subordinate to f in the unit disc, that is, $g = f \circ \varphi$ with φ analytic, $|\varphi| < 1$ and $\varphi(0) = 0$. Goluzin proved in 1951 that $M_2(r, g') \leq M_2(r, f')$ for $r \leq 1/2$, and that $M_p(r, g') \leq M_p(r, f')$ for every $p > 0$ when $r \leq \sqrt{2} - 1$. Problem 5.39 of Hayman and Lingham's *Research Problems in Function Theory*, posed by P. L. Duren, asks for the largest number r_p such that the inequality between the p -th means of the derivatives holds for $0 < r < r_p$. We show that $r_p = 1/2$ for every $0 < p \leq 2$, in the quantitative form $M_p(r, g') \leq (\alpha^2 + 4r^2(1 - \alpha^2))^{1/2} M_p(r, f')$ for $r \leq 1/2$, where $\alpha = |\varphi'(0)|$. The proof is short: Hölder's inequality between Littlewood's subordination theorem and Goluzin's theorem, applied to a power of the zero-free part of f' . For $p > 2$ the problem remains open, and we prove two-sided bounds. The function $p \mapsto r_p$ is non-increasing and left-continuous, $r_p \rightarrow 1/2$ as $p \downarrow 2$, and $r_p \rightarrow \sqrt{2} - 1 = r_\infty$ as $p \rightarrow \infty$. Moreover $r_p \geq r_1(p) > \sqrt{2} - 1$ for every finite $p > 2$, where $r_1(p)$ is the root in $(\sqrt{2} - 1, 1/2)$ of $8r^4 - 16r^3 + (p + 4)r^2 + 2pr - p = 0$; so the lower bound $\sqrt{2} - 1$ recorded with the problem is not best possible for any finite p . In the other direction, $r_p < 1/2$ for every $p \geq 12.0068$, with explicit upper bounds for larger p , for instance $r_{20} < 0.4779$ and $r_{100} < 0.4372$; these rest on 19 explicit pairs (f, φ) violating the inequality, verified in exact rational arithmetic by three independently written programs. The exact value of r_p for $p > 2$ is not determined; numerical experiments, which prove nothing, suggest that $r_p = 1/2$ up to $p \approx 12.0065$. This is an unrefereed note.

1 Introduction

1.1 The problem

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For functions f and g analytic in $\mathbb{D}$, g is *subordinate* to f , written $g \prec f$, if $g = f \circ \varphi$ for a function φ that is analytic in $\mathbb{D}$ and satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = 0$. Such a φ is called a *Schwarz function*. For a function h analytic in $\mathbb{D}$, $0 \leq r < 1$ and $0 < p < \infty$ let

$$M_p(r, h) = \left(\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta \right)^{1/p}, \quad M_\infty(r, h) = \max_{|z|=r} |h(z)|.$$

Littlewood's subordination theorem [Lit25] states that $g \prec f$ implies $M_p(r, g) \leq M_p(r, f)$ for all $p > 0$ and $0 < r < 1$. For the derivatives this is false: if $f(z) = z$ and $g(z) = z^2$, then $M_p(r, g') = 2r > 1 = M_p(r, f')$ for $r > 1/2$.

Problem 5.39 of Hayman and Lingham's collection [HL19], which is attributed there to P. L. Duren, recalls a theorem of Goluzin [Gol51a], namely that $g \prec f$ implies $M_2(r, g') \leq M_2(r, f')$ for $0 \leq r \leq 1/2$,

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together with the example above, and then asks, in the words of the source: “Find the largest number r_p , $0 < r_p < 1$, independent of f and g ” such that

$$M_p(r, g') \leq M_p(r, f') \quad (0 < r < r_p) \quad \text{whenever } g \prec f. \quad (1)$$

The definition of subordination in [HL19] is the one given above; no univalence and no normalisation is imposed on f , g or φ , and p ranges over all positive numbers. A note attached to the problem states that $|g'(z)| \leq |f'(\varphi(z))|$ for $|z| \leq \sqrt{2} - 1$, with a reference to Carathéodory [Car54, p. 19]; in other words, $|\varphi'(z)| \leq 1$ there. With Littlewood’s theorem, applied to $f' \circ \varphi \prec f'$, the note obtains $r_p \geq \sqrt{2} - 1$ for all $p > 0$, and it adds that this value need not be optimal. The update printed with the problem in [HL19] reports that no progress is known to the editors. The problem is record AMR-022-5039 of the UnsolvedMath dataset [ula26], where it is labelled open.

To fix the meaning of r_p , for $0 < p \leq \infty$ let $\mathcal{R}_p$ be the set of all $r \in (0, 1)$ such that $M_p(r, (f \circ \varphi)') \leq M_p(r, f')$ for every function f analytic in $\mathbb{D}$ and every Schwarz function φ , and let

$$r_p = \sup\{\rho \in (0, 1] : (0, \rho) \subset \mathcal{R}_p\}.$$

We shall see that $\mathcal{R}_p = (0, r_p]$ and $\sqrt{2} - 1 \leq r_p \leq 1/2$ (Lemma 2.5). The problem asks for r_p with $0 < p < \infty$; the case $p = \infty$ is included for comparison.

1.2 What Goluzin proved

We read Goluzin’s paper [Gol51a] in the original. Its Theorem 1 is Littlewood’s theorem, with a new proof, and its Theorem 2 is Rogosinski’s coefficient inequality [Rog43]: if $g = \sum b_n z^n \prec f = \sum a_n z^n$, then $\sum_{n=1}^N |b_n|^2 \leq \sum_{n=1}^N |a_n|^2$ for every N . Theorem 3 deduces $\sum_n \lambda_n |b_n|^2 \leq \sum_n \lambda_n |a_n|^2$ for every non-increasing sequence $\lambda_n \geq 0$, by summation by parts. The results on the means of the derivative are the following.

- *Theorem 5*: (1) holds for $p = 2$ and $0 < r \leq 1/2$, and not always for larger r ; for $r < 1/2$ equality holds only if $g(z) = f(\varepsilon z)$ with $|\varepsilon| = 1$. (Theorem 4 is the analogue for the areas of the images of $|z| < r$, with $r \leq 1/\sqrt{2}$; Reich [Rei54] found the sharp bound for the ratio of the areas for every r .)
- *Theorem 6*: for $0 < r \leq \sqrt{2} - 1$ and every $p > 0$, (1) holds, with equality only for $g(z) = f(\varepsilon z)$; for $p = 1$ this compares the lengths of the images of the circle $|z| = r$. The proof applies Theorem 1 to $f' \circ \varphi \prec f'$ and uses $|\varphi'(z)| \leq 1$ for $|z| \leq \sqrt{2} - 1$, for which Goluzin refers to an earlier paper of his; the inequality is due to Dieudonné [Die31]. A remark after the theorem notes that the pair z, z^2 excludes the radii $1/2 < r < 1$.

Thus the note attached to Problem 5.39 is Goluzin’s Theorem 6, and his remark leaves open exactly the range $\sqrt{2} - 1 < r \leq 1/2$ for $p \neq 2$; the paper contains no statement about (1) in that range. Its remaining theorems concern pointwise majorisation, mostly for univalent f , as do Shah’s sharp radii [Sha57] recalled in Problem 5.38 of [HL19]. The second part [Gol51b] of the paper, which we also read, determines radii of pointwise majorisation for univalent f and g by Löwner’s method; it contains nothing on integral means. The same circle of ideas is presented in Goluzin’s book [Gol69, Chapter VIII], which we have not seen.

1.3 Results

Theorem A. *Let $0 < p \leq 2$, let f be analytic in $\mathbb{D}$, let φ be a Schwarz function, $g = f \circ \varphi$ and $\alpha = |\varphi'(0)|$. Then*

$$M_p(r, g') \leq (\alpha^2 + 4r^2(1 - \alpha^2))^{1/2} M_p(r, f') \leq M_p(r, f') \quad (0 < r \leq \tfrac{1}{2}).$$

For no p can the radius $1/2$ be replaced by a larger number. Consequently $r_p = 1/2$ for every $0 < p \leq 2$.

For $p = 2$ the second inequality is Goluzin's Theorem 5. For $p = 1$, Theorem A says that the length of the image of the circle $|z| = r$ (counted with multiplicity) does not increase under subordination as long as $r \leq 1/2$; Goluzin's Theorem 6 gives this for $r \leq \sqrt{2} - 1$.

Theorem B. (1) If $0 < p < q \leq \infty$, then $\mathcal{R}_q \subset \mathcal{R}_p$. Hence $p \mapsto r_p$ is non-increasing on $(0, \infty]$.

(2) The function $p \mapsto r_p$ is left-continuous on $(0, \infty)$.

(3) $r_\infty = \sqrt{2} - 1$ and $\lim_{p \rightarrow \infty} r_p = \sqrt{2} - 1$.

(4) $\lim_{p \downarrow 2} r_p = 1/2$.

Consequently there is a number $p_* \in [2, \infty)$ such that $r_p = 1/2$ for $0 < p \leq p_*$ and $r_p < 1/2$ for $p > p_*$.

Theorem C. For $p > 2$ let $r_1(p)$ be the unique root in the interval $(\sqrt{2} - 1, 1/2)$ of

$$8r^4 - 16r^3 + (p+4)r^2 + 2pr - p = 0. \quad (2)$$

Then $r_p \geq r_1(p) > \sqrt{2} - 1$. The function r_1 is strictly decreasing,

$$r_1(p) = \frac{1}{2} - \frac{p-2}{8} + O((p-2)^2) \quad (p \downarrow 2), \quad r_1(p) = \sqrt{2} - 1 + \frac{92 - 65\sqrt{2}}{p} + O(p^{-2}) \quad (p \rightarrow \infty),$$

where $92 - 65\sqrt{2} = 0.0761 \dots$. In particular the lower bound $\sqrt{2} - 1$ in the note to Problem 5.39 is not best possible for any finite p ; by Theorem B(3) it is best possible for $p = \infty$.

Theorem D. (1) $r_p < 1/2$ for every $p \geq 12.0068$. Hence $2 \leq p_* < 12.0068$.

(2) For each of the fifteen pairs $(p_0, r_0) = (p, r)$ in rows 5–19 of Table 2 we have $r_p < r_0$ for every $p \geq p_0$. For instance, $r_p < 0.4932$ for $p \geq 14$, $r_p < 0.4779$ for $p \geq 20$, $r_p < 0.4372$ for $p \geq 100$ and $r_p < 0.4207$ for $p \geq 1000$.

(3) The inequality (1) already fails for $f(z) = z$ at $r = 1/2$ when $p = 16$: if $\varphi(z) = z(z+a)/(1+az)$ with $a = 23/100$, then $M_{16}(1/2, \varphi') > 1$.

Theorems A to C do not depend on any computation. Theorem D rests on 19 explicit pairs (f, φ) , listed in Table 2, for which a strict inequality between two rational numbers has to be checked; this is done in exact rational arithmetic by three independently written programs (Section 7). Table 1 collects the bounds that are proved.

1.4 What remains open, and organisation

The exact value of r_p for $p > 2$ is not determined: for $2 < p < 12.0068$ we only know $r_1(p) \leq r_p \leq 1/2$, and for larger p there is a gap between $r_1(p)$ and the upper bounds of Theorem D. In particular we know only $2 \leq p_* < 12.0068$, and we do not know whether $p \mapsto r_p$ is continuous from the right or strictly decreasing for $p > p_*$. Numerical experiments (Section 8) suggest that $p_* = 12.0065 \dots$, the threshold of the family $\varphi_a(z) = z(z+a)/(1+az)$, and that for larger p the value r_p is attained in the same family (Conjecture 8.1); this is supported by searches that found no further violation, and it is not proved. The argument behind Theorem C cannot give $r_p = 1/2$ for any $p > 2$ (Remark 9.1).

The proof of Theorem A rests on two tools that are, in essence, in [Gol51a]: dividing f' by the Blaschke product of the disc $|z| < r$ formed with its zeros (Lemmas 2.1 and 2.3), and the weighted coefficient inequality behind Goluzin's Theorem 5, which we use in a slightly refined form (Lemmas 3.2 and 3.3; we make no claim of novelty for these two lemmas). Sections 2 and 3 contain the reductions and the tools, Sections 4 to 7 the proofs of Theorems A to D, Section 8 the numerical evidence, and Sections 9 and 10 remarks and the description of the computations.

p	lower bound	upper bound	p	lower bound	upper bound
$(0, 2]$	$r_p = 1/2$		14	0.42003	< 0.4932
2.5	0.46608	$\leq 1/2$	16	0.41926	< 0.4872
3	0.45226	$\leq 1/2$	20	0.41820	< 0.4779
4	0.43932	$\leq 1/2$	30	0.41683	< 0.4635
6	0.42927	$\leq 1/2$	50	0.41576	< 0.4498
8	0.42498	$\leq 1/2$	100	0.41498	< 0.4372
12	0.42108	$\leq 1/2$	1000	0.41428	< 0.4207
12.0068	0.42108	$< 1/2$	∞	$r_\infty = \sqrt{2} - 1$	

Table 1: Proved bounds for r_p ; $\sqrt{2} - 1 = 0.41421 \dots$. The lower bounds for $2 < p < \infty$ are the numbers $r_1(p)$ of Theorem C, rounded down; the upper bounds are Lemma 2.5 and Theorem D.

2 Preliminaries and reductions

2.1 Notation

Throughout, φ denotes a Schwarz function. By Schwarz's lemma, $\varphi(z) = z\omega(z)$ with ω analytic in $\mathbb{D}$ and $|\omega| \leq 1$; either ω is a constant of modulus 1, or $|\omega| < 1$ in $\mathbb{D}$. We put

$$\alpha = |\varphi'(0)| = |\omega(0)|.$$

Since $|\varphi(z)| \leq |z|$, the function φ maps every closed disc $\{|z| \leq r\}$, $r < 1$, into itself.

For $0 < r < 1$ let $r\mathbb{D} = \{|z| < r\}$, and let $A(r\mathbb{D})$ be the disc algebra of $r\mathbb{D}$: the functions that are continuous on $\{|z| \leq r\}$ and analytic in $r\mathbb{D}$. Polynomials are dense in $A(r\mathbb{D})$ in the supremum norm (approximate $k(z)$ by $k(sz)$ with $s \uparrow 1$, and $k(sz)$ by its Taylor polynomials). If $K \in A(r\mathbb{D})$, then $K \circ \varphi$ and $(K \circ \varphi)\varphi'$ belong to $A(r\mathbb{D})$.

We write $dm = d\theta/2\pi$ for the normalised arc length on a circle $|z| = r$, so that $M_p(r, h)^p = \int_{|z|=r} |h|^p dm$. For $F \in A(r\mathbb{D})$ with Taylor coefficients F_n , Parseval's formula reads

$$\int_{|z|=r} |F|^2 dm = \sum_{n \geq 0} |F_n|^2 r^{2n}. \quad (3)$$

(This holds on every circle $|z| = \rho < r$; let $\rho \uparrow r$.) We abbreviate $\|F\|_r^2 = \int_{|z|=r} |F|^2 dm$.

2.2 The zero-free majorant

The following device is used in Goluzin's proof of Littlewood's theorem [Gol51a, Theorem 1].

Lemma 2.1. *Let $0 < r < 1$, and let $F \not\equiv 0$ be analytic on a neighbourhood of $\{|z| \leq r\}$. Then there is a function E , analytic on a neighbourhood of $\{|z| \leq r\}$ and without zeros in $r\mathbb{D}$, such that*

$$|E| = |F| \quad \text{on } |z| = r, \quad |F| \leq |E| \quad \text{on } |z| \leq r.$$

Moreover, for every real $s > 0$ there is a function $E_s \in A(r\mathbb{D})$ with $|E_s| = |E|^s$ on $\{|z| \leq r\}$.

Proof. Let $a_1, \dots, a_m$ be the zeros of F in $r\mathbb{D}$, repeated according to multiplicity, and put $B(z) = \prod_{j=1}^m r(z - a_j)/(r^2 - \bar{a}_j z)$ (with $B = 1$ if $m = 0$). From the identity

$$|r^2 - \bar{a}z|^2 - r^2|z - a|^2 = (r^2 - |z|^2)(r^2 - |a|^2)$$

we get $|B| = 1$ on $|z| = r$ and $|B| \leq 1$ on $|z| \leq r$. The poles $r^2/\overline{a_j}$ of B lie outside $\{|z| \leq r\}$. Hence $E = F/B$ is analytic on a neighbourhood of $\{|z| \leq r\}$, has no zeros in $r\mathbb{D}$, and satisfies $|E| = |F|$ on $|z| = r$ and $|F| = |B||E| \leq |E|$ on $|z| \leq r$.

Since $r\mathbb{D}$ is simply connected and E has no zeros there, there is an analytic function L in $r\mathbb{D}$ with $e^L = E$. Put $E_s = e^{sL}$. Then $|E_s| = e^{s\operatorname{Re} L} = |E|^s$ in $r\mathbb{D}$. It remains to show that E_s has a limit at every point ζ_0 with $|\zeta_0| = r$. If $E(\zeta_0) = 0$, then $|E_s(z)| = |E(z)|^s \rightarrow 0$ as $z \rightarrow \zeta_0$. If $E(\zeta_0) \neq 0$, choose a disc U about ζ_0 on which E is analytic and has no zeros, and an analytic branch L_U of $\log E$ on U . On the convex set $U \cap r\mathbb{D}$ the continuous function $L - L_U$ takes values in $2\pi i\mathbb{Z}$, so it is constant, and $E_s = e^{s(L_U + \text{const})}$ extends continuously to ζ_0 . A function that is continuous in $r\mathbb{D}$ and has a limit at every boundary point extends continuously to the closed disc. Thus $E_s \in A(r\mathbb{D})$ and $|E_s| = |E|^s$ on $\{|z| \leq r\}$. $\square$

2.3 Passage to the disc algebra and the Hilbert space form

Lemma 2.2. *Let $0 < p < \infty$, $0 < r < 1$, $C \geq 0$, and let φ be a Schwarz function. Suppose that*

$$\int_{|z|=r} |k \circ \varphi|^p |\varphi'|^p dm \leq C \int_{|z|=r} |k|^p dm \quad (4)$$

for every polynomial k . Then (4) holds for every $k \in A(r\mathbb{D})$.

Proof. Let $k \in A(r\mathbb{D})$ and let k_n be polynomials with $k_n \rightarrow k$ uniformly on $\{|z| \leq r\}$. Since φ maps this disc into itself and φ' is bounded on $|z| = r$, we have $(k_n \circ \varphi)\varphi' \rightarrow (k \circ \varphi)\varphi'$ uniformly on $|z| = r$. If bounded functions u_n converge uniformly to u on the circle, then $\int |u_n|^p dm \rightarrow \int |u|^p dm$, because $t \mapsto t^p$ is uniformly continuous on bounded subsets of $[0, \infty)$. Apply this to both sides of (4) for k_n . $\square$

Every polynomial k is the derivative of a polynomial f , and then $(f \circ \varphi)' = (k \circ \varphi)\varphi'$. So (4) for all polynomials k is the inequality $M_p(r, (f \circ \varphi)')^p \leq C M_p(r, f')^p$ for all polynomials f .

Lemma 2.3 (Hilbert space form). *Let $0 < p < \infty$, $0 < r < 1$, $C \geq 0$, and let φ be a Schwarz function. The following statements are equivalent.*

(i) $M_p(r, (f \circ \varphi)')^p \leq C M_p(r, f')^p$ for every function f analytic in $\mathbb{D}$.

(ii) $\int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^p dm \leq C \int_{|z|=r} |K|^2 dm$ for every $K \in A(r\mathbb{D})$.

Proof. (ii) $\Rightarrow$ (i). Let f be analytic in $\mathbb{D}$ and $g = f \circ \varphi$. If $f' \equiv 0$ there is nothing to prove. Otherwise let E be the function of Lemma 2.1 for $F = f'$, and put $K = E_{p/2} \in A(r\mathbb{D})$, so that $|K|^2 = |E|^p$ on $\{|z| \leq r\}$. For $|z| = r$ we have $|\varphi(z)| \leq r$, hence $|f'(\varphi(z))| \leq |E(\varphi(z))|$ and

$$|g'(z)|^p = |f'(\varphi(z))|^p |\varphi'(z)|^p \leq |K(\varphi(z))|^2 |\varphi'(z)|^p. \quad (5)$$

Integrating and using (ii) we get

$$\int_{|z|=r} |g'|^p dm \leq C \int_{|z|=r} |K|^2 dm = C \int_{|z|=r} |E|^p dm = C \int_{|z|=r} |f'|^p dm.$$

(i) $\Rightarrow$ (ii). Let $K \not\equiv 0$ be a polynomial. Apply Lemma 2.1 with $F = K$ (the polynomial K may have zeros on the circle $|z| = r$; the lemma allows this), and call the resulting function E . Put

$k = E_{2/p} \in A(r\mathbb{D})$, so that $|k|^p = |E|^2$ on $\{|z| \leq r\}$. On $|z| = r$ we have $|K \circ \varphi|^2 \leq |E \circ \varphi|^2 = |k \circ \varphi|^p$. Statement (i) for polynomials f is (4) for all polynomials, so by Lemma 2.2

$$\int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^p dm \leq \int_{|z|=r} |k \circ \varphi|^p |\varphi'|^p dm \leq C \int_{|z|=r} |k|^p dm = C \int_{|z|=r} |E|^2 dm = C \int_{|z|=r} |K|^2 dm.$$

For general $K \in A(r\mathbb{D})$ approximate K uniformly on $\{|z| \leq r\}$ by polynomials; both sides of (ii) are continuous under this convergence. $\square$

For a Schwarz function φ , $0 < r < 1$ and $0 < p < \infty$ let $N_p(\varphi, r)$ be the smallest constant C for which (i) holds, that is,

$$N_p(\varphi, r) = \sup_{f' \neq 0} \frac{M_p(r, (f \circ \varphi)')^p}{M_p(r, f')^p} = \sup \left\{ \int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^p dm : K \in A(r\mathbb{D}), \|K\|_r = 1 \right\}, \quad (6)$$

where the second equality is Lemma 2.3. By Lemma 3.1 below, $N_p(\varphi, r) \leq \max_{|z|=r} |\varphi'|^p < \infty$. A radius r belongs to $\mathcal{R}_p$ if and only if $N_p(\varphi, r) \leq 1$ for every Schwarz function φ .

Remark 2.4. Put $\psi(\zeta) = \varphi(r\zeta)/r$. The functions $(z/r)^j$, $j \geq 0$, are orthonormal on $|z| = r$, and for $K = \sum_j x_j (z/r)^j$ the integral in (6) is the Hermitian form $\sum_{j,k} x_j \overline{x_k} A_{jk}$ with the Gram matrix

$$A_p(\varphi, r) = \left[\int_{|\zeta|=1} \psi^j \overline{\psi^k} |\psi'|^p dm \right]_{j,k \geq 0}.$$

So $N_p(\varphi, r)$ is the norm of this positive form, and the largest eigenvalue of every finite section of $A_p(\varphi, r)$ is a lower bound for it. This is what the computations of Section 8 evaluate.

2.4 The set of admissible radii

Lemma 2.5. *For every $0 < p \leq \infty$ we have $\mathcal{R}_p = (0, r_p]$ and $\sqrt{2} - 1 \leq r_p \leq 1/2$.*

Proof. Part (c) below uses Lemma 3.1 and Lemma 3.4(a), which are proved in Section 3; their proofs do not use the present lemma.

(a) If $r \in \mathcal{R}_p$ and $0 < s < 1$, then $sr \in \mathcal{R}_p$. Given f and φ , put $f_s(w) = f(sw)$ and $\varphi_s(z) = \varphi(sz)/s$. By Schwarz's lemma $|\varphi_s(z)| \leq |z|$, so φ_s is a Schwarz function, and $(f_s \circ \varphi_s)(z) = (f \circ \varphi)(sz)$. Hence $M_p(r, (f_s \circ \varphi_s)') = s M_p(sr, (f \circ \varphi)')$ and $M_p(r, f'_s) = s M_p(sr, f')$, and the inequality at the radius r for the pair (f_s, φ_s) is the inequality at the radius sr for (f, φ) .

(b) $\mathcal{R}_p$ is closed in $(0, 1)$, because for fixed f and φ both means are continuous functions of r .

(c) $(0, \sqrt{2} - 1] \subset \mathcal{R}_p$. This is Goluzin's Theorem 6. Let $0 < r \leq \sqrt{2} - 1$. By Lemma 3.4(a), $|\varphi'| \leq 1$ on $|z| = r$, so $|g'| \leq |f' \circ \varphi|$ there. For $p = \infty$ the maximum principle gives $|f'(\varphi(z))| \leq M_\infty(r, f')$ for $|z| = r$. For $p < \infty$ we may assume $f' \neq 0$; let E be as in Lemma 2.1 for $F = f'$; then $|f' \circ \varphi| \leq |E \circ \varphi|$ on $|z| = r$ and, by Lemma 3.1 applied to $E_{p/2}$,

$$\int_{|z|=r} |g'|^p dm \leq \int_{|z|=r} |E_{p/2} \circ \varphi|^2 dm \leq \int_{|z|=r} |E_{p/2}|^2 dm = \int_{|z|=r} |f'|^p dm.$$

(d) $\mathcal{R}_p \subset (0, 1/2]$: for $f(z) = z$ and $\varphi(z) = z^2$ we have $M_p(r, (f \circ \varphi)') = 2r$ and $M_p(r, f') = 1$.

By (a)–(d), $\mathcal{R}_p$ is a non-empty subset of $(0, 1/2]$ that is closed in $(0, 1)$ and contains sr whenever it contains r . Hence $\mathcal{R}_p = (0, \rho]$ with $\rho = \max \mathcal{R}_p \in [\sqrt{2} - 1, 1/2]$, and $r_p = \rho$. $\square$

3 Tools

For a function $F = \sum_{n \geq 0} F_n z^n$ analytic in $\mathbb{D}$ we write $\|F\|^2 = \sum_{n \geq 0} |F_n|^2 = \sup_{\rho < 1} \int_{|z|=\rho} |F|^2 dm$, which may be infinite. If ω is analytic in $\mathbb{D}$ with $|\omega| \leq 1$, then $\|\omega F\| \leq \|F\|$, since $|\omega F| \leq |F|$ on every circle. If $F = cz^k + G$, where G is analytic in $\mathbb{D}$ and has a zero of order at least $k+1$ at 0, then $\|F\|^2 = |c|^2 + \|G\|^2$.

3.1 Littlewood's theorem for the exponent 2

Lemma 3.1. *If $0 < r < 1$ and $K \in A(r\mathbb{D})$, then $\int_{|z|=r} |K \circ \varphi|^2 dm \leq \int_{|z|=r} |K|^2 dm$.*

Proof. Put $\psi(\zeta) = \varphi(r\zeta)/r$. Then ψ is analytic on a neighbourhood of the closed unit disc, $\psi(0) = 0$ and $|\psi| \leq 1$ on $|\zeta| \leq 1$; the claim is that $\int_{|\zeta|=1} |k \circ \psi|^2 dm \leq \int_{|\zeta|=1} |k|^2 dm$ for $k(\zeta) = K(r\zeta) \in A(\mathbb{D})$. Let first $k(\zeta) = \sum_{j=0}^n x_j \zeta^j$ be a polynomial. Define $v_n = x_n$ and $v_j = x_j + \psi v_{j+1}$ for $j = n-1, \dots, 0$; by Horner's rule $v_0 = k \circ \psi$. Each function ψv_{j+1} is continuous on $|\zeta| \leq 1$, analytic in $\mathbb{D}$ and vanishes at 0, so its mean over $|\zeta| = 1$ is 0. Therefore

$$\int_{|\zeta|=1} |v_j|^2 dm = |x_j|^2 + \int_{|\zeta|=1} |\psi v_{j+1}|^2 dm \leq |x_j|^2 + \int_{|\zeta|=1} |v_{j+1}|^2 dm,$$

and by induction $\int_{|\zeta|=1} |k \circ \psi|^2 dm \leq \sum_{j=0}^n |x_j|^2 = \int_{|\zeta|=1} |k|^2 dm$. The general case follows by uniform approximation of k by polynomials on $|\zeta| \leq 1$. $\square$

This is the case $p = 2$ of Littlewood's theorem [Lit25] for functions in the disc algebra, with a standard proof. Together with Lemma 2.1 it gives the theorem for all exponents in the form in which we use it: if E is as in Lemma 2.1 and $0 < p < \infty$, then

$$\int_{|z|=r} |E \circ \varphi|^p dm = \int_{|z|=r} |E_{p/2} \circ \varphi|^2 dm \leq \int_{|z|=r} |E_{p/2}|^2 dm = \int_{|z|=r} |E|^p dm. \quad (7)$$

3.2 A refined Rogosinski inequality

Lemma 3.2. *Let $\mathcal{K}(w) = \sum_{n \geq 1} a_n w^n$ be a formal power series, let φ be a Schwarz function, and let $\sum_{n \geq 1} b_n z^n$ be the formal power series $\mathcal{K} \circ \varphi$. Then for every $N \geq 1$*

$$\sum_{n=1}^N |b_n|^2 \leq \sum_{n=1}^{N-1} |a_n|^2 + |\varphi'(0)|^2 |a_N|^2. \quad (8)$$

Rogosinski's theorem [Rog43] is (8) with the factor $|\varphi'(0)|^2$ replaced by 1. For $N = 1$ the lemma is the identity $b_1 = \varphi'(0)a_1$, which Goluzin uses for the equality statements of his Theorems 3–5.

Proof. Write $\varphi = z\omega$ and $c = \omega(0) = \varphi'(0)$. Since $\varphi^n = O(z^n)$, the coefficients $b_1, \dots, b_N$ depend only on $a_1, \dots, a_N$. For $N = 1$ we have $b_1 = a_1 c$, and (8) holds with equality. Let $N \geq 2$ and put

$$H = \sum_{k=1}^{N-1} a_k \varphi^k + a_N c z \varphi^{N-1},$$

a bounded analytic function in $\mathbb{D}$. Then

$$\sum_{k=1}^N a_k \varphi^k - H = a_N \varphi^{N-1}(\varphi - cz) = a_N z^N \omega^{N-1}(\omega - \omega(0)) = O(z^{N+1}),$$

so b_n is the n -th Taylor coefficient of H for $1 \leq n \leq N$, and $\sum_{n=1}^N |b_n|^2 \leq \|H\|^2$. Define

$$u_{N-1} = a_{N-1} z^{N-1} + a_N c z^N, \quad u_k = a_k z^k + \omega u_{k+1} \quad (k = N-2, \dots, 1).$$

Expanding, $\omega u_1 = a_1 \varphi + \omega^2 u_2 = \dots = \sum_{k=1}^{N-2} a_k \varphi^k + \omega^{N-1} u_{N-1} = H$ (for $N = 2$ there is no recursion, and $\omega u_1 = \omega u_{N-1} = H$ directly). By downward induction u_k is a bounded analytic function with a zero of order at least k at 0. Hence ωu_{k+1} has a zero of order at least $k+1$ at 0, and by the two facts stated at the beginning of this section

$$\|u_k\|^2 = |a_k|^2 + \|\omega u_{k+1}\|^2 \leq |a_k|^2 + \|u_{k+1}\|^2 \quad (1 \leq k \leq N-2).$$

Since $\|u_{N-1}\|^2 = |a_{N-1}|^2 + |c|^2 |a_N|^2$, we obtain

$$\|H\|^2 = \|\omega u_1\|^2 \leq \|u_1\|^2 \leq \sum_{k=1}^{N-1} |a_k|^2 + |c|^2 |a_N|^2. \quad \square$$

3.3 A quantitative form of Goluzin's theorem

Lemma 3.3. *Let $0 < r \leq 1/2$, let φ be a Schwarz function with $\alpha = |\varphi'(0)|$, and let $K = \sum_{m \geq 0} k_m z^m \in A(r\mathbb{D})$. Then*

$$\int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^2 dm \leq \sum_{m \geq 0} \left[\alpha^2 + (1 - \alpha^2) \left(\frac{m+2}{m+1} \right)^2 r^2 \right] |k_m|^2 r^{2m} \leq \gamma(\alpha, r) \int_{|z|=r} |K|^2 dm, \quad (9)$$

where

$$\gamma(\alpha, r) = \alpha^2 + 4r^2(1 - \alpha^2) \leq 1.$$

With γ replaced by 1 this is Goluzin's Theorem 5 in the form (ii) of Lemma 2.3, and the proof follows his: Rogosinski's inequality and summation by parts. The only change is that Lemma 3.2 is used in place of Rogosinski's theorem.

Proof. Let $\mathcal{K}(z) = \sum_{n \geq 1} a_n z^n$ with $a_n = k_{n-1}/n$, so that $\mathcal{K}$ is analytic in $r\mathbb{D}$ and $\mathcal{K}' = K$. The function $\mathcal{K} \circ \varphi$ is analytic in $r\mathbb{D}$; let $\sum_{n \geq 1} b_n z^n$ be its Taylor series, which is the formal composition. Since $(\mathcal{K} \circ \varphi)' = (K \circ \varphi)\varphi' \in A(r\mathbb{D})$, Parseval's formula (3) gives

$$\int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^2 dm = \sum_{n \geq 1} w_n |b_n|^2, \quad w_n = n^2 r^{2n-2}.$$

Because $w_{n+1}/w_n = ((n+1)/n)^2 r^2 \leq 4r^2 \leq 1$, the sequence (w_n) is non-increasing; this is the only place where $r \leq 1/2$ is used. Put $B_n = \sum_{j \leq n} |b_j|^2$ and $A_n = \sum_{j \leq n} |a_j|^2$, with $A_0 = 0$. By summation by parts and Lemma 3.2, that is, $B_n \leq A_{n-1} + \alpha^2 |a_n|^2$, we get for every $N \geq 1$

$$\begin{aligned} \sum_{n=1}^N w_n |b_n|^2 &= \sum_{n=1}^{N-1} (w_n - w_{n+1}) B_n + w_N B_N \\ &\leq \sum_{n=1}^{N-1} (w_n - w_{n+1}) (A_{n-1} + \alpha^2 |a_n|^2) + w_N (A_{N-1} + \alpha^2 |a_N|^2) \\ &= \sum_{m=1}^{N-1} (\alpha^2 w_m + (1 - \alpha^2) w_{m+1}) |a_m|^2 + \alpha^2 w_N |a_N|^2 \\ &\leq \sum_{m=1}^N (\alpha^2 w_m + (1 - \alpha^2) w_{m+1}) |a_m|^2. \end{aligned}$$

In the third line we used that the total coefficient of A -terms containing $|a_m|^2$ is $\sum_{n=m+1}^{N-1} (w_n - w_{n+1}) + w_N = w_{m+1}$. Now $w_m |a_m|^2 = |k_{m-1}|^2 r^{2(m-1)}$ and $w_{m+1} |a_m|^2 = ((m+1)/m)^2 r^2 |k_{m-1}|^2 r^{2(m-1)}$. Letting $N \rightarrow \infty$ and replacing $m-1$ by m gives the first inequality in (9). The second follows from $((m+2)/(m+1))^2 \leq 4$ and (3), and $\gamma \leq \alpha^2 + (1 - \alpha^2) = 1$ because $4r^2 \leq 1$. $\square$

3.4 Consequences of the Schwarz–Pick lemma

Lemma 3.4. *Let $\varphi = z\omega$ be a Schwarz function, $|z| = r < 1$ and $t = |\omega(z)|$. Then*

$$(a) \quad |\varphi'(z)| \leq D_r(t) := t + \frac{r(1-t^2)}{1-r^2}; \text{ in particular } |\varphi'(z)| \leq 1 \text{ if } r \leq \sqrt{2}-1;$$

$$(b) \quad \alpha = |\omega(0)| \leq \hat{\alpha}_r(t) := \frac{t+r}{1+rt}.$$

Proof. If ω is a constant of modulus 1, then $t = 1 = \alpha = |\varphi'(z)|$ and both statements are trivial. Otherwise ω maps $\mathbb{D}$ into $\mathbb{D}$, and the Schwarz–Pick lemma gives

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}, \quad |b| \leq |z| \quad \text{for} \quad b := \frac{\omega(0) - \omega(z)}{1 - \overline{\omega(z)}\omega(0)}.$$

(a) follows from $\varphi' = \omega + z\omega'$. Moreover $D_r(t) - 1 = (1-t)(r(1+t)/(1-r^2) - 1) \leq (1-t)(2r/(1-r^2) - 1)$, which is ≤ 0 when $r^2 + 2r - 1 \leq 0$, that is, when $r \leq \sqrt{2} - 1$.

(b) With $w = \omega(z)$ we have $\omega(0) = (w+b)/(1+\bar{w}b)$ and

$$1 - |\omega(0)|^2 = \frac{(1-|w|^2)(1-|b|^2)}{|1+\bar{w}b|^2} \geq \frac{(1-|w|^2)(1-|b|^2)}{(1+|w||b|)^2} = 1 - \left(\frac{|w|+|b|}{1+|w||b|} \right)^2.$$

Hence $|\omega(0)| \leq (t+|b|)/(1+t|b|) \leq (t+r)/(1+tr)$, because $x \mapsto (t+x)/(1+tx)$ is non-decreasing on $[0, 1]$. $\square$

Both inequalities are classical. Part (a) is due to Dieudonné [Die31, pp. 351–352], including the consequence for $r \leq \sqrt{2} - 1$ that is used in Goluzin’s Theorem 6; the two Schwarz–Pick estimates also appear in the proof of [Gol51a, Theorem 10].

4 The exponents $0 < p \leq 2$: proof of Theorem A

Proof of Theorem A. Let $0 < p \leq 2$ and $0 < r \leq 1/2$, and write $\gamma = \gamma(\alpha, r)$. By Lemma 2.3 with $C = \gamma^{p/2}$ it suffices to show that

$$I_p := \int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^p dm \leq \gamma^{p/2} \|K\|_r^2 \quad \text{for every } K \in A(r\mathbb{D}). \quad (10)$$

For $p = 2$ this is Lemma 3.3. For $0 < p < 2$ write

$$|K \circ \varphi|^2 |\varphi'|^p = (|K \circ \varphi|^2 |\varphi'|^2)^{p/2} (|K \circ \varphi|^2)^{1-p/2}$$

and apply Hölder’s inequality with the exponents $2/p$ and $2/(2-p)$:

$$I_p \leq \left(\int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^2 dm \right)^{p/2} \left(\int_{|z|=r} |K \circ \varphi|^2 dm \right)^{1-p/2} \leq (\gamma \|K\|_r^2)^{p/2} (\|K\|_r^2)^{1-p/2} = \gamma^{p/2} \|K\|_r^2,$$

by Lemma 3.3 for the first factor and Lemma 3.1 for the second. This proves (10), hence $M_p(r, g') \leq \gamma^{1/2} M_p(r, f') \leq M_p(r, f')$ for $0 < r \leq 1/2$, and $(0, 1/2] \subset \mathcal{R}_p$. By Lemma 2.5(d) no radius larger than $1/2$ belongs to $\mathcal{R}_p$. So $r_p = 1/2$. $\square$

Remark 4.1. The factor $\gamma(\alpha, r)^{1/2}$ is attained for $f(z) = z$, $\varphi(z) = z^2$, where $\alpha = 0$ and both sides equal $2r$, and for rotations $\varphi(z) = \varepsilon z$, where $\alpha = 1$. Let $0 < p \leq 2$ and $0 < r < 1/2$. If φ is not a rotation, then $\alpha < 1$ by Schwarz's lemma, hence $\gamma(\alpha, r) < 1$, and $M_p(r, g') = M_p(r, f')$ together with Theorem A gives $M_p(r, f') \leq \gamma^{1/2} M_p(r, f')$, that is, $M_p(r, f') = 0$ and $f' \equiv 0$. So equality $M_p(r, g') = M_p(r, f')$ holds only if f is constant, in which case both sides vanish, or $g(z) = f(\varepsilon z)$ with $|\varepsilon| = 1$. For $p = 2$ this is the equality statement in Goluzin's Theorem 5.

5 Monotonicity, left-continuity and the limits: proof of Theorem B

Proposition 5.1. *Let φ be a Schwarz function and $0 < r < 1$. Then $N_p(\varphi, r)^{1/p} \leq N_q(\varphi, r)^{1/q}$ for $0 < p < q < \infty$.*

Proof. Let $C = N_q(\varphi, r)$. Let f be analytic in $\mathbb{D}$ with $f' \not\equiv 0$, $g = f \circ \varphi$, let E be as in Lemma 2.1 for $F = f'$, and put $k = E_{p/q} \in A(r\mathbb{D})$, so that $|k|^q = |E|^p$. On $|z| = r$ we have, as in (5),

$$|g'|^p \leq |E \circ \varphi|^p |\varphi'|^p = (|k \circ \varphi|^q |\varphi'|^q)^{p/q} (|E \circ \varphi|^p)^{1-p/q}.$$

Hölder's inequality with the exponents q/p and $q/(q-p)$ gives

$$\int_{|z|=r} |g'|^p dm \leq \left(\int_{|z|=r} |k \circ \varphi|^q |\varphi'|^q dm \right)^{p/q} \left(\int_{|z|=r} |E \circ \varphi|^p dm \right)^{1-p/q}.$$

By the definition of C , inequality (4) holds with the exponent q for all polynomials, hence for k by Lemma 2.2; so the first integral is at most $C \int_{|z|=r} |k|^q dm = C \int_{|z|=r} |f'|^p dm$. The second integral is at most $\int_{|z|=r} |E|^p dm = \int_{|z|=r} |f'|^p dm$ by (7). Hence $\int_{|z|=r} |g'|^p dm \leq C^{p/q} \int_{|z|=r} |f'|^p dm$, that is, $N_p(\varphi, r) \leq N_q(\varphi, r)^{p/q}$. $\square$

Proof of Theorem B(1)–(3). (1) Let $0 < p < q < \infty$ and $r \in \mathcal{R}_q$. Then $N_q(\varphi, r) \leq 1$ for every Schwarz function φ , hence $N_p(\varphi, r) \leq 1$ by Proposition 5.1, and $r \in \mathcal{R}_p$. With Lemma 2.5 we get $r_q \leq r_p$. For $q = \infty$ the inclusion follows from part (3) and Lemma 2.5: $\mathcal{R}_\infty = (0, \sqrt{2} - 1] \subset \mathcal{R}_p$.

(2) Let $0 < p < \infty$. By (1) the limit $r^* = \lim_{q \uparrow p} r_q$ exists and $r^* \geq r_p$. Let $0 < r < r^*$. Then $r < r_q$, that is, $r \in \mathcal{R}_q$, for every $q < p$. For fixed f and φ the functions $q \mapsto \int_{|z|=r} |g'|^q dm$ and $q \mapsto \int_{|z|=r} |f'|^q dm$ are continuous on $(0, \infty)$ by dominated convergence (the integrands are bounded uniformly for q in a compact interval and depend continuously on q). Hence the inequality $\int_{|z|=r} |g'|^q dm \leq \int_{|z|=r} |f'|^q dm$, valid for $q < p$, persists for $q = p$. So $r \in \mathcal{R}_p$ for every $r < r^*$, and $r_p \geq r^*$.

(3) By Lemma 2.5, $r_p \geq \sqrt{2} - 1$ for all $p \leq \infty$. Let $\sqrt{2} - 1 < r < 1$ and

$$b = \frac{3r^2 - 1}{r(1 + r^2)}, \quad 1 - b = \frac{(1 - r)(1 + 2r - r^2)}{r(1 + r^2)} > 0, \quad 1 + b = \frac{(1 + r)(r^2 + 2r - 1)}{r(1 + r^2)} > 0.$$

So $-1 < b < 1$, and $\varphi(z) = z(z - b)/(1 - bz)$ is a Schwarz function. We have $\varphi'(z) = (2z - b - bz^2)/(1 - bz)^2$, and with $2r - b(1 + r^2) = (1 - r^2)/r$ and $1 - br = 2(1 - r^2)/(1 + r^2)$ we find

$$\varphi'(r) = \frac{(1 + r^2)^2}{4r(1 - r^2)} = 1 + \frac{(r^2 + 2r - 1)^2}{4r(1 - r^2)} > 1. \quad (11)$$

For $f(z) = z$ this gives $M_\infty(r, (f \circ \varphi)') \geq \varphi'(r) > 1 = M_\infty(r, f')$, so $r \notin \mathcal{R}_\infty$. Hence $r_\infty = \sqrt{2} - 1$. Moreover $M_p(r, \varphi') \rightarrow \max_{|z|=r} |\varphi'| > 1$ as $p \rightarrow \infty$, so $r \notin \mathcal{R}_p$, that is, $r_p < r$, for all sufficiently large p . Since r was arbitrary in $(\sqrt{2} - 1, 1)$, $\lim_{p \rightarrow \infty} r_p = \sqrt{2} - 1$. $\square$

The value $r_\infty = \sqrt{2} - 1$ is classical in substance: the lower bound is the case $p = \infty$ of the argument in Goluzin's Theorem 6, and the function φ above is, up to a rotation, the extremal function of Dieudonné [Die31, formula (26)]; the value (11) is the maximum of the function D_r of Lemma 3.4, attained at $t = (1 - r^2)/(2r) = \omega(r)$.

Part (4) of Theorem B follows from Theorem C and is proved at the end of Section 6. The last assertion of Theorem B follows from the other parts: the set $\{p : r_p = 1/2\}$ contains $(0, 2]$ by Theorem A, it is an interval with left end point 0 by (1), it is bounded by (3), and it contains its supremum p_* by (2).

6 A lower bound for $p > 2$: proof of Theorem C

For $\sqrt{2} - 1 < r < 1/2$ put

$$P_1(r) = 2 + \frac{2(1 - 4r^2)(1 - r)^2}{r^2 + 2r - 1}. \quad (12)$$

On this interval the numerator $2(1 - 4r^2)(1 - r)^2$ is positive and decreasing and the denominator $r^2 + 2r - 1$ is positive and increasing. So P_1 is continuous and strictly decreasing, $P_1(r) \rightarrow \infty$ as $r \downarrow \sqrt{2} - 1$ and $P_1(r) \rightarrow 2$ as $r \uparrow 1/2$.

Proposition 6.1. *Let $\sqrt{2} - 1 < r < 1/2$ and $2 < p \leq P_1(r)$. Then $N_p(\varphi, r) \leq 1$ for every Schwarz function φ ; that is, $M_p(r, g') \leq M_p(r, f')$ whenever $g \prec f$.*

Proof. Let $\varphi = z\omega$ be a Schwarz function and $K \in A(r\mathbb{D})$. Let $\Lambda = \max_{|z|=r} |\varphi'| = |\varphi'(z_1)|$ with $|z_1| = r$, and $t_1 = |\omega(z_1)|$. By Lemma 3.3,

$$\int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^p dm \leq \Lambda^{p-2} \int_{|z|=r} |K \circ \varphi|^2 |\varphi'|^2 dm \leq \Lambda^{p-2} \gamma(\alpha, r) \|K\|_r^2. \quad (13)$$

Lemma 3.4, applied at the point z_1 , gives $\Lambda \leq D_r(t_1)$ and $\alpha \leq \hat{\alpha}_r(t_1)$. Since $\gamma(\alpha, r) = 4r^2 + (1 - 4r^2)\alpha^2$ is non-decreasing in α , we get $\Lambda^{p-2} \gamma(\alpha, r) \leq H(t_1)$, where for $0 \leq t \leq 1$

$$H(t) = D_r(t)^{p-2} \Gamma(t), \quad \Gamma(t) = 4r^2 + (1 - 4r^2) \hat{\alpha}_r(t)^2 \leq 1.$$

By (6) it suffices to show that $H \leq 1$ on $[0, 1]$.

The function $D_r(t) = t + r(1 - t^2)/(1 - r^2)$ is a concave quadratic polynomial with $D_r(0) > 0$ and $D_r(1) = 1$, and $D_r(t) - 1 = (1 - t)(r(1 + t)/(1 - r^2) - 1)$ vanishes exactly at $t = 1$ and at

$$t_0 = \frac{1 - r - r^2}{r}.$$

As r increases from $\sqrt{2} - 1$ to $1/2$, t_0 decreases from 1 to $1/2$; so $1/2 < t_0 < 1$. By concavity, $0 < D_r \leq 1$ on $[0, t_0]$ and $D_r \geq 1$ on $[t_0, 1]$. On $[0, t_0]$ we therefore have $H \leq \Gamma \leq 1$.

It remains to consider $t \in [t_0, 1]$. First, $\log D_r$ is concave (as the logarithm of a positive concave function), $\log D_r(1) = 0$, and

$$(\log D_r)'(1) = D_r'(1) = 1 - \frac{2r}{1 - r^2} = -A_0, \quad A_0 := \frac{r^2 + 2r - 1}{1 - r^2} > 0.$$

A concave function lies below its tangents, so $\log D_r(t) \leq A_0(1 - t)$. Second, a direct computation gives

$$\hat{\alpha}_r'(t) = \frac{1 - r^2}{(1 + rt)^2}, \quad \Gamma''(t) = (1 - 4r^2) \frac{2(1 - r^2)(1 - 3r^2 - 2rt)}{(1 + rt)^4},$$

and for $t \geq t_0$ we have $1 - 3r^2 - 2rt \leq 1 - 3r^2 - 2rt_0 = -(1 - r)^2 < 0$. So Γ is concave on $[t_0, 1]$; it is positive, hence $\log \Gamma$ is concave there. Moreover $\Gamma(1) = 1$ and $\Gamma'(1) = 2(1 - 4r^2)(1 - r)/(1 + r)$, so $\log \Gamma(t) \leq -\Gamma'(1)(1 - t)$. Adding the two estimates we get, for $t_0 \leq t \leq 1$,

$$\log H(t) \leq (1 - t) \left[(p - 2)A_0 - \frac{2(1 - 4r^2)(1 - r)}{1 + r} \right],$$

and the bracket is ≤ 0 if and only if

$$p - 2 \leq \frac{2(1 - 4r^2)(1 - r)}{(1 + r)A_0} = \frac{2(1 - 4r^2)(1 - r)^2}{r^2 + 2r - 1} = P_1(r) - 2.$$

Thus $H \leq 1$ on $[0, 1]$ when $p \leq P_1(r)$. $\square$

Proof of Theorem C. The equation $P_1(r) = p$ means $(p - 2)(r^2 + 2r - 1) = 2(1 - 4r^2)(1 - r)^2$, and expanding the difference of the two sides gives (2). Since P_1 decreases strictly from ∞ to 2 on $(\sqrt{2} - 1, 1/2)$, for every $p > 2$ equation (2) has exactly one root $r_1(p)$ in this interval, and r_1 is strictly decreasing. If $\sqrt{2} - 1 < r \leq r_1(p)$, then $P_1(r) \geq p$, so $r \in \mathcal{R}_p$ by Proposition 6.1. With Lemma 2.5 we get $(0, r_1(p)] \subset \mathcal{R}_p$, that is, $r_p \geq r_1(p) > \sqrt{2} - 1$.

For the expansions, let $r_0 = \sqrt{2} - 1$. Then $r^2 + 2r - 1 = (r - r_0)(r + 1 + \sqrt{2})$, and at $r = r_0$ we have $(1 - 4r_0^2)(1 - r_0)^2 = (8\sqrt{2} - 11)(6 - 4\sqrt{2}) = 92\sqrt{2} - 130$ and $r_0 + 1 + \sqrt{2} = 2\sqrt{2}$. Hence $(p - 2)(r_1(p) - r_0) \rightarrow 2(92\sqrt{2} - 130)/(2\sqrt{2}) = 92 - 65\sqrt{2}$ as $p \rightarrow \infty$. More precisely, $1/(p - 2) = (r - r_0)u(r)$ with a function u that is analytic and non-zero at r_0 , so $r_1(p) - r_0$ is an analytic function of $1/(p - 2)$ near 0, which gives the error term $O(p^{-2})$. Near $r = 1/2$, with $r = 1/2 - v$,

$$P_1(r) - 2 = \frac{8v(1 - v)(1 + 2v)^2}{1 - 12v + 4v^2} = 8v + O(v^2),$$

which gives $r_1(p) = 1/2 - (p - 2)/8 + O((p - 2)^2)$ as $p \downarrow 2$. $\square$

Proof of Theorem B(4). By Theorem C and Lemma 2.5, $r_1(p) \leq r_p \leq 1/2$ for $p > 2$, and $r_1(p) \rightarrow 1/2$ as $p \downarrow 2$. $\square$

Values of $r_1(p)$, rounded down to five decimals, are given in Tables 1 and 3; for example $r_1(2.1) \geq 0.48941$ and $r_1(10) \geq 0.42260$.

Remark 6.2 (Where the argument loses). The proof of Proposition 6.1 loses in two places. First, in (13) the weight $|\varphi'|^{p-2}$ is replaced by its maximum. Second, the bounds $\Lambda \leq D_r(t_1)$ and $\alpha \leq \hat{\alpha}_r(t_1)$ of Lemma 3.4 are used together although they are sharp in opposite configurations: if equality holds in Lemma 3.4(a) at z_1 and $0 < t_1 < 1$, then ω is an automorphism of $\mathbb{D}$, and after rotations $z_1 = r$, $\omega(r) = t_1$ and $\omega(0) = (t_1 - r)/(1 - rt_1)$, whereas equality in (b) requires $|\omega(0)| = (t_1 + r)/(1 + rt_1)$. For example, for $\omega(z) = (z + a)/(1 + az)$ with $a = 1 - \varepsilon$ and small $\varepsilon > 0$ the maximum of $|\varphi'|$ on $|z| = r$ is attained at $z_1 = r$, and one finds $1 - t_1 = \varepsilon(1 - r)/(1 + r) + O(\varepsilon^2)$ and $1 - \hat{\alpha}_r(t_1) = \varepsilon((1 - r)/(1 + r))^2 + O(\varepsilon^2)$, whereas $1 - \alpha = \varepsilon$: the bound (b) underestimates $1 - \alpha$ by the factor $((1 + r)/(1 - r))^2$, which lies between 5.8 and 9 on our interval. For this family $\log(\Lambda^{p-2}\gamma(\alpha, r)) = \varepsilon[(p - 2)(r^2 + 2r - 1)/(1 + r)^2 - 2(1 - 4r^2)] + O(\varepsilon^2)$, which is negative for small ε as long as $p < 2 + ((1 + r)/(1 - r))^2(P_1(r) - 2)$. Numerical experiments in one of the verification runs (Blaschke products ω of degree at most 3, ten radii between 0.416 and 0.495) suggest that for $r \leq 0.49$ the product $\Lambda^{p-2}\gamma(\alpha, r)$ stays ≤ 1 up to about this exponent. This is an unproved numerical observation, and it is not used here; even a proof of it would not give $r_p = 1/2$ for any $p > 2$ (Remark 9.1).

7 Upper bounds: certified violations and proof of Theorem D

All examples in this section use the Schwarz functions

$$\varphi_a(z) = \frac{z(z+a)}{1+az} \quad (0 < a < 1), \quad \varphi'_a(z) = \frac{az^2 + 2z + a}{(1+az)^2},$$

which are analytic in $|z| < 1/a$, and functions f with $f' = h$, where h is a polynomial with $h(0) = 1$. Thus $g = f \circ \varphi_a$ has the derivative $g' = (h \circ \varphi_a) \varphi'_a$. The following two lemmas express the two means through Taylor coefficients, so that a violation of (1) becomes a strict inequality between two sums that can be bounded in exact arithmetic.

Lemma 7.1 (even exponents). *Let $p = 2k$ with a positive integer k , $0 < r < 1$, and let c_n and d_n be the Taylor coefficients of $(g')^k$ and of h^k . Then for every $N \geq 0$*

$$M_p(r, g')^p = \sum_{n \geq 0} |c_n|^2 r^{2n} \geq \sum_{n=0}^N |c_n|^2 r^{2n}, \quad M_p(r, f')^p = \sum_{n=0}^{k \deg h} |d_n|^2 r^{2n}.$$

Proof. $(g')^k$ and h^k are analytic on a neighbourhood of $\{|z| \leq r\}$, $|g'|^p = |(g')^k|^2$ and $|h|^p = |h^k|^2$; apply (3). $\square$

Lemma 7.2 (general exponents). *Let $0 < p < \infty$, $0 < r < 1$, $0 < a < 1$, $s = \sqrt{1-a^2}$, and let*

$$z_- = -\frac{a}{1+s}, \quad z_+ = \frac{1}{z_-}$$

be the zeros of $az^2 + 2z + a$; so $-1 < z_- < 0$ and $|z_+| > 1$. Assume $|z_-| \neq r$ and put

$$e = \varphi'_a \quad \text{if } |z_-| > r, \quad e = \frac{\varphi'_a}{b} \quad \text{with } b(z) = \frac{r(z-z_-)}{r^2 - z_-z} \quad \text{if } |z_-| < r.$$

Let $h(z) = 1 + \sum_{n=1}^d h_n z^n$ be a polynomial with $\eta := \sum_{n=1}^d |h_n| < 1$. Then:

- (i) *e is analytic and has no zeros on a neighbourhood of $\{|z| \leq r\}$, $|e| = |\varphi'_a|$ on $|z| = r$, and $e(0) = a$ in the first case, $e(0) = r(1+s)$ in the second.*
- (ii) *The function $S = (h \circ \varphi_a) e / e(0)$ is analytic and has no zeros on a neighbourhood of $\{|z| \leq r\}$, and $S(0) = 1$. Let $T = S^{p/2}$ and $t = h^{p/2}$ be the branches with the value 1 at 0, with Taylor coefficients T_n and t_n . Then for every $N \geq 0$*

$$M_p(r, g')^p = e(0)^p \sum_{n \geq 0} |T_n|^2 r^{2n} \geq e(0)^p \sum_{n=0}^N |T_n|^2 r^{2n},$$

$$M_p(r, f')^p = \sum_{n \geq 0} |t_n|^2 r^{2n} \leq \sum_{n=0}^N |t_n|^2 r^{2n} + (1+\eta)^p \frac{r^{2N+2}}{1-r^2}.$$

- (iii) *$T_0 = 1$ and $nT_n = \sum_{j=1}^n \left(\frac{p}{2}j - (n-j)\right) S_j T_{n-j}$ for $n \geq 1$, where S_j are the Taylor coefficients of S ; the same recursion with h_j in place of S_j gives the t_n .*

Proof. (i) We have $\varphi'_a(z) = a(z - z_-)(z - z_+)/(1 + az)^2$, with the pole $-1/a$ outside $\mathbb{D}$. If $|z_-| > r$, both zeros lie outside $\{|z| \leq r\}$ and $e(0) = \varphi'_a(0) = a$. Let $|z_-| < r$. For $|z| = r$ we have $r^2 - z_-z = z(\bar{z} - z_-)$, and since z_- is real, $|r^2 - z_-z| = r|z - z_-|$; so $|b| = 1$ on $|z| = r$. Further $e(z) = a(z - z_+)(r^2 - z_-z)/(r(1 + az)^2)$ has the zeros z_+ and r^2/z_- , both of modulus greater than r , and $e(0) = -az_+r = r(1 + s)$.

(ii) Since $|h - 1| \leq \eta < 1$ on $\{|z| \leq 1\}$, the polynomial h has no zeros on a neighbourhood of the closed unit disc, and neither has $h \circ \varphi_a$ on a neighbourhood of $\{|z| \leq r\}$, because $|\varphi_a(z)| \leq |z|$ in $\mathbb{D}$. With (i) this gives the first sentence. Hence T is analytic on a neighbourhood of $\{|z| \leq r\}$ and $|T|^2 = |S|^p$. On $|z| = r$ we have $|g'| = |h \circ \varphi_a| |\varphi'_a| = |h \circ \varphi_a| |e| = e(0)|S|$, so $|g'|^p = e(0)^p |T|^2$, and (3) gives the first line. Similarly t is analytic on a neighbourhood of the closed unit disc, $|t|^2 = |h|^p$, and (3) gives $M_p(r, f')^p = \sum_n |t_n|^2 r^{2n}$. Since $|t| \leq (1 + \eta)^{p/2}$ on $|z| = 1$, Cauchy's estimate yields $|t_n| \leq (1 + \eta)^{p/2}$, and the tail $\sum_{n > N} |t_n|^2 r^{2n}$ is at most $(1 + \eta)^p r^{2N+2}/(1 - r^2)$.

(iii) Compare the coefficients of z^{n-1} in $T'S = \frac{p}{2} S'T$, using $S_0 = 1$. $\square$

What the programs check. The data p, r, a and the coefficients of h are rational numbers in all cases; they are listed in Table 2.

Rows 1 and 5–19 have even $p = 2k$. The program `c1_cert_exact.py` computes, with integers, the coefficients c_n of Lemma 7.1 for $n \leq N$ (with $N = 160$ in row 1 and $N = 120$ in rows 5–19) and all coefficients d_n , and tests the inequality

$$\sum_{n=0}^N c_n^2 r^{2n} > \sum_{n=0}^{k \deg h} d_n^2 r^{2n} \quad (14)$$

between two rational numbers. By Lemma 7.1, (14) implies $M_p(r, g') > M_p(r, f')$.

Rows 2–4 have $p = P/Q$ with integers P, Q , and a is chosen such that s is rational: $a = 55/73$, $s = 48/73$, $z_- = -5/11$, $e(0) = 121/146$ in row 2, and $a = 171/221$, $s = 140/221$, $z_- = -9/19$, $e(0) = 361/442$ in rows 3 and 4. In all three rows $|z_-| < r = 1/2$, and $\eta = 13/50, 79/250, 3177/10000$, respectively. Then all S_n, T_n and t_n are rational, and the program computes them for $n \leq N = 70$ by Lemma 7.2(iii). It takes a positive rational number ℓ (proposed by a floating-point evaluation of $e(0)^p$), verifies exactly that $\ell^Q \leq e(0)^P$, so that $\ell \leq e(0)^p$, and tests

$$\ell \sum_{n=0}^N T_n^2 r^{2n} > \sum_{n=0}^N t_n^2 r^{2n} + (1 + \eta)^{13} \frac{r^{2N+2}}{1 - r^2}, \quad (15)$$

where $13 = \lceil p \rceil \geq p$ in all three rows. By Lemma 7.2(ii), (15) implies $M_p(r, g') > M_p(r, f')$.

All 19 tests are passed. The comparisons are made between Python integers or `Fraction` objects; no floating-point number enters any of them. The last column of Table 2 is the quotient of the two sides of (14) or (15), rounded down; it is a lower bound for $M_p(r, g')^p / M_p(r, f')^p$.

A second program, `v2_cert.py`, was written independently in one of the verification runs, from the statements of Lemmas 7.1 and 7.2 and the data of Table 2 only. It uses a different method: S is a constant times $U(z)/(1 + az)^{d+2}$ with a polynomial U , so $T = S^{p/2}$ satisfies $U(z)(1 + az)T' = \frac{p}{2}[U'(z)(1 + az) - (d + 2)aU(z)]T$, which gives a linear recurrence of fixed order for the T_n ; it is run in integer arithmetic, with truncation orders N between 120 and 4000. This program confirms all 19 strict inequalities, its certified lower bounds agree with those in Table 2 in all digits shown, and for $n \leq 120$ it obtains the same rational numbers T_n from the recursion of Lemma 7.2(iii).

A third program, `r3_exact.py`, was written in a further verification run from the data of Table 2 and the inequalities to be certified, before the other two programs were read. For even p it forms $g = f \circ \varphi_a$ as a composition of power series, differentiates termwise and takes the k -th power in integer arithmetic. In rows 2–4 it obtains $T = \exp(\frac{p}{2} \log S)$, where $\log S$ is the sum of the series

of $\log(1 - z/z_+)$, $\log(1 - z_-z/r^2)$, $-2\log(1 + az)$ and of $\log h$ composed with φ_a . It certifies all 19 inequalities, with the eight decimals of Table 2 in every row, for the truncation orders named above and also for $N = 640$ (row 1), $N = 200$ (rows 2–4) and $N = 480$ (rows 5–19).

no.	p	r	a	$h(z) = f'(z)$	ratio $\geq$
1	16	1/2	23/100	1	1.00047206
2	12.1	1/2	55/73	$1 + \frac{13}{50}z$	1.00115523
3	12.01	1/2	171/221	$1 + \frac{1}{25}z - \frac{3}{100}z^2 + \frac{3}{500}z^3$	1.00006207
4	12.0068	1/2	171/221	$1 + 0.2796z - 0.0308z^2 + 0.0055z^3 + 0.0018z^4$	1.00000451
5	14	0.4932	0.823	$1 + 0.278z$	1.00120913
6	16	0.4872	0.857	$1 + 0.274z$	1.00119192
7	18	0.4821	0.883	$1 + 0.268z$	1.00090728
8	20	0.4779	0.898	$1 + 0.257z$	1.00109198
9	24	0.4710	0.923	$1 + 0.240z$	1.00084451
10	30	0.4635	0.945	$1 + 0.216z$	1.00067576
11	40	0.4553	0.962	$1 + 0.184z$	1.00071031
12	50	0.4498	0.973	$1 + 0.162z$	1.00051894
13	76	0.4415	0.9846	$1 + 0.125z$	1.00041777
14	100	0.4372	0.989	$1 + 0.105z$	1.00037374
15	150	0.4322	0.992	$1 + 0.080z$	1.00041786
16	200	0.4294	0.9932	$1 + 0.066z$	1.00046202
17	300	0.4263	0.9938	$1 + 0.051z$	1.00065953
18	500	0.4234	0.994	$1 + 0.037z$	1.00111540
19	1000	0.4207	0.9938	$1 + 0.0246z$	1.00255273

Table 2: The 19 certificates: for $\varphi = \varphi_a$ and $f' = h$ the inequality $M_p(r, g') > M_p(r, f')$ holds. All decimal numbers are exact rational data. The last column is the certified lower bound for $M_p(r, g')^p / M_p(r, f')^p$, rounded down to eight decimals. Rows 1 and 5–19 use Lemma 7.1; rows 2–4 use Lemma 7.2 in the case $|z_-| < r$.

Proof of Theorem D. Let (p_0, r_0) be the pair (p, r) of one of the rows of Table 2. By the test described above, $M_{p_0}(r_0, g') > M_{p_0}(r_0, f')$ for the pair (f, φ_a) of that row, so $r_0 \notin \mathcal{R}_{p_0}$. Since $\mathcal{R}_{p_0} = (0, r_{p_0}]$ by Lemma 2.5, this means $r_{p_0} < r_0$, and $r_p \leq r_{p_0} < r_0$ for all $p \geq p_0$ by Theorem B(1). Row 4 gives (1), and then $p_* < 12.0068$ because $r_{p_*} = 1/2$; rows 5–19 give (2); and row 1, where $f(z) = z$, gives (3). $\square$

8 Numerical evidence and a conjecture

Nothing in this section is proved, and nothing in it is used in the proofs of Theorems A to D. All computations here are in floating-point arithmetic.

8.1 What is computed

By Remark 2.4, $N_p(\varphi, r)$ is the norm of the Gram matrix $A_p(\varphi, r)$. The programs evaluate the largest eigenvalue $\lambda_n(\varphi, r, p)$ of its section with $0 \leq j, k < n$, with the integrals replaced by the trapezoidal rule with M nodes; typically $40 \leq n \leq 220$ and $M = 2048$. Up to the quadrature error, λ_n is a lower bound for $N_p(\varphi, r)$: a value above 1 indicates a violation of (1), while values below 1 are only evidence. The Schwarz functions are $\varphi = z\omega$ with ω a finite Blaschke product. Since $N_p(\varphi, r)$ does not change when $\varphi(z)$ is replaced by $\varepsilon_1\varphi(\varepsilon_2z)$ with $|\varepsilon_1| = |\varepsilon_2| = 1$, for ω of degree one it suffices to consider the family φ_a , $0 \leq a < 1$, of Section 7. At $r = 1/2$ its end points are trivial: $N_p(\varphi_0, 1/2) = 1$

for every p , since $\varphi_0(z) = z^2$ gives $A_p = \text{diag}(4^{-j})$, and $N_p(\varphi_a, 1/2) \rightarrow 1$ as $a \rightarrow 1$. The question is whether the value 1 is exceeded in between.

8.2 The threshold at $r = 1/2$ inside the family

By Proposition 5.1 there is a number $p_1 \in (0, \infty]$ such that $\sup_a N_p(\varphi_a, 1/2) \leq 1$ for $p < p_1$ and > 1 for $p > p_1$; clearly $p_* \leq p_1$, and $p_1 \leq 12.0068$ by row 4 of Table 2. Numerically, the maximum of $\lambda_{60}(\varphi_a, 1/2, p)$ over $0.3 \leq a \leq 0.95$, a range that excludes the two trivial end points, equals 0.99988 for $p = 12$, 0.999991 for $p = 12.006$, 1.000009 for $p = 12.007$, 1.000064 for $p = 12.01$ and 1.00176 for $p = 12.1$. Bisection gives $p_1 \approx 12.006505$, attained near $a = 0.7745$. One of the verification runs recomputed this with its own code and three discretisations and obtained 12.0065046 and $a = 0.77447$ each time; its scan of 1301 values of $a \in [0.0005, 0.9998]$ found $\lambda_{120} \leq 1$ throughout for $p \in \{11.9, 12.0, 12.006, 12.0065\}$, and $\lambda_{120} > 1$ on a short interval around $a = 0.7744$ for $p = 12.0066$ and $p = 12.0068$. So the exponent 12.0068 of Theorem D(1) is the numerical threshold of the family rounded up, with a margin of $3 \cdot 10^{-4}$. With $f(z) = z$ alone the family violates (1) at $r = 1/2$ only for $p > 15.95$ (numerically), so the functions f of Table 2 are essential.

8.3 Searches outside the family at $r = 1/2$

The following searches for a Schwarz function $\varphi = z\omega$ with $\lambda_n(\varphi, 1/2, p) > 1$ at exponents $p \leq p_1$ found no value above 1.

- *First-order test at the critical member of the family* ($p = p_1$, $a \approx 0.7745$): ω is multiplied by one more Blaschke factor with zero $(1-\varepsilon)e^{i\beta}$, or by the singular inner function $\exp(-\varepsilon(1+u)/(1-u))$, $u = e^{-i\beta}z$, with $\varepsilon \in \{0.02, 0.005\}$ and seven values of $\beta \in [0, \pi]$, and a is optimised again. In all 28 cases the maximum decreases (by $3.9 \cdot 10^{-4}$ to 0.19). This tests seven directions with two kinds of factors and does not establish local optimality among all Schwarz functions. A verification run repeated the test with 73 values of β , $\varepsilon \in \{0.02, 0.005, 0.001\}$ and $p \in \{11.9, 12.0, 12.0065, 12.0068\}$; the maximum decreased in every case (for $p = 12.0068 > p_1$ the maximum of the family itself exceeds 1).
- *Blaschke products of degree 2 and 3 at $p = 12$* , zeros of modulus at most 0.97, Nelder–Mead method with 14 random starts for each degree: the best values are 0.99813 and 0.99738, below the value 0.99988 of the family. Exploratory runs of the same kind: degrees 2 and 3 without a bound on the moduli at $p \in \{3, 6, 9, 10, 11, 11.5, 12\}$ (every run converged to the trivial value 1, with all zeros on the unit circle), and with moduli at most 0.97 for degrees 2, 3 at $p \in \{11, 11.5, 11.9, 12\}$ and degree 4 at $p \in \{11, 11.9\}$ (all values below 1).
- *Denser searches in a verification run*, zeros of modulus at most 0.985: a grid of 42,768 points for degree 2, 50,000 random points for degree 3 and 40,000 for degree 4, each followed by the Nelder–Mead method from the best 24 points; exponents $p \in \{3, 6, 10, 11.9\}$ for degrees 2, 3, 4 and $p \in \{12.0, 12.006\}$ for degrees 2 and 3. The largest value found is 0.99894 ($p = 12.006$, degree 2). In addition, 288,000 random configurations of degree 2 and 3 in which all zeros, or all but one, lie at distance between 10^{-4} and 0.015 from the unit circle ($p \in \{3, 6, 10, 11.9\}$), and local searches without a bound on the moduli (60 starts for degree 2, 40 for degree 3; $p \in \{3, 6, 10, 11.9, 12.0, 12.006\}$), all with values below 1.

Degrees greater than 4 and Schwarz functions not of this form were not searched.

8.4 Critical radii for larger p , and the first-order test at the identity

For $p > p_1$ let $r_c(p)$ be the infimum of the radii r for which $\max_a \lambda_{160}(\varphi_a, r, p) > 1$, and let $\rho_c(p)$ be the same quantity when only $f(z) = z$ is admitted. A third radius comes from Schwarz functions near the identity: $\varphi^\varepsilon(z) = z \exp(-\varepsilon(1+z)/(1-z))$, $\varepsilon > 0$, is a Schwarz function, and for a polynomial $G = \sum g_n z^n$ a direct expansion gives, with $x = r^2$,

$$\frac{d}{d\varepsilon} \int_{|z|=r} |G \circ \varphi^\varepsilon|^2 |(\varphi^\varepsilon)'|^p dm \Big|_{\varepsilon=0} = -p Q(G),$$

$$Q(G) = \sum_{m,n} x^{\max(m,n)} (1 + \max(m,n) - (1 - \frac{2}{p}) \min(m,n)) g_m \overline{g_n}.$$

If $Q(G) < 0$ for some G , then $N_p(\varphi^\varepsilon, r) > 1$ for small ε , so $r \notin \mathcal{R}_p$. Let $x^*(p)$ be the supremum of the x for which the form Q is positive semidefinite, and $r_{\text{id}}(p) = \sqrt{x^*(p)}$. Computed from sections of size 80, $x^*(2) = 1/2$, $x^*(5) = 1/3$ (where $G = (1-z)^2$ lies in the kernel), $r_{\text{id}}(p) = 1/2$ for $p \approx 13.256$, and $x^*(p)$ decreases towards $3 - 2\sqrt{2} = (\sqrt{2} - 1)^2$ as p grows.

p	$r_1(p)$ (proved)	$r_c(p)$	maximising a	$\rho_c(p)$	$r_{\text{id}}(p)$
12.1	0.42102	0.499635	0.7785	0.5	0.505335
14	0.42003	0.492819	0.8359	0.5	0.496958
16	0.41926	0.486814	0.8710	0.499982	0.489961
20	0.41820	0.477513	0.9110	0.496393	0.479557
30	0.41683	0.463165	0.9519	0.487614	0.464157
50	0.41576	0.449358	0.9762	0.476302	0.449791
76	0.41522	0.440913	0.9861	0.467672	0.441145
100	0.41498	0.436490	0.9901	0.462464	0.436648

Table 3: Numerical critical radii of the family φ_a (floating point; not proved), next to the proved lower bound $r_1(p)$ of Theorem C, rounded down. An entry 0.5 for ρ_c means that the test with $f(z) = z$ shows no violation for $r \leq 1/2$.

Table 3 lists some of the computed values. The radii r_0 of Theorem D(2) exceed $r_c(p_0)$ by less than 10^{-3} for $14 \leq p_0 \leq 100$. Three observations, all numerical. First, $r_c(p) < \rho_c(p)$: general functions f give smaller critical radii than $f(z) = z$. Second, $r_c(p) < r_{\text{id}}(p)$, and $r_{\text{id}}(p) > 1/2$ for $p < 13.256$, although the family violates (1) at $r = 1/2$ already for $p > 12.0065$; so the critical Schwarz functions are not small perturbations of the identity. Third, the maximising parameter tends to 1, and $(r_c(p) - (\sqrt{2} - 1)) p^{2/3}$ lies between 0.466 and 0.480 for $20 \leq p \leq 100$. As far as these computations go, $r_p \leq r_c(p)$, so in this range $r_p - (\sqrt{2} - 1)$ is at most about $0.48 p^{-2/3}$, while Theorem C shows that it is at least about $0.076/p$; the true order of decay is not known.

8.5 A conjecture

Conjecture 8.1. (a) $p_* = p_1$; that is, $r_p = 1/2$ for all $p \leq p_1 = 12.0065 \dots$ (b) For $p > p_1$ the number r_p is the critical radius $r_c(p)$ of the family φ_a .

The evidence for (a) is the searches listed above, which found no violation at $r = 1/2$ for $p < p_1$; this is not a proof, and it concerns only the Schwarz functions that were searched. Part (b) rests on thinner evidence. For $p > p_1$ the computations described above concern the family alone. A verification run added tests outside the family, none of which found a larger value: a first-order test with one more Blaschke factor at $(p, r_c(p))$ for $p \in \{14, 20, 50, 100, 200, 500\}$, where the change of the maximum was negative, with a margin that shrinks as p grows; products of two factors with zeros

on the same ray for $p \in \{14, 20, 30, 50, 100, 200, 500\}$; and a search over degrees 2 and 3 (about 22,000 configurations each) at the radius $r_c(p) - 0.0004$ for $p \in \{14, 20, 50\}$. What is proved is only $r_1(p) \leq r_p \leq 1/2$ for all $p > 2$, together with Theorem D.

9 Remarks

Remark 9.1 (Why the argument of Theorem C cannot reach $1/2$). The estimate (13) bounds $|\varphi'|^{p-2}$ by its maximum. Near the extremal map z^2 this is too costly. Take $r = 1/2$ and φ_a with small $a > 0$. For $z = e^{i\theta}/2$ we have $\varphi'_a(z) = 2z(1+u)/(1+v)^2$ with $u = a(e^{i\theta}/4 + e^{-i\theta})$ and $v = ae^{i\theta}/2$, so $\log |\varphi'_a| = \operatorname{Re} L$ with $L = u - 2v - u^2/2 + v^2 + O(a^3)$. Since $\operatorname{Re}(u - 2v) = (a/4) \cos \theta$, the mean of $\operatorname{Re} L$ is $-a^2/4 + O(a^3)$ and the mean of $(\operatorname{Re} L)^2$ is $a^2/32 + O(a^3)$. Hence

$$\max_{|z|=1/2} |\varphi'_a| = 1 + \frac{a}{4} + O(a^2), \quad \int_{|z|=1/2} |\varphi'_a|^p dm = 1 + \frac{p(p-16)}{64} a^2 + O(a^3).$$

So (13) loses the factor $1 + (p-2)a/4$ to first order in a , while the gain available from Lemma 3.3 is only of second order (at $r = 1/2$ we have $\gamma = 1$, and for $K = 1$ the left-hand side of (9) is $1 - \frac{7}{16}a^2 + O(a^3)$). The true mean has no term of first order: the excess of $|\varphi'_a|$ over 1 averages out. A proof of $r_p = 1/2$ for some $p > 2$ has to capture this cancellation, uniformly in f .

Remark 9.2 (Univalence). Problem 5.39 imposes no univalence, in contrast to Problem 5.38 of [HL19], which assumes that f belongs to the class S of normalised univalent functions. Suppose that f is restricted to S in the definition of r_p . The lower bounds of Theorems A and C remain valid, and so does $r_p = 1/2$ for $0 < p \leq 2$, since the pair z, z^2 has $f \in S$. Every function f in Table 2, normalised by $f(0) = 0$, belongs to S : $f'(0) = 1$ and $\operatorname{Re} f' = \operatorname{Re} h \geq 1 - \eta > 0$ in $\mathbb{D}$, and a function with $\operatorname{Re} f' > 0$ in $\mathbb{D}$ is univalent, because $f(z_2) - f(z_1) = (z_2 - z_1) \int_0^1 f'(z_1 + t(z_2 - z_1)) dt \neq 0$ for $z_1 \neq z_2$. So the violations of Table 2 are violations within S , at the exponents listed there. The passage from p_0 to all $p \geq p_0$ in Theorem D, however, uses Theorem B(1), whose proof applies the hypothesis to functions that need not belong to S . If instead φ is required to be univalent, the maps z^2 and φ_a are excluded and the problem changes further. We do not treat these variants.

Remark 9.3 (The limit $p \rightarrow 0$). For $h \neq 0$ analytic on a neighbourhood of $|z| = r$ let $M_0(r, h) = \exp \int_{|z|=r} \log |h| dm$ be the geometric mean, the limit of $M_p(r, h)$ as $p \downarrow 0$. Letting $p \downarrow 0$ in Theorem A we get $M_0(r, g') \leq \gamma(\alpha, r)^{1/2} M_0(r, f') \leq M_0(r, f')$ for $g \prec f$ with $g' \neq 0$ and $0 < r \leq 1/2$, and the pair z, z^2 shows that the critical radius for the geometric mean is again $1/2$.

10 Computations and reproducibility

Theorems A to C and all lemmas of Sections 2 and 3 are proved without any computation. Theorem D depends on the 19 strict inequalities (14) and (15), and on nothing else that is computed. What is certified is, for each row of Table 2, a strict inequality between two explicitly computed rational numbers, which by Lemmas 7.1 and 7.2 implies $M_p(r, g') > M_p(r, f')$. It is certified by three programs, written independently of one another: `c1_cert_exact.py` (Python standard library only; about two seconds); `v2_cert.py`, from the second verification run (`mpmath` is used only to propose the rational number ℓ , which is then verified exactly; about one minute); and `r3_exact.py`, from the third (standard library only; about four seconds).

The source files of this note contain the three programs with their recorded outputs, and in addition: a double-precision quadrature of the two means for the 19 rows (`n5_crosscheck.py`), which reproduces the ratios of Table 2 to ten digits, and two quadratures with 50 digits from the verification runs (`v2_quad.py`, `r3_quad.py`); a program that verifies the rounding of the values of $r_1(p)$ in exact

rational arithmetic (`r1_table.py`); a symbolic check of the algebraic identities used in Lemmas 3.4 and 7.2 and in Sections 5 and 6; floating-point consistency tests of Theorem A, Lemmas 3.2 and 3.3 and Proposition 6.1; the scripts for Section 8; and the programs and outputs of the verification runs. The exact programs and the quick numerical scripts were run again from the source archive of this note and reproduced the recorded outputs, up to lines that report running times; a file `README.md` describes how to run each program.

Verification. The results were checked as follows.

1. The exact program of this note (`c1_cert_exact.py`) certifies the 19 inequalities behind Theorem D, as described in Section 7. On data that should not give a certificate it reports failure; for example, with the data of row 4 and $p = 12.006$ the test (15) fails.
2. A first independent verification run, AI-assisted, re-derived the proofs of all lemmas in Sections 2 and 3 and of Theorems A to C, and compared the statement of the problem with the source. It checked Lemma 3.2 and the finite form of the chain in the proof of Lemma 3.3 in exact rational arithmetic on 2,440 Schwarz functions with rational data (541,680 comparisons, no failure), verified the table of $r_1(p)$ by exact root isolation, and searched for violations of Theorem A at $r = 1/2$ and of Theorem C at $r = r_1(p)$ over Blaschke products of degree up to 6 and several families of functions f ; no violation was found, while three control cases with known violations were detected. It found no mathematical error. It pointed out an inaccuracy in an earlier version of Remark 6.2 and asked for clarifications in the proofs of Lemmas 2.1, 2.3 and 2.5, which have been made; the quantitative form of Theorem A was added at its suggestion.
3. A second independent verification run, also AI-assisted, re-derived Lemmas 7.1 and 7.2 and the deduction of Theorem D, and wrote the program `v2_cert.py` from the statements alone, before the first program was read. It confirmed all 19 inequalities, reproduced the ratios by quadrature with 50 digits, ran the programs of the note again with identical results, recomputed the numbers of Section 8 with its own code, read Goluzin's paper [Gol51a] and searched the literature. It found no mathematical error and asked for corrections to the description of the numerical searches and of the literature, which have been made.
4. A third independent verification run, AI-assisted, read the final text completely and re-derived every proof. With its own exact program `r3_exact.py`, written before the other two were read, it certified all 19 inequalities and reproduced the eight decimals of every row of Table 2. It confirmed the ratios by quadrature with 50 digits, all printed values of $r_1(p)$ by exact sign tests of (2), and Table 3 and the thresholds of Section 8 with its own code. It ran the programs of the note again from the source archive, compared Theorems 1–6 of [Gol51a] and pages 351–353 of [Die31] with what is said here, read [Gol51b], and verified the DOIs of the references through Crossref. It found no mathematical error. The changes that it asked for concern wording and credit and are incorporated: the wording of the note attached to the problem in Section 1, the reason in Remark 4.1, two sentences of Section 8 (the exponents covered by the searches of Section 8.3, and the third observation after Table 3, which had claimed more than the computations show), the mention of [Gol51b], and this paragraph.

The numerical searches of the verification runs cannot replace the proofs, and the proofs do not rely on them.

Scope and priority. This note gives a partial answer to Problem 5.39 of [HL19]: the number r_p is determined for $0 < p \leq 2$ and bounded from both sides for $p > 2$. The exact value of r_p for $p > 2$ remains open.

We read the problem and its update in the arXiv version of [HL19] (printed pages 98–100), both parts of Goluzin’s paper [Gol51a, Gol51b] completely in the scan of the Göttingen Digitisation Centre (pages 214–223 of the first part at reduced resolution), the introduction and the statement of the main theorem of Reich’s paper [Rei54], the zbMATH reviews of [Rog43, Rei54, Sha57] and of the first edition of Goluzin’s book, and the zbMATH summary of [DK24]. Of Dieudonné’s memoir [Die31] we read pages 351–353. We did not read [Lit25, Rog43, Car54, Sha57, Gol69]. The forms of Littlewood’s and Rogosinski’s theorems and of the Schwarz–Pick estimates that we use are proved in Section 3; what we say about [Car54, Sha57] is taken from [HL19]. A word of caution: the review of [Gol51a] displayed by zbMATH (Zbl 0044.30501), a machine conversion of a scanned review, shows the integrals of Theorem 5 without the exponent 2; Goluzin’s paper has the squares, so this display must not be read as a statement for $p = 1$.

Theorem A has a short proof, built from tools that were available in 1951. Searches of arXiv, zbMATH Open, Crossref and OpenAlex and eight web searches, made on 8 and 9 October 2026 (UTC) for this note and in its three verification runs, found no statement of Theorem A or of Theorems B to D, and the two parts of Goluzin’s paper [Gol51a, Gol51b] contain none. However, Goluzin’s book [Gol69], Duren’s books [Dur70, Dur83], Pommerenke’s book [Pom75], Pavlović’s books [Pav04, Pav19] and MathSciNet were not checked. An earlier statement, in particular of Theorem A, can therefore not be excluded. We claim no priority, and Theorem A may be known to experts. Lemmas 3.2 and 3.3 are stated with proofs and without a claim of novelty; a sharpening of Littlewood’s principle and of Rogosinski’s theorem of a different kind, for univalent φ , is due to Dmitrović and Karapetrović [DK24]. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and the examples, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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