

On a Question of Erdős on Lemniscates in $\mathbb{R}^m$: Connected Lemniscates Lie in a Ball of Radius 2

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Abstract

For points $x_1, \dots, x_n$ in $\mathbb{R}^m$ let $E = \{x \in \mathbb{R}^m : \prod_{i=1}^n |x - x_i| \leq 1\}$. A question of Erdős, posed in Hayman's collection of problems of 1974 and recorded as Problem 4.23 by Hayman and Lingham, asks how large the volume of E can be and whether a connected E can be covered by a ball of radius 2; for $m = 2$ the covering statement is a theorem of Pommerenke (1959). We answer the covering question: if E is connected, then $|x - c| < 2$ for every $x \in E$, where c is the centroid of $x_1, \dots, x_n$, for all m and n . Projecting E onto the line through c and a point of E reduces the question to Pommerenke's planar theorem for a polynomial with real zeros; we prove that theorem again, in the closed-set form needed. Collinear Chebyshev configurations give connected sets E with $\max_{x \in E} |x - c| = 2 \cdot 2^{-1/n}$, so the constant 2 cannot be lowered uniformly in n , already for $m = 1$. The same projection shows that for each n the best radius about the centroid is the same in every dimension; it equals $2 \cdot 2^{-1/n}$ for $n \leq 2$, and whether it does for all n , as numerical experiments suggest, is open, as is the volume question. The question is Erdős's and the planar case is Pommerenke's; the note contributes the reduction, which gives the covering statement in every dimension and the dimension-independence of the best radius about the centroid. We found no proof of the covering statement for $m \geq 3$ in the sources we searched. This is an unrefereed note.

1 Introduction

1.1 The problem

Let $m \geq 1$ and $n \geq 1$, let $x_1, \dots, x_n \in \mathbb{R}^m$ (repetitions are allowed), and put

$$P(x) = \prod_{i=1}^n |x - x_i|, \quad E = \{x \in \mathbb{R}^m : P(x) \leq 1\}, \quad c = \frac{x_1 + \dots + x_n}{n},$$

where $|\cdot|$ is the Euclidean norm. The set E is closed, contains the points x_i , and lies in the union of the closed unit balls about them, because $P(x) > 1$ when $|x - x_i| > 1$ for every i ; so E is compact. For $m = 2$ we identify $\mathbb{R}^2$ with $\mathbb{C}$; then $E = \{z : |f(z)| \leq 1\}$ is the lemniscate of the monic polynomial $f(z) = \prod_i (z - x_i)$, the set studied in Problems 4.7–4.12 of the collection of Hayman and Lingham [HL18]. Problem 4.23 there, attributed to Erdős, proposes the sets E as the extension of these problems to higher dimensions [HL18, PDF p. 80, printed p. 79]. It asks two questions. The first concerns the volume of E : when is its maximum attained, and how large can it be; the source adds Piranian's observation that the ball is not extremal for $m = 3$, $n = 2$. The second is the covering

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question, “If $E_n^{(m)}$ is connected, can it be covered by a ball of radius 2?” (the source writes $E_n^{(m)}$ for E). The source adds that Pommerenke [Pom59a] proved this for $m = 2$, and Update 4.23 in the arXiv version v2 (2018) of [HL18] reports that no progress had been reported to the authors. According to the page [Blo25b] of erdosproblems.com, Erdős asked this higher-dimensional generalisation as Problem 4.23 in Hayman’s collection of 1974 [Hay74], and Table 2 of [HL18] (PDF p. 254) lists Problems 4.22–4.24 among the problems proposed in that collection. The question is record AMR-022-4023 of the UnsolvedMath corpus [ula26], whose text agrees with the source except that the citation label of Pommerenke’s paper is missing.

Connectedness cannot be dropped: if $n = 2$ and $|x_1 - x_2| = d > 2$, then $P > 1$ on the hyperplane of points equidistant from x_1 and x_2 , so E is disconnected, and $\text{diam } E = (d^2 + 4)^{1/2}$, which exceeds 4 when $d > 2\sqrt{3}$. A covering ball may have any centre; we prove the statement with the centroid as centre.

1.2 Results

Theorem 1.1. *Let $m \geq 1$, $n \geq 1$ and $x_1, \dots, x_n \in \mathbb{R}^m$. If E is connected, then $|x - c| < 2$ for every $x \in E$. In particular, E lies in the open ball of radius 2 about the centroid c of $x_1, \dots, x_n$.*

So the answer to the covering question is yes, in every dimension. The constant cannot be improved uniformly in n :

Proposition 1.2. *Let $n \geq 1$, $\rho = 2^{-1/n}$ and $t_k = 2\rho \cos \frac{(2k-1)\pi}{2n}$ for $k = 1, \dots, n$. Let $m \geq 1$ and $x_k = t_k e_1 \in \mathbb{R}^m$, where e_1 is the first unit vector. Then $c = 0$, the set E is connected, $\max_{x \in E} |x| = 2\rho = 2 \cdot 2^{-1/n}$, and $\text{diam } E = 4 \cdot 2^{-1/n}$. In particular, no ball of radius less than $2 \cdot 2^{-1/n}$ covers E .*

Since $2 \cdot 2^{-1/n} \rightarrow 2$, no constant smaller than 2 works for all n , already for $m = 1$. For fixed n we put

$$R_m(n) = \sup \left\{ \max_{x \in E} |x - c| : x_1, \dots, x_n \in \mathbb{R}^m, E \text{ connected} \right\}.$$

Corollary 1.3. *For all $m, n \geq 1$ we have $R_m(n) = R_1(n)$, and $2 \cdot 2^{-1/n} \leq R_1(n) \leq 2$.*

So the best radius about the centroid depends only on n , and it is determined by n points on a line. We do not know its value.

Question 1.4. *Is $R_1(n) = 2 \cdot 2^{-1/n}$ for every $n \geq 1$? Equivalently, by Proposition 1.2 and Corollary 1.3: if E is connected, does E lie in the closed ball of radius $2 \cdot 2^{-1/n}$ about c , in every dimension?*

The answer is yes for $n \leq 2$, and numerical experiments support it for $n \leq 12$ (Section 5). The volume question of Problem 4.23 is not treated in this note.

1.3 Method and credit

For $m = 2$, Theorem 1.1 is the closed-set form of Pommerenke’s theorem [Pom59a, Theorem 3] (Remark 2.3). For general m we project E orthogonally onto the line through c and a point $x \in E$. The image is an interval that contains the projections of the points x_i , and by the Cauchy–Schwarz inequality it lies in the real sublevel set of the monic polynomial q with these real zeros, whose mean is 0. So the lemniscate of q is connected (Lemma 2.1), and Pommerenke’s theorem, in the closed-set form of Lemma 2.2, gives $|x - c| < 2$. Both lemmas are proved in full. A bound for the diameter alone does not suffice: every connected E has diameter at most $4 \cdot 2^{-1/n}$, but for $m \geq 2$ Jung’s theorem then gives only a covering ball whose radius exceeds 2 for large n (Remark 3.4).

The question is Erdős’s, and the planar case is Pommerenke’s. What the note adds is the projection argument, which gives the covering statement in every dimension and the dimension-independence of

the best radius about the centroid (Corollary 1.3). The example of Proposition 1.2 consists of the zeros of a Chebyshev polynomial, placed on a line; its proof is included, but for $m \leq 2$ the example is not claimed as new. The sources that were read and the searches that were made are described in the paragraph “Scope and priority” in Section 6.

2 Two planar lemmas

For a monic polynomial q of degree $n \geq 1$ with zeros $t_1, \dots, t_n \in \mathbb{C}$, repeated according to multiplicity, we write

$$L(q) = \{z \in \mathbb{C} : |q(z)| \leq 1\}, \quad c_q = \frac{t_1 + \dots + t_n}{n}.$$

The set $L(q)$ is compact, and it contains a disc about each zero of q , because $|q|$ is continuous. A domain is a non-empty connected open set.

Lemma 2.1. *Let q be a monic polynomial of degree $n \geq 1$. If some connected set $J \subseteq L(q)$ contains all zeros of q , then $L(q)$ is connected.*

Proof. Fix $r > 1$ and let $\Omega_r = \{z : |q(z)| < r\}$, a bounded open set that contains $L(q)$. Let G be a component of Ω_r ; it is open, and it is bounded. A boundary point z of G does not lie in G , since G is open, and it does not lie in Ω_r either: otherwise a disc about z contained in Ω_r would meet G and hence lie in G . By continuity, $|q| = r$ on ∂G , while $|q| < r$ on G ; and $\partial G \neq \emptyset$, because G is a bounded non-empty open subset of $\mathbb{C}$. Hence the minimum of $|q|$ over the compact set $\overline{G}$ is attained at a point $z_0 \in G$. If $q(z_0) \neq 0$, then $1/q$ is holomorphic near z_0 and $|1/q|$ has a local maximum at z_0 , so q is constant near z_0 by the maximum modulus principle, and hence constant; this is impossible for $n \geq 1$. Therefore every component of Ω_r contains a zero of q . All zeros lie in the connected set $J \subseteq \Omega_r$, hence in one component; so Ω_r has only one component, and Ω_r and its closure K_r are connected.

Since $L(q) \subseteq \Omega_r \subseteq K_r \subseteq \{|q| \leq r\}$ for every $r > 1$, $L(q)$ is the intersection K of the decreasing sequence of compact connected sets $K_{1+1/k}$. If $K = A \cup B$ with A, B disjoint, closed and non-empty, take disjoint open sets $U \supseteq A$, $V \supseteq B$; each $K_{1+1/k}$ meets U and V and is connected, so the compact sets $K_{1+1/k} \setminus (U \cup V)$ are non-empty and decreasing, and their intersection $K \setminus (U \cup V)$ is non-empty, a contradiction. Hence $L(q)$ is connected. $\square$

Lemma 2.2. *Let q be a monic polynomial of degree $n \geq 1$ such that $L(q)$ is connected. Then $|z - c_q| < 2$ for every $z \in L(q)$.*

Proof. Step 1: the exterior map. Let $W = \mathbb{C} \setminus L(q) = \{|q| > 1\}$. If G were a bounded component of W , then, as in the proof of Lemma 2.1, $|q| \leq 1$ on ∂G and $|q| > 1$ on G , which contradicts the maximum modulus principle on $\overline{G}$. The set $\{|z| > R\}$ lies in W for large R and is connected, so it lies in one component of W ; every component is unbounded and therefore meets it. Hence W is connected, and $\Omega = W \cup \{\infty\}$ is a domain in the Riemann sphere $\widehat{\mathbb{C}}$ with connected complement $L(q)$. Let t_1 be a zero of q and $M(z) = 1/(z - t_1)$. Then $M(\Omega)$ is a plane domain whose complement $M(L(q))$ in $\widehat{\mathbb{C}}$ is connected, so $M(\Omega)$ is simply connected [Rud87, Theorem 13.11]; and $M(\Omega) \neq \mathbb{C}$, because $L(q)$ has interior points. By the Riemann mapping theorem [Rud87, Theorem 14.8] there is a conformal map φ of the unit disc onto $M(\Omega)$ with $\varphi(0) = 0 = M(\infty)$. Then $\psi(w) = M^{-1}(\varphi(1/w))$ maps $\Delta = \{|w| > 1\} \cup \{\infty\}$ conformally onto Ω , with $\psi(\infty) = \infty$, and

$$\psi(w) = t_1 + \frac{1}{\varphi(1/w)} = aw + b_0 + b_1w^{-1} + b_2w^{-2} + \dots \quad (|w| > 1), \quad a = 1/\varphi'(0) \neq 0.$$

Step 2: $q \circ \psi$ is a power. Every point of ∂W lies in $L(q)$ and is a limit of points of W , so $|q| = 1$ on ∂W . Let $|w_k| > 1$ and $|w_k| \rightarrow 1$. Every limit point of $\psi(w_k)$ in $\widehat{\mathbb{C}}$ lies in ∂W : it is not ∞ and

not in Ω , since otherwise $w_k = \psi^{-1}(\psi(w_k))$ would accumulate in Δ . Hence $|q(\psi(w))| \rightarrow 1$ as $|w| \rightarrow 1$. The function $h(w) = q(\psi(w))/w^n$ has no zeros on $\{|w| > 1\}$, and since $q(\psi(w)) = a^n w^n (1 + O(1/w))$, it extends holomorphically to ∞ with $h(\infty) = a^n \neq 0$. The functions $H(\zeta) = h(1/\zeta)$ and $1/H$ are holomorphic in the unit disc, and $|H(\zeta)| \rightarrow 1$ as $|\zeta| \rightarrow 1$; by the maximum modulus principle $|H| \leq 1$ and $|1/H| \leq 1$, so $|H| \equiv 1$, and H is constant by the open mapping theorem. Thus $h \equiv a^n$ and $|a| = 1$. The map $\psi_1(w) = \psi(w/a)$ is again a conformal map of Δ onto Ω , and

$$\psi_1(w) = w + b_0 + O(1/w), \quad q(\psi_1(w)) = a^n (w/a)^n = w^n.$$

Since $q(z) = z^n - nc_q z^{n-1} + O(z^{n-2})$ as $z \rightarrow \infty$, we have $q(\psi_1(w)) = w^n + n(b_0 - c_q)w^{n-1} + O(w^{n-2})$, and comparing with w^n gives

$$b_0 = c_q.$$

Step 3: the coefficient estimate. Fix $z_0 \in L(q)$. Since $\psi_1(\Delta) = \Omega \not\ni z_0$, the function $F(\zeta) = 1/(\psi_1(1/\zeta) - z_0)$ is holomorphic and injective in $0 < |\zeta| < 1$ and has no zeros there, and

$$F(\zeta) = \frac{\zeta}{1 + (b_0 - z_0)\zeta + O(\zeta^2)} = \zeta - (b_0 - z_0)\zeta^2 + O(\zeta^3).$$

So F extends to a univalent function in the unit disc with $F(0) = 0$ and $F'(0) = 1$, and Bieberbach's theorem [Dur83, Theorem 2.2] gives $|z_0 - c_q| = |b_0 - z_0| \leq 2$. If equality held, F would be a rotation of the Koebe function, $F(\zeta) = \zeta/(1 - e^{i\alpha}\zeta)^2$ for some real α [Dur83, Theorem 2.2]. Then $1/F(\zeta) = 1/\zeta - 2e^{i\alpha} + e^{2i\alpha}\zeta$, that is, $\psi_1(w) = e^{i\alpha}(v + 1/v) + \beta$ with $v = e^{-i\alpha}w$ and $\beta = z_0 - 2e^{i\alpha}$. The map $v \mapsto v + 1/v$ takes $\{|v| > 1\}$ onto $\mathbb{C} \setminus [-2, 2]$, so $\Omega \cap \mathbb{C}$ would be the complement of a segment and $L(q)$ a segment, which has no interior points. Hence $|z_0 - c_q| < 2$. $\square$

Remark 2.3. Pommerenke [Pom59a, Theorem 3] proves: if f is monic, ζ is the mean of its zeros and the open set $\{|f| < 1\}$ is connected, then the curve $\{|f| = 1\}$ lies in the disc $|z - \zeta| < 2$. Problem #1046 of erdosproblems.com [Blo25a], a question that the page attributes to Erdős, Herzog and Piranian [EHP58], is stated for the open set as well, and the page records Pommerenke's answer with the mean of the zeros as centre. Lemma 2.2 is the closed-set form, and the difference matters here. If q has a critical point on the curve $|q| = 1$, then $\{|q| < 1\}$ does not contain all zeros of q' and is disconnected by the criterion recalled below, while $L(q)$ can be connected. This is the case for the Chebyshev polynomials of Section 4 with $n \geq 2$, which occur as q in the proof of Theorem 1.1 for the configurations of Proposition 1.2 and $x = \pm 2 \cdot 2^{-1/n} e_1$. The non-strict form of Lemma 2.2 can be derived from [Pom59a, Theorem 3]: for $s > 1$ the monic polynomial $\tilde{q}(z) = s^{-n}q(sz)$ has the connected open lemniscate $s^{-1}\{|q| < s^n\}$ (first paragraph of the proof of Lemma 2.1), so the curve $\{|\tilde{q}| = 1\}$, which contains the boundary of $\{|\tilde{q}| \leq 1\}$, lies in the disc $|z - c_q/s| < 2$; hence $L(q) \subseteq s\{|\tilde{q}| \leq 1\} \subseteq \{|w - c_q| < 2s\}$, and one lets $s \rightarrow 1$. The direct proof given above follows the route of Pommerenke's proof (the inverse of a branch of $q^{1/n}$ on the exterior of the lemniscate, which is ψ_1 , and an estimate for univalent functions in $|w| > 1$); it gives the strict inequality, which the limiting argument does not. On p. 222 of [Pom59a] it is also stated, with a reference to [EHP58], that $\{|f| < 1\}$ is connected if and only if it contains all zeros of f' , and the page [Blo25a] records the same equivalence; this criterion is used only in this remark.

3 Proof of Theorem 1.1, and remarks

Proof of Theorem 1.1. Let $x \in E$. If $x = c$ there is nothing to prove. Otherwise let $u = (x - c)/|x - c|$, let $\pi(y) = \langle y - c, u \rangle$ for $y \in \mathbb{R}^m$, and let

$$q(t) = \prod_{i=1}^n (t - \pi(x_i)).$$

This is a monic polynomial of degree n with real zeros $\pi(x_1), \dots, \pi(x_n)$, and $c_q = \frac{1}{n} \sum_i \pi(x_i) = \pi(c) = 0$, because π is affine.

(i) Let $y \in E$. For every i , the Cauchy–Schwarz inequality and $|u| = 1$ give $|\pi(y) - \pi(x_i)| = |\langle y - x_i, u \rangle| \leq |y - x_i|$, so $|q(\pi(y))| \leq P(y) \leq 1$. Hence $J := \pi(E) \subseteq L(q) \cap \mathbb{R}$.

(ii) The set E is compact and connected, so J is a compact interval. It contains $\pi(x_1), \dots, \pi(x_n)$, because the points x_i lie in E , and it contains $\pi(x) = |x - c|$.

(iii) By (i) and (ii), J is a connected subset of $L(q)$ that contains all zeros of q . By Lemma 2.1, $L(q)$ is connected.

(iv) By Lemma 2.2, $|z| = |z - c_q| < 2$ for every $z \in L(q)$. Since $|x - c| = \pi(x) \in J \subseteq L(q)$, we obtain $|x - c| < 2$. $\square$

Since E is compact, also $\max_{x \in E} |x - c| < 2$. In step (iv) only the case of real zeros of Lemma 2.2 is used; for $m = 2$, the proof therefore derives the planar theorem for arbitrary zeros from its case of real zeros.

Remark 3.1. The connectedness of E enters only in step (ii). The same proof shows: if $K \subseteq E$ is connected and contains $x_1, \dots, x_n$, then $|x - c| < 2$ for every $x \in K$ (take $J = \pi(K)$).

Remark 3.2. Let N be any norm on $\mathbb{R}^m$ and $E_N = \{x : \prod_i N(x - x_i) \leq 1\}$. If E_N is connected, then $N(x - c) < 2$ for every $x \in E_N$. Indeed, E_N is compact and contains the points x_i . For $x \in E_N$, $x \neq c$, the Hahn–Banach theorem gives a linear functional ℓ with $\ell(x - c) = N(x - c)$ and $|\ell| \leq N$. With $\pi(y) = \ell(y - c)$ we have $|\pi(y) - \pi(x_i)| \leq N(y - x_i)$, $c_q = 0$ and $\pi(x) = N(x - c)$, and steps (i)–(iv) go through unchanged.

Remark 3.3. In the plane, every component of $\{|q| < \kappa\}$ contains a zero of q , for every $\kappa > 0$ (proof of Lemma 2.1). In $\mathbb{R}^3$ the analogous statement fails for E and for the sets $\{P < \kappa\}$, and this is why the proof passes to the plane. Let $a > 0$ and let $x_1, \dots, x_6$ be the points $\pm a e_k$ ($k = 1, 2, 3$) of $\mathbb{R}^3$. For $x = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$P(x)^2 = \prod_{k=1}^3 \left((|x|^2 + a^2)^2 - 4a^2 \xi_k^2 \right).$$

On the sphere $|x| = r < a$, put $y_k = \xi_k^2$; then $y_k \geq 0$, $y_1 + y_2 + y_3 = r^2$, each factor is at least $(a^2 - r^2)^2 > 0$, and $\log P^2 = \sum_k \log((r^2 + a^2)^2 - 4a^2 y_k)$ is a concave function of (y_1, y_2, y_3) . Its minimum on this triangle is attained at a vertex, so $P(x) \geq (a^2 - r^2)(a^2 + r^2)^2$ for $|x| = r$, and for $r = a/\sqrt{3}$ this bound is $\frac{32}{27}a^6$. If $\frac{27}{32} < a^6 \leq 1$, the sphere $|x| = a/\sqrt{3}$ does not meet E , while $0 \in E$ (as $P(0) = a^6$) and the points x_i lie outside the sphere. So the component of E that contains 0 contains none of the points x_i ; the same holds for $\{P < \kappa\}$ when $a^6 < \kappa < \frac{32}{27}a^6$. Bauer, Catanese and Di Scala show that already in $\mathbb{R}^3$ the function $\prod_i |x - x_i|^2$ can have arbitrarily many local minima other than the points x_i [BCD17, Theorem 0.3(2)]; in the plane it has none, by the minimum modulus principle.

Remark 3.4. The covering question is not settled by a bound for the diameter. By Chebyshev’s theorem, a monic real polynomial p of degree n satisfies $\max_{[\alpha, \beta]} |p| \geq M := 2((\beta - \alpha)/4)^n$: otherwise $p^* - p$, where $p^*(t) = MT_n((2t - \alpha - \beta)/(\beta - \alpha))$ is monic and takes the values $\pm M$ alternately at $n + 1$ points, would be a non-zero polynomial of degree at most $n - 1$ with n sign changes (T_n as in Section 4). Steps (i) and (ii) of the proof of Theorem 1.1 hold for every unit vector u , and they show that $J = \pi(E)$ is an interval on which $|q| \leq 1$. Hence the projection of a connected set E to any line has length at most $4 \cdot 2^{-1/n}$, so $\text{diam } E \leq 4 \cdot 2^{-1/n}$, with equality in Proposition 1.2. For $m = 1$ this already shows that a connected E is an interval of length at most $4 \cdot 2^{-1/n}$. For $m \geq 2$, Jung’s theorem [Jun01] gives only a covering ball of radius $4 \cdot 2^{-1/n} (m/(2m + 2))^{1/2}$, which exceeds 2 for $m = 2$ and $n \geq 5$ and, for larger m , from some n on. The centroid form of Pommerenke’s theorem is what gives the radius 2.

4 Sharpness: proof of Proposition 1.2

Let T_n be the Chebyshev polynomial of the first kind of degree $n \geq 1$, so that $T_n(\cos \zeta) = \cos n\zeta$ for all $\zeta \in \mathbb{C}$; its leading coefficient is 2^{n-1} . Let $\rho = 2^{-1/n}$, so that $2\rho^n = 1$, and $q_n(t) = T_n(t/(2\rho))$.

Proof of Proposition 1.2. (a) *The polynomial.* With $\theta_k = \frac{(2k-1)\pi}{2n}$ we have $\cos n\theta_k = 0$, so the n distinct numbers $\cos \theta_k$ ($k = 1, \dots, n$) are the zeros of T_n , and $T_n(s) = 2^{n-1} \prod_k (s - \cos \theta_k)$. Hence

$$q_n(t) = \frac{2^{n-1}}{(2\rho)^n} \prod_{k=1}^n (t - t_k) = \prod_{k=1}^n (t - t_k),$$

because $2^{n-1}/(2\rho)^n = 1/(2\rho^n) = 1$. Since $\theta_{n+1-k} = \pi - \theta_k$, the zeros t_k are symmetric about 0, and $c = 0$.

(b) *Reduction to the plane.* For $y \in \mathbb{R}^m$ write $y = te_1 + s$ with $s \perp e_1$, and put $z = t + i|s| \in \mathbb{C}$. Then $|y - x_k|^2 = (t - t_k)^2 + |s|^2 = |z - t_k|^2$ for every k , so $P(y) = |q_n(z)|$ and $|y| = |z|$. Thus $y \in E$ if and only if $z \in L(q_n)$.

(c) *An upper bound.* Let $z \in L(q_n)$ and $w = z/(2\rho)$, so that $|T_n(w)| \leq 1$. Write $w = \cos \zeta$ with $\zeta = \alpha + i\beta$ (the cosine maps $\mathbb{C}$ onto $\mathbb{C}$). From $|\cos(\alpha + i\beta)|^2 = \cos^2 \alpha + \sinh^2 \beta$ we get

$$\cos^2 n\alpha + \sinh^2 n\beta = |T_n(w)|^2 \leq 1, \quad \text{that is,} \quad \sinh^2 n\beta \leq \sin^2 n\alpha.$$

Now $|\sinh n\beta| \geq n|\sinh \beta|$, since the Taylor coefficients $(n^{2j+1} - n)/(2j+1)!$ of $\sinh ny - n \sinh y$ are non-negative, and $|\sin n\alpha| \leq n|\sin \alpha|$, by induction from $\sin(k+1)\alpha = \sin k\alpha \cos \alpha + \cos k\alpha \sin \alpha$. Hence $\sinh^2 \beta \leq \sin^2 \alpha$ and $|w|^2 = \cos^2 \alpha + \sinh^2 \beta \leq 1$. So $|z| \leq 2\rho$ on $L(q_n)$, and by (b), $|y| \leq 2\rho$ for every $y \in E$.

(d) *The real points.* For real s we have $|T_n(s)| \leq 1$ if $|s| \leq 1$, and $|T_n(s)| = \cosh(n \operatorname{arccosh} |s|) > 1$ if $|s| > 1$. Hence $L(q_n) \cap \mathbb{R} = [-2\rho, 2\rho]$, and $\pm 2\rho e_1 \in E$. Together with (c), $\max_{x \in E} |x| = 2\rho$ and $\operatorname{diam} E = 4\rho$; a ball that covers E contains $\pm 2\rho e_1$, so its radius is at least 2ρ .

(e) *Connectedness.* Let $y = te_1 + s \in E$. For $0 \leq \lambda \leq 1$,

$$P(te_1 + \lambda s) = \prod_{k=1}^n ((t - t_k)^2 + \lambda^2 |s|^2)^{1/2} \leq P(y) \leq 1,$$

so the segment from te_1 to y lies in E ; in particular $te_1 \in E$, and $t \in [-2\rho, 2\rho]$ by (d). So E is the union of the segment $[-2\rho, 2\rho]e_1 \subseteq E$ and of segments that meet it, and E is connected. $\square$

The critical points of T_n are the numbers $\cos(j\pi/n)$, $j = 1, \dots, n-1$, because $T'_n(\cos \theta) \sin \theta = n \sin n\theta$, and there $T_n = \pm 1$; so all critical values of q_n have modulus 1, as used in Remark 2.3.

5 Fixed n : proof of Corollary 1.3, and open questions

For points $p_1, \dots, p_n \in \mathbb{R}$ we write $E_1 = \{t \in \mathbb{R} : \prod_i |t - p_i| \leq 1\}$.

Proof of Corollary 1.3. $R_m(n) \geq R_1(n)$: let $p_1, \dots, p_n \in \mathbb{R}$ with E_1 connected, and put $x_i = p_i e_1 \in \mathbb{R}^m$. The argument of step (e) in the proof of Proposition 1.2 shows that every point $te_1 + s$ of E is joined to $te_1 \in E$ by a segment in E , and that $t \in E_1$. So E is the union of the connected set $E_1 e_1 \subseteq E$ and of segments that meet it, hence connected; its centroid is $(\frac{1}{n} \sum_i p_i) e_1$, and it contains $E_1 e_1$.

$R_m(n) \leq R_1(n)$: let $x_1, \dots, x_n \in \mathbb{R}^m$ with E connected, let $x \in E \setminus \{c\}$, and let π and q be as in the proof of Theorem 1.1. The real configuration $p_i = \pi(x_i)$ has centroid 0, and its set $E_1 = L(q) \cap \mathbb{R}$

contains the interval J , which contains all p_i and the point $\pi(x) = |x - c|$. For $t \geq \max_i p_i$ the function $\prod_i |t - p_i|$ is increasing, and similarly on the left of $\min_i p_i$; hence E_1 is an interval, and $|x - c| \leq \max_{t \in E_1} |t| \leq R_1(n)$.

Finally, $R_1(n) \leq 2$ by Theorem 1.1, and $R_1(n) \geq 2 \cdot 2^{-1/n}$ by Proposition 1.2 with $m = 1$. $\square$

Question 1.4. By Corollary 1.3, Question 1.4 asks whether every monic real-rooted polynomial q of degree n with $|q| \leq 1$ between its smallest and its largest zero satisfies $|t - c_q| \leq 2 \cdot 2^{-1/n}$ whenever t is real and $|q(t)| \leq 1$. For $n = 1$ this is clear, with equality. For $n = 2$, let the zeros be $c_q \pm d/2$; the maximum of $|q|$ between them is $d^2/4$, so $d \leq 2$, and the real points with $|q(t)| \leq 1$ satisfy $(t - c_q)^2 \leq 1 + d^2/4 \leq 2$; so $R_1(2) = \sqrt{2} = 2 \cdot 2^{-1/2}$. For $n \geq 3$ we know only $2 \cdot 2^{-1/n} \leq R_1(n) \leq 2$, and $\max_{x \in E} |x - c| < 2$ for each configuration by Theorem 1.1. Numerical experiments, which are evidence and not proof, support a positive answer. The second verification run (Section 6) maximised $\max_{t \in E_1} |t - c|$ over real configurations under the constraints $|q| \leq 1$ at the critical points, with 400 starting points for each $n = 2, \dots, 12$ and two seeds; the largest values agree with $2 \cdot 2^{-1/n}$ to within $2 \cdot 10^{-10}$ (the constraints were accepted up to 10^{-9}). The third verification run, with exhaustive grids for $n = 3, 4$ and differential evolution for $3 \leq n \leq 10$, found no feasible value above $2 \cdot 2^{-1/n}$, and its differential evolution reached $2 \cdot 2^{-1/n}$ to within 10^{-12} for every such n . In its search over complex zeros in the plane ($n \leq 8$) the best runs ended near the real Chebyshev configurations, and the searches in $\mathbb{R}^3$ and $\mathbb{R}^4$ of both runs found nothing above $2 \cdot 2^{-1/n}$ beyond their numerical tolerance. The largest values occurred for nearly collinear configurations close to those of Proposition 1.2; this is an observation, not a proved statement. The number $2 \cdot 2^{-1/n} = 2^{1-1/n}$ also appears in Update 4.7 of [HL18], as the sharp constant for fixed n in an attempted proof of the conjecture of Problem 4.7 on Cartan's covering lemma, which according to [HL18] has gaps. We do not know whether the one-variable question above has been considered; Pommerenke's paper on polynomials with real zeros [Pom59b], which Erdős lists among the references for these problems [Erd61, p. 247], was not read.

The volume question. The first question of Problem 4.23, when the volume of E is maximal and how large it can be, is not addressed here.

6 Computations and reproducibility

No statement of the note depends on a computation. The program of this note tests Theorem 1.1 on configurations whose set E is certified to be connected by the following sufficient condition.

Lemma 6.1. *Let $p \in \mathbb{R}^m$. If for every i the closed ball with diameter $[p, x_i]$ lies in E , then E is star-shaped with respect to p ; in particular, E is connected.*

Proof. The point p lies in each of these balls, so $p \in E$. Let $y \in E$, $y \neq p$, $u = (y - p)/|y - p|$ and $\tau = \max(0, \max_i \langle x_i - p, u \rangle)$. If $0 \leq t \leq \tau$ and j attains the inner maximum, then

$$\left| p + tu - \frac{p + x_j}{2} \right|^2 - \frac{|x_j - p|^2}{4} = t^2 - t \langle x_j - p, u \rangle \leq 0,$$

so $p + tu$ lies in the ball with diameter $[p, x_j]$, hence in E . For $t \geq \tau$, each factor $|p + tu - x_i| = ((t - \langle x_i - p, u \rangle)^2 + d_i^2)^{1/2}$, where d_i is the distance of x_i from the line $p + \mathbb{R}u$, is non-decreasing in t , so $P(p + tu) \leq P(y) \leq 1$ for $\tau \leq t \leq |y - p|$. Hence the segment $[p, y]$ lies in E . $\square$

The program of this note. The program `check_note.py` (Python with NumPy, SciPy, SymPy and mpmath; about 35 seconds on a laptop) prints a report and writes no files. (i) For $n \leq 12$ it checks in exact arithmetic the identities used in the proof of Proposition 1.2: the recurrences of T_n and of the Chebyshev polynomials U_{n-1} of the second kind; that q_n is monic with zeros t_k of mean 0; $T_n(\pm 1) = (\pm 1)^n$; $T_n^2 - (s^2 - 1)U_{n-1}^2 = 1$ and $T'_n = nU_{n-1}$, with a Sturm count of the zeros of U_{n-1} in $(-1, 1)$, so that $\{t \in \mathbb{R} : |q_n(t)| \leq 1\}$ is $[-2\rho, 2\rho]$ and all critical values of q_n have modulus 1; the identity for $|\cos(\alpha + i\beta)|^2$; $\sin n\alpha = \sin \alpha U_{n-1}(\cos \alpha)$ and $|U_{n-1}| \leq n$ on $[-1, 1]$; and the Taylor coefficients of $\sinh ny - n \sinh y$ up to y^{25} . Numerically, $\max |w| = 1$ on $4096n$ points of the curve $|T_n(w)| = 1$. (ii) As a numerical sanity check of Theorem 1.1 it takes 87 configurations with $2 \leq n \leq 8$ in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ (37, 28, 22; random, nearly collinear, nearly planar, and symmetric ones), each scaled up by bisection as long as the hypothesis of Lemma 6.1 with $p = c$ is verified. As $\log |x - x_i|$ is subharmonic on $\mathbb{R}^m$ for $m \geq 2$, it suffices that $P \leq 1$ on the boundary spheres; this is verified on adaptively refined coverings of the spheres by cells of radius h , with $P(y) \leq \prod_j (|z - x_j| + h)$ for $|y - z| \leq h$, in floating point with a safety margin; an undecided cell counts as failure. For each certified configuration the program computes $\max_{x \in E} |x - c|$ by ray casting over at least 1500 directions with local refinement, and it checks the inequality behind the proof for every sampled direction u : the exit distance along u is at most the right end $B(u) < 2$ of $\{t \in \mathbb{R} : |q_u(t)| \leq 1\}$, $q_u(t) = \prod_i (t - \langle x_i - c, u \rangle)$, and $|q_u| \leq 1$ between the zeros of q_u . For $m = 2$ it also checks the critical values and recomputes the radius from the curve $|q| = 1$. The largest radii are 1.3643, 1.3324 and 1.2988 in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$; Lemma 6.1 certifies only configurations well inside the connected range. Two controls behave as they must: a disconnected configuration fails the test, and $\{P \leq 3\}$ for the Chebyshev configuration with $n = 10$ scaled by $3^{1/10}$ has radius $3^{1/10}2^{9/10} = 2.0828 > 2$. (iii) For Lemma 2.2 it checks exactly, for $2 \leq n \leq 8$, that the only critical value of $z^n - 1$ has modulus 1 and that $|q(2^{1/n})| = 1$; since $|z|^n \leq |z^n - 1| + 1 \leq 2$ on the lemniscate, this is connected and its largest distance from the mean 0 of the zeros is $2^{1/n}$. It also checks numerically seven polynomials with real, complex or multiple zeros and critical values at most 1 (computed to 50 digits); the largest value of $|z - c_q|$ is 1.8353, for 0.99 times the nine Chebyshev zeros. (iv) It checks exactly the formula for $n = 2$ in Section 5 and the identities of Remark 3.3. All checks pass.

Verification. The results were first obtained, with proofs and with the programs in `original/`; the first account contained Theorem 1.1, Lemmas 2.1 and 2.2, and Proposition 1.2 with the lower bound $\text{diam } E \geq 4 \cdot 2^{-1/n}$, the value $\max_{x \in E} |x| = 2 \cdot 2^{-1/n}$ being observed numerically. Two independent verification runs, both AI-assisted, followed; each wrote its own programs, and neither read the other's report. These programs and those of the first computation are in the package (folders `original/`, `verification_run_A/` and `verification_run_B/`) with their recorded outputs. Their tests of connectedness are not certificates: a connected graph of sampled segments in E (first computation) or a labelling of grid cells (both runs) shows only that the points x_i and the sampled points or cells lie in one component, at the resolution used; their results are evidence.

The first run re-derived Theorem 1.1, Lemmas 2.1 and 2.2 and Proposition 1.2 line by line, including the degenerate cases, and proved the upper bound in part (c) of the proof of Proposition 1.2, the extension to other norms (Remark 3.2) and the diameter bound of Remark 3.4. It checked the statement against [HL18] and [Blo25b, Blo25a], read [Erd61, pp. 246–247], checked the bibliographic data, recomputed Proposition 1.2 for $n \leq 20$ in exact and 60-digit arithmetic, re-ran the original program for Proposition 1.2 (identical output), and tested Theorem 1.1 on 1,400 configurations in $\mathbb{R}^3$ and $\mathbb{R}^4$ at their largest connected scale, connectedness being decided by voxel labelling with two adjacencies and two mesh sizes (largest radii 1.6742 in $\mathbb{R}^3$, 1.6399 in $\mathbb{R}^4$). It found [JMR16].

The second run wrote its programs from the definitions before reading those of the original computation. It certified the identities of Proposition 1.2 exactly for $n \leq 40$, proved the upper bound for the

Chebyshev configurations independently, made the searches for Question 1.4 described in Section 5, tested Lemma 2.2 on 600 random connected lemniscates and Theorem 1.1 in $\mathbb{R}^3$ by grid labelling and near-collinear searches (largest radius 1.8001, for $n = 8$, below $2 \cdot 2^{-1/8} = 1.8340$), checked that wrong inputs are detected, and re-ran the five original programs from a copy, with byte-identical outputs. It found Corollary 1.3, Remark 3.1, the extension to other norms, the example of Remark 3.3 (numerically) and [EH78].

Both runs confirmed every statement and found no mathematical error. They required corrections of descriptions: page numbers (PDF p. 80, printed p. 79 of [HL18]; pp. 246–247 of [Erd61]); the origin in [Hay74], as stated in [Blo25b]; no statements on the contents of papers that had not been read; the open-set form on [Blo25a]; the removal of an unproved remark on coplanar configurations; an accurate description of the numerical tests of connectedness and of what the original program for Proposition 1.2 checks exactly; the label “observation” for the remark on nearly collinear configurations; and the citation of [EH78, JMR16]. All were made. When the note was written, [Pom59a] and [CDL96] were read and Remark 2.3 was written; Lemma 6.1, the proof in Remark 3.3 and the program of this note were added then and were not examined by the two runs.

A third independent verification run, also AI-assisted, then examined the final text of the note, the package and the sources, as if nothing had been checked before. It read Problem 4.23 with its update and Tables 1 and 2 of [HL18], the pages [Blo25b, Blo25a] and [EH78] again, and the cited parts of [BCD17, JMR16, Erd61]; it re-derived every proof line by line, in particular Lemma 6.1, the proof in Remark 3.3, the case $n \leq 2$ of Question 1.4 and Remarks 3.2 and 3.4, and it read the program of this note line by line. Its own programs (folder `independent_run_2/`) check Proposition 1.2 exactly for $n \leq 20$, with a second proof of the upper bound in part (c) (for $n \geq 2$ the polynomial $|T_n(e^{i\theta})|^2 - 1$ in $\cos \theta$ has no zero in $(-1, 1)$, so $|T_n| \geq 1$ on the unit circle, and the maximum modulus principle for $1/T_n$ on $\{|w| > 1\}$ gives $|T_n| > 1$ there), and by ray casting in $\mathbb{R}^3$ at 40 digits for $n \leq 12$. They give a second proof of the bound in Remark 3.3 without concavity, compute the minimum of P on the sphere for four values of a , and find the component of 0 by a voxel labelling (seven components in all). They test Lemma 6.1 on 64,800 segments in 108 random configurations, the one-dimensional problem of Question 1.4 as described in Section 5, the example of disconnected sets in Section 1, Remarks 2.3, 3.2 and 3.4, and Theorem 1.1 in $\mathbb{R}^3$ on 30 configurations; all checks pass. The run also ran every program of the package again from the source archive; all outputs agree with the recorded ones up to running times. It found no error in a proof or in a stated number. It required corrections of wording, which were made: the origin of the question in the abstract (now also with Table 2 of [HL18]), “best radius about the centroid” for “best constant”, the credit for the example of Proposition 1.2, the letter for the level in Remark 3.3, the description of part (iii) of the program, and the tolerance of the constraints in Section 5.

Scope and priority. The note answers the covering question of Problem 4.23 of [HL18] (Theorem 1.1, sharp by Proposition 1.2) and reduces the problem for fixed n to the line (Corollary 1.3); Question 1.4 and the volume question remain open. The question is Erdős’s. According to [Blo25b] and to Table 2 of [HL18] it was posed as Problem 4.23 in Hayman’s collection [Hay74], which we did not see; we read it in [HL18], in the arXiv version v2 of 2018, where Problem 4.23 and Update 4.23 are on PDF p. 80 (printed p. 79); the published edition [HL19] was not consulted. The planar case is Pommerenke’s theorem [Pom59a, Theorem 3]; Lemma 2.2 is its closed-set form and is proved here in full, so that the note depends on [Pom59a] only for the credit.

What was read: [Pom59a] (Theorems 3 and 4, the latter on the diameter and the width, and the criterion on p. 222), which does not consider sets in $\mathbb{R}^m$; [CDL96], on the size (capacity, Lebesgue measure in $\mathbb{C}$ or $\mathbb{C}^n$) of lemniscates of polynomials in one and several complex variables, with no covering statement for the sets E ; [Erd61, pp. 246–247], where the question in $\mathbb{R}^m$ does not occur;

[EH78] in full, on inclusions between two such sets in $\mathbb{R}^3$ (equality is forced when the points of the larger set are coplanar, and an example with six points shows that coplanarity is needed); Section 5 of [JMR16], on star-shapedness and components of the sets $\{\prod_i \gamma_i(x - p_i) \leq \alpha\}$ for gauges γ_i , without a covering statement (the papers on three-dimensional lemniscates in computer-aided design that it cites were not read); the abstract, definitions and main theorems of [BCD17], whose full text the first run searched for Erdős, Pommerenke, Hayman, Piranian and covering statements, without result; and the abstract of [MPT26], on the convexity of high sublevel sets of $\prod_i |x - x_i|^2$. Not read: [EHP58], [Pom59b], [Hay74], [HL19].

The searches were made on 16 October 2026 by the first computation and the three verification runs: queries to the arXiv interface (the third run screened the titles of all 311 entries that it returns for “lemniscate” and “lemniscates”), zbMATH Open, Crossref and OpenAlex (including the lists of works citing [Pom59a] and [EHP58]; when the third run repeated these requests, they were refused because of rate limits), Semantic Scholar (partly refused because of rate limits; it gave the third run the works citing [Pom59a] and [Pom59b]), the pages [Blo25b, Blo25a] with their discussion threads and the lists of problems tagged “analysis” and “polynomials” on erdosproblems.com, and nine web searches; the works citing [HL19] were not screened. No printed proof of the covering statement in $\mathbb{R}^m$, $m \geq 3$, was found in the sources searched. The works on these sets in higher dimensions that we found [EH78, JMR16, BCD17, MPT26] contain no covering result. The argument is short and uses only classical tools, so a published version may exist that these searches missed. A search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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