

The Equation $J_0(z) = 1$ Has at Most One Solution on Each Ray: An Answer to a Question of Zalcman

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Abstract

L. Zalcman asked whether the equation $J_0(z) = 1$, where J_0 is the Bessel function of order zero, has at most one solution on each ray from the origin. The question is Problem 2.45 in Hayman and Lingham’s *Research Problems in Function Theory*, where it is noted that an affirmative answer would show that the exceptional set in a two-circle theorem of Delsarte and Lions is void. We prove that the answer is yes. Apart from the double solution $z = 0$, the solutions are $\pm z_m$ and $\pm \overline{z_m}$ ($m \geq 1$), all simple, where z_m lies in the open first quadrant and $\pi/2 > \arg z_1 > \arg z_2 > \dots > 0$. Hence every open ray from 0 contains at most one solution. (On closed rays the statement fails trivially, because 0 is a solution.) For $\operatorname{Re} z \geq 60$ the proof uses Olver’s explicit error bounds for Hankel’s expansion, a contraction mapping, Rouché’s theorem and a differential inequality for the arguments of the approximate solutions. For the rectangle $[0, 60] \times [0, 5]$ it uses a certified computation in exact integer ball arithmetic (an argument-principle count, Rouché discs and exact comparisons of arguments), which was reproduced by two independent interval-arithmetic computations. Consequently, the set of positive quotients of nonzero solutions of $J_0(z) = 1$ is $\{1\}$, and Delsarte’s two-circle theorem in the plane holds for every pair of distinct radii: a continuous function on $\mathbb{R}^2$ with the circle mean-value property for two distinct radii is harmonic. This is an unrefereed note.

1 Introduction

Let $J_0(z) = \sum_{k \geq 0} (-1)^k (z/2)^{2k} / (k!)^2$ be the Bessel function of the first kind of order zero. It is an even entire function, real on the real axis. Problem 2.45 of the collection of research problems in function theory of Hayman and Lingham [HL19], attributed to L. Zalcman, asks whether the equation

$$J_0(z) = 1 \tag{1}$$

“has at most one solution on each ray from the origin” [HL19, Problem 2.45]. The problem text adds two remarks: an affirmative answer would show that the exceptional set in a theorem of Delsarte and Lions [DL59] is void, and asymptotic estimates allow at most finitely many solutions on each ray. According to the tables in [HL19], the problem first appeared among the new problems collected at the 1973 Canterbury symposium [CH74]. In the arXiv version of [HL19] (2018), Update 2.45 states that no progress on the problem had been reported to the authors. The problem is record AMR-022-2045 of the UnsolvedMath dataset [ula26].

Since $J_0(z) - 1 = -z^2/4 + O(z^4)$, the point $z = 0$ is a double solution of (1), and it lies on every closed ray. Read with closed rays, the question would have the trivial answer “no” for every ray

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through a nonzero solution. This is not the intended reading, because the exceptional set is defined by quotients of nonzero solutions (see below). We therefore consider open rays $\{te^{i\theta} : t > 0\}$, that is, nonzero solutions.

Theorem 1.1. *The solutions of (1) are $z = 0$, which is a double solution, and the points $\pm z_m, \pm \overline{z_m}$ ($m \geq 1$), which are simple solutions. Here $z_1, z_2, \dots$ are the solutions in the open first quadrant, numbered by increasing real part, and*

$$\frac{\pi}{2} > \arg z_1 > \arg z_2 > \arg z_3 > \dots > 0. \quad (2)$$

Consequently every open ray from 0 contains at most one solution of (1); it contains one exactly when its direction is $\pm \arg z_m$ or $\pi \pm \arg z_m$ for some $m \geq 1$. Moreover, for $m \geq 10$ we have $|z_m - F_m| < 0.576/(2\pi m)$, where F_m is the unique solution with $\operatorname{Re} F \geq 60$ of $F = 2\pi m + \pi/4 + \frac{i}{2} \operatorname{Log}(2\pi F)$.

Table 1: The solutions z_m of $J_0(z) = 1$ in the open first quadrant with $\operatorname{Re} z_m < 80$, rounded. For $m \leq 9$ the values are centres of certified discs of radius 2^{-150} (Section 4); for $10 \leq m \leq 12$ they are centres of certified discs of radius 2^{-100} from the first independent computation of Section 4.1. The second independent computation of Section 4.1 confirms every digit shown.

m	$\operatorname{Re} z_m$	$\operatorname{Im} z_m$	$\arg z_m$
1	6.947506242533	1.880440653101	0.26433072
2	13.276697958613	2.205702959318	0.16462978
3	19.579253925689	2.401190563940	0.12203017
4	25.873373935358	2.541420474435	0.09791124
5	32.163666085293	2.650840397869	0.08223138
6	38.451875827822	2.740576973117	0.07115259
7	44.738818789432	2.816641954352	0.06287447
8	51.024930018196	2.882654594139	0.05643503
9	57.310463905467	2.940963696532	0.05127137
10	63.595579655298	2.993180391125	0.04703115
11	69.880382255084	3.040457948947	0.04348204
12	76.164943883334	3.083650344697	0.04046438

The two-circle theorem. For $r > 0$ and $f \in C(\mathbb{R}^2)$ let

$$M_r f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r(\cos \theta, \sin \theta)) d\theta$$

be the mean of f over the circle of radius r about x . Harmonic functions satisfy $M_r f = f$ for every r . For $\lambda \in \mathbb{C}$ let $e_\lambda(x) = e^{i\lambda x_1}$. By [DLM26, (10.9.1)],

$$M_r e_\lambda = J_0(\lambda r) e_\lambda. \quad (3)$$

Let

$$H_2 = \{\alpha/\beta : \alpha, \beta \in \mathbb{C} \setminus \{0\}, J_0(\alpha) = J_0(\beta) = 1, \alpha/\beta \in (0, \infty)\}$$

be the set of positive quotients of nonzero solutions of (1) (the notation is that of Berenstein and Gay [BG86]; see below). Clearly $1 \in H_2$. If $r_1/r_2 = \alpha/\beta \in H_2$, then $\lambda = \alpha/r_1 = \beta/r_2$ satisfies $J_0(\lambda r_1) = J_0(\lambda r_2) = 1$. So, by (3), $M_{r_1} e_\lambda = M_{r_2} e_\lambda = e_\lambda$, while $\Delta e_\lambda = -\lambda^2 e_\lambda \neq 0$. By Delsarte's theorem, these examples are the only obstruction.

Theorem 1.2 (Delsarte’s two-circle theorem). *Let $r_1, r_2 > 0$ with $r_1/r_2 \notin H_2$. If $f \in C(\mathbb{R}^2)$ satisfies $M_{r_1}f = M_{r_2}f = f$ on $\mathbb{R}^2$, then f is harmonic.*

This is the planar case of Delsarte’s two-radius theorem [Del58]; see also [DL59, Fla65, Zal80, NV94]. We have not seen [Del58] or [DL59], and we quote Theorem 1.2 in the form given in the zbMATH summary of the paper [BG86] of Berenstein and Gay. That summary states a local version: if $r_1/r_2 \notin H_n$ and $r_1 + r_2 < R$, then a continuous function on the ball of radius R about 0 in $\mathbb{R}^n$ is harmonic if and only if it has the mean-value property for the spheres of radii r_1 and r_2 about every centre x with $|x| + r_j < R$. Here H_n is the set of positive quotients of nonzero solutions of the n -dimensional analogue of (1) (Remark 6.2), and the summary records that H_n is finite and contains 1. Applying the local version on the discs of radius $R > r_1 + r_2$ about all points of $\mathbb{R}^2$ gives Theorem 1.2. The summary prints the function whose zeros define H_n as Λ_n of Remark 6.2, with an unclosed parenthesis and without the term -1 . We read it as $\Lambda_n - 1$: this is the only reading compatible with the finiteness of H_n stated there (the positive zeros of J_0 have infinitely many quotients), with the examples e_λ and with the remark in [HL19]. The examples e_λ show that the hypothesis $r_1/r_2 \notin H_2$ is necessary, and Theorem 1.2 shows that it is sufficient. Zalcman’s question is whether $H_2 = \{1\}$.

Corollary 1.3. $H_2 = \{1\}$.

Corollary 1.4. *Let $r_1, r_2 > 0$ be distinct. Then every $f \in C(\mathbb{R}^2)$ with $M_{r_1}f = M_{r_2}f = f$ is harmonic. For $r_1 = r_2 = r$ this fails: the function $e_{z_1/r}$ satisfies $M_re_{z_1/r} = e_{z_1/r}$ but is not harmonic.*

The new input to Corollary 1.4 is Corollary 1.3; the harmonicity comes from Theorem 1.2. In the same way, Corollary 1.3 shows that the hypothesis $r_1/r_2 \notin H_2$ of the local two-circle theorem of Berenstein and Gay [BG86] reduces to $r_1 \neq r_2$.

Method. By symmetry it suffices to locate the solutions in the closed first quadrant, and none of them lies on the axes (Lemma 2.1). We split this quadrant into the half-strip $\Omega = \{\operatorname{Re} z \geq 60, \operatorname{Im} z \geq 0\}$, the half-strip $S = \{0 \leq \operatorname{Re} z \leq 60, \operatorname{Im} z \geq 5\}$ and the rectangle $B = \{0 \leq \operatorname{Re} z \leq 60, 0 \leq \operatorname{Im} z \leq 5\}$. On $\Omega \cup S$ we use Hankel’s expansion of J_0 with Olver’s explicit error bounds [Olv97, DLM26]. In S they show that there are no solutions. In Ω they show three things. Every solution lies within distance $0.576/(2\pi m)$ of the fixed point F_m of an explicit contraction, for some $m \geq 10$. Each disc $D(F_m, 1/(2\pi m))$ contains exactly one solution (Rouché’s theorem). The arguments of these discs decrease with m (a differential inequality for $\arg F_m$ as a function of m). On B we use a certified computation in exact integer ball arithmetic. The argument principle shows that B contains exactly nine nonzero solutions, Rouché’s theorem encloses them in discs of radius 2^{-150} , and exact comparisons order their arguments and compare the last one with the first disc of the tail. The computation takes about one second. Two independent verification runs reproduced it with their own code and interval arithmetic, the second one with a different counting method (Section 4.1).

Related work. Two-radius theorems have a large literature. Besides [Del58, DL59, Zal80, BG86], see for example [Zal72] for two-radius theorems of Pompeiu type, the survey [NV94], Volchkov’s refinements [Vol95, Vol96], and [PS10] for Damek–Ricci spaces. In the works we consulted (several only through reviews or summaries; see “Scope and priority” at the end), the hypothesis that r_1/r_2 is not a quotient of solutions is kept, and we found no statement that $H_2 = \{1\}$. Two recent papers, [VV23a] and [VV23b], have titles that refer to a problem of Zalcman. By its zbMATH summary, [VV23b] concerns a different question of Zalcman: the reconstruction of a distribution f on $\mathbb{R}^n$ from its spherical means $f * \sigma_{r_1}$ and $f * \sigma_{r_2}$ over two radii, a problem that is correctly posed only when r_1/r_2 is not a quotient of positive zeros of $J_{n/2-1}$. Of [VV23a] we could see only the bibliographic record in zbMATH (title, classification and reference list), which does not show which problem of

Zalcman is meant. Neither record indicates that Problem 2.45 is addressed: it is not mentioned in the zbMATH summary of [VV23b], and the reference list of [VV23a] recorded in zbMATH does not include the problem collections [CH74, HL19].

2 Preliminaries

Throughout, $g(z) = J_0(z) - 1$, $z = x + iy$ and $r = |z|$. Log is the principal logarithm and $\arg z \in (-\pi, \pi]$. $D(c, \rho)$ and $\overline{D}(c, \rho)$ are the open and the closed disc of radius ρ about c . We write $\overline{Q} = \{x \geq 0, y \geq 0\}$ for the closed first quadrant and $I_0(y) = \sum_{k \geq 0} (y/2)^{2k} / (k!)^2 = J_0(iy)$.

Lemma 2.1. (a) $g(-z) = g(z)$ and $g(\bar{z}) = \overline{g(z)}$.

(b) $g(x) < 0$ for real $x \neq 0$, and $g(iy) > 0$ for real $y \neq 0$.

(c) Every nonzero solution of (1) lies in an open quadrant. If two of them lie on the same open ray, then one of the maps $z \mapsto z, -z, \bar{z}, -\bar{z}$ sends them to two solutions in the open first quadrant with the same argument.

Proof. (a) The power series of g is even and has real coefficients. (b) By [DLM26, (10.9.1)], $J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) d\theta$. For $x \neq 0$ the continuous integrand is less than 1 except at finitely many θ , so $J_0(x) < 1$. Also $g(iy) = \sum_{k \geq 1} (y/2)^{2k} / (k!)^2 > 0$. (c) The first claim follows from (b). The four maps permute the open quadrants, map open rays to open rays, and preserve the set of solutions by (a). $\square$

Lemma 2.2. For $|z| \leq 1$ we have $g(z) = -\frac{z^2}{4} S(\frac{z^2}{4})$, where $S(w) = \sum_{j \geq 0} (-w)^j / ((j+1)!)^2$ satisfies $|S(w) - 1| < 0.0643$ for $|w| \leq 1/4$. Hence $z = 0$ is a double zero of g , and g has no other zero in $\overline{D}(0, 1)$. Along the arc $z = e^{i\theta}$, with θ decreasing from $\pi/2$ to 0, the continuous argument of g decreases by exactly π .

Proof. For $|w| \leq 1/4$ we have $|S(w) - 1| \leq \sum_{j \geq 1} 4^{-j} / ((j+1)!)^2 < 0.0643$ [Q], so $S(w) \neq 0$ and $|\arg S(w)| < 0.07$. On the arc, a continuous argument of $g(e^{i\theta})$ is $\pi + 2\theta + \arg S(e^{2i\theta}/4)$. At $\theta = \pi/2$ and at $\theta = 0$ the number $S(e^{2i\theta}/4) = S(\mp 1/4)$ is positive. So the change from $\theta = \pi/2$ to $\theta = 0$ is $\pi - 2\pi = -\pi$. $\square$

Lemma 2.3. For $n \in \{0, 1, 2\}$ and $z \in \mathbb{C}$ we have $|J_n(z)| \leq I_0(|\operatorname{Im} z|)$. Consequently $|g'(z)| \leq I_0(|\operatorname{Im} z|)$ and $|g''(z)| \leq I_0(|\operatorname{Im} z|)$.

Proof. By [DLM26, (10.9.2)], $J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta$, and $|\cos(a + ib)| \leq \cosh b$ for real a, b . Hence $|J_n(z)| \leq \frac{1}{\pi} \int_0^\pi \cosh(y \sin \theta) d\theta = I_0(|y|)$ by [DLM26, (10.32.1)]. Finally $g' = -J_1$ and $g'' = -J_1' = -(J_0 - J_2)/2$ [DLM26, (10.6.3), (10.6.1)]. $\square$

Lemma 2.4. Let $z \in \overline{Q} \setminus \{0\}$ and $r = |z|$. Then

$$J_0(z) = (2\pi z)^{-1/2} \left[e^{-i(z-\pi/4)} (1 + R^-(z)) + e^{i(z-\pi/4)} (1 + R^+(z)) \right] \quad (4)$$

with the principal branch of the square root, where

$$|R^+(z)| \leq \varepsilon_+(r) := \frac{1}{4r} e^{1/(4r)}, \quad |R^-(z)| \leq \varepsilon_-(r) := \frac{\pi}{8r} e^{\pi/(8r)}. \quad (5)$$

Proof. By [DLM26, (10.4.4)], $J_0 = \frac{1}{2}(H_0^{(1)} + H_0^{(2)})$. By [DLM26, (10.17.13)] with $\nu = 0$ and $\ell = 1$, $H_0^{(1)}(z) = (2/(\pi z))^{1/2} e^{i(z-\pi/4)} (1 + R^+(z))$ and $H_0^{(2)}(z) = (2/(\pi z))^{1/2} e^{-i(z-\pi/4)} (1 + R^-(z))$, where

$R^\pm = R_1^\pm(0, z)$. This gives (4). Since $a_1(0) = -1/8$ [DLM26, (10.17.1)] and $|\nu^2 - \frac{1}{4}| = \frac{1}{4}$, Olver's error bound [Olv97, pp. 266–267], [DLM26, (10.17.14)] reads

$$|R^\pm(z)| \leq \frac{1}{4} \mathcal{V}_\pm \exp\left(\frac{1}{4} \mathcal{V}_\pm\right), \quad (6)$$

where $\mathcal{V}_\pm = \int |t|^{-2} |dt|$ is the total variation of t^{-1} along a path from z to $\pm i\infty$. The admissible paths are those along which $\text{Im } t$ varies monotonically: nondecreasing for $\mathcal{V}_+$ and nonincreasing for $\mathcal{V}_-$. This is what keeps the kernel of the Volterra integral equation behind (6), which is $1 - e^{\mp 2i(z-t)}$ times a constant of modulus $\frac{1}{2}$, bounded by 1 in modulus. The text of [DLM26, §10.17(iv)] states the condition with $|\text{Im } t|$ in place of $\text{Im } t$. For $\mathcal{V}_-$ and $\text{Im } z > 0$ that version cannot be meant literally, because every path from z to $-i\infty$ crosses the real axis; the same holds for the bound for $\mathcal{V}_+$ that [DLM26, (10.17.15)] gives when $-\pi/2 \leq \arg z < 0$.

The bounds $\mathcal{V}_+ \leq |z|^{-1}$ and $\mathcal{V}_- \leq \chi(1)|z|^{-1} = \pi/(2|z|)$ on $\overline{Q} \setminus \{0\}$ are [DLM26, (10.17.15)]. For completeness we exhibit admissible paths. Write $z = re^{i\theta}$ with $0 \leq \theta \leq \pi/2$, and let $\rho > r$ and $X > 0$. For $\mathcal{V}_+$, go along the ray from z to $\rho e^{i\theta}$, then along the arc $|t| = \rho$ to $i\rho$, then up the imaginary axis. Along this path $\text{Im } t$ is nondecreasing, and the variation of t^{-1} is $(1/r - 1/\rho) + (\pi/2 - \theta)/\rho + 1/\rho$, which tends to $1/r$ as $\rho \rightarrow \infty$. For $\mathcal{V}_-$, go along the horizontal segment from z to $z + X$, then vertically down from $z + X$. Along this path $\text{Im } t$ is nonincreasing, and the variation is at most

$$\int_x^\infty \frac{du}{u^2 + y^2} + \int_{-\infty}^\infty \frac{ds}{(x + X)^2 + s^2} = \frac{\theta}{r \sin \theta} + \frac{\pi}{x + X},$$

where $\theta/\sin \theta$ is read as 1 if $\theta = 0$. As $X \rightarrow \infty$ this tends to $\theta/(r \sin \theta) \leq \pi/(2r)$, since $\theta/\sin \theta$ increases on $(0, \pi/2]$. Since (6) holds for every admissible path and its right side increases with $\mathcal{V}_\pm$, we obtain (5). $\square$

For $z \in \overline{Q} \setminus \{0\}$ put

$$\phi(z) = -i\left(z - \frac{\pi}{4}\right) - \frac{1}{2} \text{Log}(2\pi z), \quad u(z) = e^{\phi(z)}.$$

Then $u = e^{-i(z-\pi/4)}(2\pi z)^{-1/2}$ and $e^{i(z-\pi/4)}(2\pi z)^{-1/2} = 1/(2\pi z u)$, so (4) becomes

$$g(z) = (u - 1) + E, \quad E = u R^- + \frac{1 + R^+}{2\pi z u}, \quad |u| = \frac{e^y}{\sqrt{2\pi r}}. \quad (7)$$

Note that $\varepsilon_\pm(r)$ and $r\varepsilon_\pm(r)$ decrease in r .

3 The region $\text{Re } z \geq 60$

Let $\Omega = \{z : \text{Re } z \geq 60, \text{Im } z \geq 0\}$ and $S = \{z : 0 \leq \text{Re } z \leq 60, \text{Im } z \geq 5\}$. The numerical inequalities marked [T1]–[T7] below are verified in exact rational arithmetic by the program `tail_constants_2045.py`. It encloses π in $[3.14159265358979, 3.14159265358980]$ and bounds $\exp$ and $\ln$ by truncated series with remainder estimates. They were checked again in interval arithmetic in two independent verification runs (Section 4.1). Each of them concerns a quantity that is monotone in r or in m , for the reasons given in the text, and is checked at the extreme value.

Lemma 3.1. *Let $z \in \overline{Q}$.*

- (A) *If $r \geq 5$ and $|u(z)| \geq 2$, then $|g(z)| > 0.81$.*
- (B) *If $r \geq 60$ and $|u(z)| \leq 1/2$, then $|g(z)| > 0.44$.*

(C) g has no zeros in S .

Proof. (A) By (7) and (5), $|g| \geq |u|(1 - \varepsilon_-) - 1 - (1 + \varepsilon_+)/ (2\pi r|u|)$. The right side increases in $|u|$ and in r , so $|g| \geq 1 - 2\varepsilon_-(5) - (1 + \varepsilon_+(5))/ (20\pi) > 0.81$ [T1]. (B) Since $y \geq 0$, we have $|u| \geq (2\pi r)^{-1/2}$. Hence $|g| \geq 1 - (1 + \varepsilon_-)|u| - (1 + \varepsilon_+)/ (2\pi r|u|) \geq 1 - \frac{1}{2}(1 + \varepsilon_-(60)) - (1 + \varepsilon_+(60))/\sqrt{120\pi} > 0.44$ [T2]. (C) For $z \in S$ we have $5 \leq y \leq r \leq 60 + y$, so $|u|^2 \geq e^{2y}/(2\pi(60 + y))$. This increases in y , and at $y = 5$ it is at least 4, because $e^{10} > 520\pi$ [T1']. So $|u| \geq 2$ and $r \geq 5$, and (A) applies. $\square$

Lemma 3.2. For real $t \geq 10$ let $T_t(z) = 2\pi t + \pi/4 + \frac{i}{2} \text{Log}(2\pi z)$.

- (a) T_t maps Ω into Ω , and $|T_t(z) - T_t(z')| \leq |z - z'|/120$ for $z, z' \in \Omega$. Hence T_t has a unique fixed point $F(t) \in \Omega$.
- (b) $2\pi t \leq \text{Re } F(t) \leq 2\pi t + \pi/4$ and $\frac{1}{2} \ln(4\pi^2 t) \leq \text{Im } F(t) < \text{Re } F(t)$.
- (c) $t \mapsto F(t)$ is real-analytic, and $F' = 2\pi F/(F - i/2)$.
- (d) For $z, z' \in \Omega$ we have $|\phi(z) - \phi(z') + i(z - z')| \leq |z - z'|/120$. In particular $\frac{119}{120}|z - z'| \leq |\phi(z) - \phi(z')| \leq \frac{121}{120}|z - z'|$.
- (e) Let $m \geq 10$ be an integer and $z \in \Omega$. Then $\phi(z) = -2\pi im + w$ if and only if $z = T_m(z) + iw$. In particular $F_m := F(m)$ satisfies $\phi(F_m) = -2\pi im$ and $u(F_m) = 1$.

Proof. (a) For $z \in \Omega$ we have $\arg z \in [0, \pi/2]$, so $\text{Re } T_t(z) = 2\pi t + \pi/4 - \frac{1}{2} \arg z \in [2\pi t, 2\pi t + \pi/4]$, and $2\pi t \geq 20\pi > 60$ [T3']. Also $\text{Im } T_t(z) = \frac{1}{2} \ln(2\pi|z|) > 0$. The set Ω is convex and $|z| \geq 60$ on it, so $|\text{Log } z - \text{Log } z'| \leq |z - z'|/60$. (b) The bounds on $\text{Re } F$ come from (a). Next, $\text{Im } F = \frac{1}{2} \ln(2\pi|F|) \geq \frac{1}{2} \ln(4\pi^2 t)$. If $\text{Im } F \geq \text{Re } F$, then $2\text{Im } F = \ln(2\pi|F|) \leq \ln(2\sqrt{2}\pi \text{Im } F)$, which is impossible for $\text{Im } F \geq 20\pi$. (c) $\partial_z(z - T_t(z)) = 1 - i/(2z) \neq 0$ on Ω , so the implicit function theorem applies. Differentiating $F = T_t(F)$ gives $F' = 2\pi + \frac{i}{2} F'/F$. (d) $\phi(z) - \phi(z') + i(z - z') = -\frac{1}{2}(\text{Log } z - \text{Log } z')$. (e) $i\phi(z) = z - \pi/4 - \frac{i}{2} \text{Log}(2\pi z)$, so $\phi(z) = -2\pi im + w$ means $z = 2\pi m + \pi/4 + \frac{i}{2} \text{Log}(2\pi z) + iw$. $\square$

Lemma 3.3. If $z \in \Omega$ and $g(z) = 0$, then $|z - F_m| < 0.576/(2\pi m)$ for some integer $m \geq 10$.

Proof. By Lemma 3.1, $1/2 < |u| < 2$. By (7), $|u - 1| = |E| \leq |u|\varepsilon_- + (1 + \varepsilon_+)/ (2\pi r|u|) < \delta_1(r) := 2\varepsilon_-(r) + (1 + \varepsilon_+(r))/(\pi r)$. So $1 - \delta_1 < |u| < 1 + \delta_1$, and using (7) once more,

$$|u - 1| \leq \delta(r) := (1 + \delta_1)\varepsilon_- + \frac{1 + \varepsilon_+}{2\pi r(1 - \delta_1)}.$$

At $r = 60$ we have $\delta_1 < 0.0186$ and $\delta < 0.0095$ [T3], and δ_1, δ and $r\delta$ decrease in r . Write $u = 1 + \eta$ with $|\eta| \leq \delta(r) < 1$. Then $w := \text{Log}(1 + \eta)$ satisfies $|w| \leq -\ln(1 - \delta) \leq \lambda(r) := \delta/(1 - \delta)$. Here $r\lambda(r)$ decreases in r , $60\lambda(60) < 0.5709$ and $\frac{120}{119}\lambda(60) < 0.0097$ [T3]. Since $e^{\phi(z)} = e^w$, we have $\phi(z) = -2\pi im + w$ for some integer m . Taking imaginary parts, $2\pi m = x - \pi/4 + \frac{1}{2} \arg z + \text{Im } w \geq 60 - \pi/4 - \lambda(60) > 18\pi$ [T3'], so $m \geq 10$. By Lemma 3.2(e) and (a), $|z - F_m| \leq |T_m(z) - T_m(F_m)| + |w| \leq |z - F_m|/120 + |w|$. Hence

$$|z - F_m| \leq \frac{120}{119}\lambda(|z|) \leq \frac{120}{119} \cdot \frac{60\lambda(60)}{|z|}.$$

In particular $|z - F_m| \leq \frac{120}{119}\lambda(60) < 0.0097$, so $|z| \geq |F_m| - 0.0097 \geq 2\pi m - 0.0097$. Therefore $(2\pi m)|z - F_m| \leq \frac{120}{119} \cdot 60\lambda(60) \cdot 2\pi m / (2\pi m - 0.0097) < 0.576$ [T4]; the middle expression decreases in m . $\square$

Proposition 3.4. Let $m \geq 10$, $\rho_m = 1/(2\pi m)$ and $D_m = D(F_m, \rho_m)$. Then $\overline{D}_m \subset \Omega$, the discs $\overline{D}_m$ are pairwise disjoint, and g has exactly one zero z_m in D_m , which is simple. The zeros of g in Ω are exactly the points z_m , $m \geq 10$.

Proof. We have $\operatorname{Re} F_m - \rho_m \geq 20\pi - 1/(20\pi) > 60$ and $\operatorname{Im} F_m - \rho_m \geq \frac{1}{2} \ln(40\pi^2) - 1/(20\pi) > 0$ [T5'], [T5''], so $\overline{D}_m \subset \Omega$. The discs are disjoint because $\operatorname{Re} F_{m+1} - \operatorname{Re} F_m \geq 2\pi - \pi/4 > 2\rho_m$.

Let $z \in \partial D_m$ and $w = \phi(z) - \phi(F_m)$. By Lemma 3.2(d), $\sigma_- := \frac{119}{120}\rho_m \leq |w| \leq \sigma_+ := \frac{121}{120}\rho_m < 0.0161$. Since $u(F_m) = 1$, we have $u(z) = e^w$ and

$$|u(z) - 1| \geq |w| - \frac{1}{2}|w|^2 e^{|w|} \geq \sigma_- - \frac{1}{2}\sigma_-^2 e^{\sigma_-},$$

because $t \mapsto t - \frac{1}{2}t^2 e^t$ increases on $[0, 0.4]$ [T5''']. Also $|u| \leq e^{\sigma_+}$, $|u|^{-1} \leq e^{\sigma_+}$ and $|z| \geq r' := 2\pi m - \rho_m$, so by (7) $|E| \leq e^{\sigma_+} [\varepsilon_-(r') + (1 + \varepsilon_+(r'))/(2\pi r')]$. Multiplying by $2\pi m$, we get on ∂D_m

$$(2\pi m)|E| < 0.5641 < 0.9837 < (2\pi m)|u - 1|$$

[T5]. The upper bound for $|E|$ decreases in m and the lower bound for $|u - 1|$ increases, so it suffices to check $m = 10$. By Rouché's theorem, $g = (u - 1) + E$ and $u - 1$ have the same number of zeros in D_m . If $u(z) = 1$ with $z \in \overline{D}_m$, then $\phi(z) - \phi(F_m) \in 2\pi i\mathbb{Z}$ and $|\phi(z) - \phi(F_m)| \leq \sigma_+ < 2\pi$, so $\phi(z) = \phi(F_m)$, and $z = F_m$ by Lemma 3.2(d). Since $(u - 1)'(F_m) = \phi'(F_m) = -i - 1/(2F_m) \neq 0$, the function $u - 1$ has exactly one zero in D_m , and it is simple. Hence the same holds for g . Finally, by Lemma 3.3, every zero of g in Ω lies in some D_m , because $0.576 < 1$, and is therefore equal to z_m . $\square$

Proposition 3.5. *For $m \geq 10$, every point of $\overline{D}_m$ has a larger argument than every point of $\overline{D}_{m+1}$. In particular $\arg z_m > \arg z_{m+1}$.*

Proof. By Lemma 3.2(c),

$$\frac{d}{dt} \arg F(t) = \operatorname{Im} \frac{F'}{F} = \operatorname{Im} \frac{2\pi}{F - i/2} = -\frac{2\pi b}{a^2 + b^2}, \quad a = \operatorname{Re} F(t), \quad b = \operatorname{Im} F(t) - \frac{1}{2}.$$

Let $v(t) = \frac{1}{2}(\ln(4\pi^2 t) - 1)$, so that $v(10) > 0$. By Lemma 3.2(b), for $t \in [m, m+1]$ we have $0 < v(m) \leq v(t) \leq b < a \leq 2\pi(m+1) + \pi/4$. Since $b/(a^2 + b^2)$ increases in $b \in (0, a)$ and decreases in $a > 0$, integration over $[m, m+1]$ gives

$$\arg F_m - \arg F_{m+1} \geq G(m) := \frac{2\pi v(m)}{(2\pi(m+1) + \pi/4)^2 + v(m+1)^2}.$$

The function $m^2 G(m)$ increases for $m \geq 10$, because v increases while $(2\pi(m+1) + \pi/4)/m$ and $v(m+1)/m$ decrease (the latter since $\ln(44\pi^2) > 2$ [T6''']). Hence $m^2 G(m) \geq 100 G(10) > 0.3196$ [T6]. On the other hand, every point of $\overline{D}_m$ has argument within $\arcsin(\rho_m/|F_m|)$ of $\arg F_m$, and $\rho_m/|F_m| \leq 1/(4\pi^2 m^2) \leq 0.0003$, so this angle is at most $1.0001/(4\pi^2 m^2)$ [T6'], [T6'']. Therefore

$$\inf_{\overline{D}_m} \arg - \sup_{\overline{D}_{m+1}} \arg \geq G(m) - \frac{2 \cdot 1.0001}{4\pi^2 m^2} \geq \frac{0.3196 - 0.0507}{m^2} > 0. \quad \square$$

Lemma 3.6. $\operatorname{Im} F_{10} < 3.0193$. Consequently, with $\pi_- = 3.14159265358979 < \pi$ and $b^* = (20\pi_- - 0.016) + 3.0353i$, every $b \in \overline{D}_{10}$ satisfies $0 < \arg b \leq \arg b^* < 0.04829$.

Proof. Let $y = \operatorname{Im} F_{10}$. By Lemma 3.2(b), $|F_{10}| \leq \operatorname{Re} F_{10} + y \leq 20.25\pi + y$, so $y = \frac{1}{2} \ln(2\pi|F_{10}|) \leq \frac{1}{2} \ln(2\pi(20.25\pi + y))$. The function $y \mapsto \frac{1}{2} \ln(2\pi(20.25\pi + y)) - y$ is decreasing and negative at $y = 3.0193$ [T7], so $y < 3.0193$. For $b \in \overline{D}_{10}$ we have $\operatorname{Re} b \geq 20\pi - 1/(20\pi) > 20\pi_- - 0.016$ and $0 < \operatorname{Im} b \leq y + 1/(20\pi) < 3.0353$, so $\operatorname{Im} b / \operatorname{Re} b < 3.0353/(20\pi_- - 0.016)$ and hence $\arg b < \arg b^*$. Finally $\arg b^* = \arctan(3.0353/(20\pi_- - 0.016)) < 0.04829$. $\square$

4 The rectangle $[0, 60] \times [0, 5]$

Proposition 4.1 (computer-assisted). *The function g has exactly nine zeros in $B \setminus \{0\}$, where $B = \{x + iy : 0 \leq x \leq 60, 0 \leq y \leq 5\}$. They are simple and lie in the open first quadrant. Numbered by increasing real part, they lie in discs of radius 2^{-150} about centres whose rounded values are listed in Table 1, and*

$$\arg z_1 > \arg z_2 > \cdots > \arg z_9 > \sup\{\arg b : b \in \overline{D}_{10}\}.$$

The proof is a computation. It is carried out by two programs in the source files, `ball_2045.py` and `cert_bounded_2045.py`, and we describe what they verify.

Arithmetic. A real number is represented by a pair of integers (m, e) with $e \geq 0$, which stands for the interval $[(m - e)2^{-320}, (m + e)2^{-320}]$, and a complex number by a pair of such intervals. Additions are exact. Each multiplication, and each division by an integer, rounds the midpoint to the nearest integer and adds to e a rigorous bound for the rounding error and the propagated radii. J_0 and J_1 are evaluated from their power series [DLM26, (10.2.2)]. Once the ratio of consecutive terms is at most $1/2$ in modulus for all later terms, the tail is bounded by the modulus of the last term computed, which is added to the radius. An exact rational upper bound $M = 27.2398 \dots$ for $I_0(5)$ is computed from its series in the same way. No floating-point number enters any decision.

Counting. Let Γ be the boundary of $B \setminus \overline{D}(0, 1)$, traversed positively: the segment from 1 to 60, the right side from 60 to $60 + 5i$, the top side from $60 + 5i$ to $5i$, the segment from $5i$ to i , and the arc of $|z| = 1$ from i to 1. By Lemmas 2.1 and 2.2, g has no zeros on the two segments and the arc. Along the segments g is real with constant sign, so its argument is constant there, and along the arc it changes by $-\pi$. The program divides the right side and the top side into 31 and 256 segments $[a, b]$ with dyadic end points, each of which satisfies

$$|b - a| |g'(a)| + \frac{1}{2} |b - a|^2 M \leq \frac{1}{2} |g(a)|; \quad (8)$$

this is checked rigorously from the enclosures of $g(a)$ and $g'(a) = -J_1(a)$. By Taylor's formula and Lemma 2.3 ($|g''| \leq I_0(5) \leq M$ for $0 \leq \operatorname{Im} z \leq 5$), every value of g on $[a, b]$ lies in the closed disc of radius $\frac{1}{2} |g(a)|$ about $g(a)$. Hence g has no zero on $[a, b]$, and the continuous argument of g changes by at most $\arcsin \frac{1}{2} = \pi/6$ from a to b . Except at the end points 60 and $5i$, where g is real, the enclosure of each value $g(a)$ lies in an open quadrant. Since consecutive arguments differ by at most $\pi/6$, the quadrant of the continuous argument changes by 0 or ± 1 at each step, and it can be followed exactly. Starting from $\arg g(60) = \pi$ (as $g(60) < 0$), the program finds that the continuous argument reaches 20π at $5i$, where $g(5i) > 0$. Thus the argument of g changes by 19π along the right and top sides and by $0 + 19\pi + 0 - \pi = 18\pi$ along Γ . By the argument principle, g has exactly nine zeros inside Γ , counted with multiplicity. By Lemmas 2.1 and 2.2 and the above, these are all the zeros of g in $B \setminus \{0\}$.

Localisation. Newton's method, run in exact rational arithmetic from floating-point starting values, gives centres $\hat{z}_1, \dots, \hat{z}_9$; these are exact dyadic rationals, recorded in the output files. With $\rho = 2^{-150}$ the program verifies

$$|g(\hat{z}_m)| + \frac{1}{2} \rho^2 M < \rho |g'(\hat{z}_m)| \quad (1 \leq m \leq 9), \quad (9)$$

with $|g(\hat{z}_m)| \leq 2.2 \cdot 10^{-73}$ and $|g'(\hat{z}_m)| \geq 0.94$. On the circle $|z - \hat{z}_m| = \rho$, Taylor's formula and Lemma 2.3 give $|g(z) - g'(\hat{z}_m)(z - \hat{z}_m)| \leq |g(\hat{z}_m)| + \frac{1}{2} \rho^2 M < |g'(\hat{z}_m)(z - \hat{z}_m)|$. So, by Rouché's theorem, g has exactly one zero in $D(\hat{z}_m, \rho)$, and it is simple. The nine discs are disjoint and lie inside Γ , so they contain all nine zeros.

Arguments. For points a, b of the open first quadrant, $\arg a > \arg b$ if and only if $\operatorname{Im}(a\bar{b}) > 0$. If $a \in D(\hat{z}_j, \rho)$ and $b \in D(\hat{z}_{j+1}, \rho)$, then $\operatorname{Im}(a\bar{b}) \geq \operatorname{Im}(\hat{z}_j \overline{\hat{z}_{j+1}}) - \rho(|\hat{z}_j| + |\hat{z}_{j+1}|) - \rho^2$. The program finds $\operatorname{Im}(\hat{z}_j \overline{\hat{z}_{j+1}}) \geq 9.64$ for $1 \leq j \leq 8$, while the correction is below 10^{-43} . Finally, it finds $\operatorname{Im}(\hat{z}_9 \overline{b^*}) > 10.78$, with b^* as in Lemma 3.6; this exceeds the correction $\rho|b^*|$, so $\arg z_9 > \arg b^* \geq \sup_{\overline{D}_{10}} \arg$.

The computation takes about one second. Floating point is used only to produce the starting values for Newton's method, and the certification does not depend on them.

4.1 Independent reproductions

Two independent verification runs (see “Verification” at the end) reproduced Proposition 4.1 with their own programs. Both use the interval arithmetic of mpmath [mpm26] with outward rounding, at 420 and at 240 bits, and their own power series with tail bounds.

First run. Its programs do the following.

- They apply the argument principle to the whole rectangle $[-X, X] \times [-5, 5]$, without using Lemmas 2.1 and 2.2 or any symmetry. Each step $[a, b]$ is certified by $I_0(\eta) |b - a| \leq \frac{1}{2}|g(a)|$, where η is the largest value of $|\operatorname{Im} z|$ on $[a, b]$; this uses $|g'| = |J_1| \leq I_0(|\operatorname{Im} z|)$ (Lemma 2.3). The increments of the argument are enclosed in intervals and summed. The winding number is enclosed in $[38, 38]$ for $X = 60$ and in $[50, 50]$ for $X = 80$. Since 0 is a double zero and the zeros in the four open quadrants correspond to each other (Lemma 2.1), this gives $38 = 2 + 4 \cdot 9$ and $50 = 2 + 4 \cdot 12$. So g has nine zeros in $(0, 60) \times (0, 5)$ and twelve in $(0, 80) \times (0, 5)$.
- They certify Rouché discs of radius 2^{-100} about $z_1, \dots, z_{12}$, using $|g''| \leq I_0(|\operatorname{Im} z|)$. The certified gaps between consecutive arguments decrease from 0.0997 (between $\arg z_1$ and $\arg z_2$) to 0.0030 (between $\arg z_{11}$ and $\arg z_{12}$). They enclose F_{10}, F_{11}, F_{12} rigorously and find $(2\pi m)|z_m - F_m| \leq 0.1995, 0.1998, 0.2000$ for $m = 10, 11, 12$. They also find $\arg z_9 > 0.05127 > 0.04733 \geq \sup_{\overline{D}_{10}} \arg$.
- They re-derive the inequalities [T1]–[T7] and [Q] and check them with interval enclosures of π , $\exp$ and $\ln$. They also check the inequalities of Lemma 3.3 and Propositions 3.4 and 3.5 directly, without the monotonicity arguments, for 450 values of m between 10 and 10^{10} .

Second run. It counts the zeros in B without the argument principle, without Lemmas 2.1 and 2.2 and without symmetry, by covering the closed rectangle B with sets on which the number of zeros is known. It uses the bound $|g''(x + iy)| \leq I_0'(|y|)$, which follows by differentiating the integral of Lemma 2.1 twice.

- On the square $[0, 1]^2$ it uses $h(z) = (J_0(z) - 1)/z^2$, an entire function with $h(0) = -\frac{1}{4}$, and checks $|h(c)| > s \sup_{|z| \leq \sqrt{2}} |h'(z)|$, where c is the centre and s the half-diagonal of the square. So 0 is the only zero of g in $[0, 1]^2$.
- Nine discs of radius $\frac{1}{8}$ about approximations of $z_1, \dots, z_9$ each contain exactly one zero, which is simple (Rouché's theorem).
- The rest of B is covered by 969 squares Q with $|g(c)| - |g'(c)|s - \frac{1}{2}s^2 I_0''(\eta) > 0$, where c is the centre and s the half-diagonal of Q and η is the largest value of $\operatorname{Im} z$ on Q . By Taylor's formula g has no zero on such a square.

Hence g has exactly nine zeros in $B \setminus \{0\}$, all simple, and none on the boundary of B . The second run also certifies Rouché discs of radius 2^{-110} about $z_1, \dots, z_{12}$, which confirm every digit of

Table 1 and the order $\arg z_1 > \cdots > \arg z_{12}$. It encloses F_{10}, F_{11}, F_{12} and finds $(2\pi m)|z_m - F_m| \leq 0.19950, 0.19976, 0.19998$ for $m = 10, 11, 12$ and $\sup_{\overline{D}_{10}} \arg \leq 0.047323 < 0.051271 \leq \arg z_9$. It re-checks [T1]–[T7] and [Q] in interval arithmetic, checks the inequalities of Lemma 3.3 and Propositions 3.4 and 3.5 directly for 3038 values of m between 10 and 10^{15} , and checks $\arg F_m - \arg F_{m+1} \geq G(m)$ for $10 \leq m \leq 59$ with rigorous enclosures of F_m .

5 Proofs of the main results

Proof of Theorem 1.1. Since $g(z) = -z^2/4 + O(z^4)$, the point 0 is a double zero of g . The closed first quadrant is $B \cup S \cup \Omega$. By Proposition 4.1, the zeros of g in $B \setminus \{0\}$ are $z_1, \dots, z_9$. By Lemma 3.1(C) there are none in S , and by Proposition 3.4 the zeros in Ω are the z_m with $m \geq 10$. All of them are simple, and by Lemma 2.1(b) they lie in the open quadrant. We have $\operatorname{Re} z_9 < 60$, and for $m \geq 10$ the real part of $z_m \in D_m$ lies between $2\pi m - \rho_m$ and $2\pi m + \pi/4 + \rho_m$. So the numbering is by increasing real part. The inequalities (2) follow from Proposition 4.1 for $m \leq 9$, from Proposition 4.1 and $z_{10} \in \overline{D}_{10}$ for $\arg z_9 > \arg z_{10}$, and from Proposition 3.5 for $m \geq 10$. By Lemma 2.1(a) the solutions are 0 and $\pm z_m, \pm \overline{z_m}$, all simple. If two nonzero solutions lay on the same open ray, then by Lemma 2.1(c) two of the z_m would have the same argument, contradicting (2). The direction statement follows, since $\overline{z_m}, -z_m$ and $-\overline{z_m}$ have arguments $-\arg z_m, \arg z_m - \pi$ and $\pi - \arg z_m$. The bound on $|z_m - F_m|$ is Lemma 3.3. Finally, a solution F of $F = 2\pi m + \pi/4 + \frac{i}{2} \operatorname{Log}(2\pi F)$ with $\operatorname{Re} F \geq 60$ satisfies $\operatorname{Im} F = \frac{1}{2} \ln(2\pi|F|) > 0$, so it lies in Ω , and there it is unique by Lemma 3.2(a). $\square$

Proof of Corollary 1.3. Let α, β be nonzero solutions of (1) with $\alpha/\beta = t > 0$. Then $\alpha = t\beta$ lies on the open ray through β , so $\alpha = \beta$ by Theorem 1.1, and $t = 1$. Conversely $1 = z_1/z_1 \in H_2$. $\square$

Proof of Corollary 1.4. If $r_1 \neq r_2$, then $r_1/r_2 \neq 1$, so $r_1/r_2 \notin H_2$ by Corollary 1.3, and Theorem 1.2 applies. If $r_1 = r_2 = r$, let $\lambda = z_1/r$. By (3), $M_r e_\lambda = J_0(z_1)e_\lambda = e_\lambda$, while $\Delta e_\lambda = -\lambda^2 e_\lambda \neq 0$. $\square$

6 Remarks

Remark 6.1. The bound $|z_m - F_m| < 0.576/(2\pi m)$ is far from sharp. The next term of Hankel's expansion suggests that $(z_m - F_m)F_m \rightarrow \frac{1}{8} - \frac{i}{2\pi}$ as $m \rightarrow \infty$, so that $(2\pi m)|z_m - F_m| \rightarrow |\frac{1}{8} - \frac{i}{2\pi}| \approx 0.2024$. Floating-point computations, which are not part of the proof, agree. For $10 \leq m \leq 20000$ the computed values of $(2\pi m)|z_m - F_m|$ are at most 0.2024. The computed arguments decrease, with consecutive differences at least $2.5 \cdot 10^{-9}$ (near $m = 20000$); this is consistent with the lower bound $0.3196/m^2$ of Proposition 3.5.

Remark 6.2. For functions on $\mathbb{R}^n$ the circle means are replaced by sphere means, and J_0 by $\Lambda_n(z) = \Gamma(n/2) (2/z)^{n/2-1} J_{n/2-1}(z)$; thus $\Lambda_2 = J_0$ and $\Lambda_3(z) = (\sin z)/z$. The corresponding exceptional set H_n consists of the positive quotients of nonzero solutions of $\Lambda_n(z) = 1$ (see [BG86] and the discussion after Theorem 1.2). The method of this note should apply to each fixed n , but we have not carried this out, and nothing is claimed here for $n \neq 2$.

Verification. The claims were checked as follows.

1. The certified computation of Section 4 (`ball_2045.py` and `cert_bounded_2045.py`; exact integer and rational arithmetic, about one second), and the check of the inequalities [T1]–[T7] and [Q] in exact rational arithmetic (`tail_constants_2045.py`). Both were re-run from the copies in the source files and reproduced the recorded outputs.

2. Floating-point cross-checks with SciPy (`sanity_scipy_2045.py`), which are not used in the proof:
 - the zeros for $m \leq 20000$ (Remark 6.1);
 - a floating-point argument principle on $B \setminus \overline{D}(0, 1)$, which gives 9.000000;
 - the ratios $|R^\pm(z)|/\varepsilon_\pm(|z|)$ at 4000 random points of $\overline{Q}$ with $|z| \geq 5$, which are at most 0.50 and 0.34;
 - on the circles ∂D_m , $10 \leq m \leq 14$, the values $(2\pi m) \min |u-1| \approx 0.99$ and $(2\pi m) \max |E| \approx 0.20$, against the bounds 0.9837 and 0.5641 used in Proposition 3.4.
3. A first independent verification run, which was AI-assisted, re-derived the arguments of Lemma 2.4 and Section 3, re-ran the programs of item 1, and wrote its own interval-arithmetic programs, described in Section 4.1. It also made non-rigorous checks with `mpmath` at 34–40 digits:
 - the ratios $|R^\pm|/\varepsilon_\pm$ on a grid that includes the edges of the quadrant, below 0.50 and 0.34;
 - the zeros z_m for $10 \leq m \leq 3000$, computed from a 30-term Hankel expansion, with decreasing arguments;
 - the formula of Lemma 3.2(c), compared with finite differences.

It found no mathematical error. It asked for corrections to the presentation, which have been made: the statement of the path condition in Lemma 2.4, the precise form of the two-circle theorem used (Theorem 1.2) and the remark on necessity, the publication of the programs, and the discussion of [VV23a, VV23b].

4. A second independent verification run, also AI-assisted, read the problem again in the source of [HL19] and §10.17 of [DLM26], re-derived the proofs, re-ran all programs from the source files (the recorded outputs were reproduced), and wrote its own interval-arithmetic programs, described in Section 4.1. It also made non-rigorous checks with `mpmath` at 40 digits: the ratios $|R^\pm|/\varepsilon_\pm$ on a polar grid with $5 \leq |z| \leq 400$ that includes both edges of the quadrant (at most 0.4997 and 0.3383); the variation of t^{-1} along the paths of Lemma 2.4, by quadrature; and the statements of Remark 6.1 at sample values of m up to 20001. It found no mathematical error. It asked for corrections to the presentation and to the sourcing, which have been made: the wording of Update 2.45, the source and the form of Theorem 1.2, the description of [VV23a], the way Olver’s bound was used, and one clause in the proof of Theorem 1.1.

All programs and their outputs are included in the source files.

Scope and priority. Theorem 1.1 answers Problem 2.45 of [HL19] affirmatively for open rays, which is the reading intended there; for closed rays the statement fails only because 0 is a solution. The proof is computer-assisted on the rectangle $[0, 60] \times [0, 5]$. All other steps are analytic, and their numerical constants are checked in exact arithmetic. The only external input beyond standard facts about Bessel functions is Olver’s error bound for Hankel’s expansion, which we use in the form stated in [DLM26, §10.17(iv)]; the page reference to [Olv97] is taken from there, and we have not seen [Olv97] itself. Corollary 1.4 combines Corollary 1.3 with Delsarte’s theorem. We did not see [Del58] or [DL59]. Our formulation of Theorem 1.2 and our description of [BG86] follow the zbMATH summary of [BG86], as explained after Theorem 1.2; we know [BG86, Vol95, Vol96] only from their zbMATH reviews or summaries. We read the problem in the arXiv version of [HL19]. We did not see the problem list [CH74], Zalcman’s survey [Zal80], the papers [Fla65, Zal72] or the survey [NV94],

and we know [VV23a] only from its bibliographic record. We searched arXiv, Crossref, OpenAlex, zbMATH, Semantic Scholar and the web for work on the solutions of $J_0(z) = 1$ and on the exceptional set of the two-circle theorem. This included the works citing [DL59] that are listed by OpenAlex. We found no answer to the problem, and the arXiv version of [HL19] (2018) reports none. The method, asymptotic expansions with explicit error bounds combined with a finite certified computation, is standard, and the result may well be within reach of experts. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments and the certified computation, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [BG86] Carlos A. Berenstein and Roger Gay. A local version of the two-circles theorem. *Israel Journal of Mathematics*, 55(3):267–288, 1986. doi:10.1007/BF02765026.
- [CH74] J. Clunie and W. K. Hayman, editors. *Proceedings of the Symposium on Complex Analysis, Canterbury 1973*, volume 12 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1974. Problem list “New problems”, pp. 155–180, doi:10.1017/CBO9780511662263.034. doi:10.1017/CBO9780511662263.
- [Del58] Jean Delsarte. Note sur une propriété nouvelle des fonctions harmoniques. *Comptes Rendus de l’Académie des Sciences, Paris*, 246:1358–1360, 1958. Zbl 0084.09403.
- [DL59] J. Delsarte and J. L. Lions. Moyennes généralisées. *Commentarii Mathematici Helvetici*, 33:59–69, 1959. doi:10.1007/BF02565907.
- [DLM26] DLMF. NIST Digital Library of Mathematical Functions. <https://dlmf.nist.gov/>, 2026. Release 1.2.8 of 2026-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [Fla65] Leopold Flatto. The converse of Gauss’s theorem for harmonic functions. *Journal of Differential Equations*, 1(4):483–490, 1965. doi:10.1016/0022-0396(65)90006-9.
- [HL19] Walter K. Hayman and Eleanor F. Lingham. *Research Problems in Function Theory: Fiftieth Anniversary Edition*. Problem Books in Mathematics. Springer, Cham, 2019. Problem 2.45 (L. Zalcman) and Update 2.45; arXiv version titled *Research Problems in Function Theory (New Edition)*. arXiv:1809.07200, doi:10.1007/978-3-030-25165-9.
- [mpm26] mpmath development team. mpmath: a Python library for arbitrary-precision floating-point arithmetic. <https://mpmath.org/>, 2026. Version 1.4.1.
- [NV94] Ivan Netuka and Jiří Veselý. Mean value property and harmonic functions. In *Classical and Modern Potential Theory and Applications*, pages 359–398. Springer Netherlands, Dordrecht, 1994. doi:10.1007/978-94-011-1138-6_29.
- [Olv97] F. W. J. Olver. *Asymptotics and Special Functions*. A K Peters, Wellesley, MA, 1997. Reprint of the 1974 Academic Press edition. doi:10.1201/9781439864548.

- [PS10] Norbert Peyerimhoff and Evangelia Samiou. Spherical spectral synthesis and two-radius theorems on Damek–Ricci spaces. *Arkiv för Matematik*, 48(1):131–147, 2010. [arXiv:0803.1657](#), doi:10.1007/s11512-009-0105-5.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-022-2045.
- [Vol95] V. V. Volchkov. New two-radii theorems in the theory of harmonic functions. *Russian Academy of Sciences. Izvestiya Mathematics*, 44(1):181–192, 1995. doi:10.1070/IM1995v044n01ABEH001588.
- [Vol96] V. V. Volchkov. The final version of the mean value theorem for harmonic functions. *Mathematical Notes*, 59(3):247–252, 1996. doi:10.1007/BF02308535.
- [VV23a] Valery Volchkov and Vitaly Volchkov. Zalcman’s problem and related two-radii theorems. *Analysis and Mathematical Physics*, 13(5):Paper No. 72, 2023. doi:10.1007/s13324-023-00835-5.
- [VV23b] Natalia P. Volchkova and Vitaliy V. Volchkov. On one Zalcman problem for the mean value operator. *Ural Mathematical Journal*, 9(1):187–200, 2023. doi:10.15826/umj.2023.1.017.
- [Zal72] Lawrence Zalcman. Analyticity and the Pompeiu problem. *Archive for Rational Mechanics and Analysis*, 47(3):237–254, 1972. doi:10.1007/BF00250628.
- [Zal80] Lawrence Zalcman. Offbeat integral geometry. *The American Mathematical Monthly*, 87(3):161–175, 1980. doi:10.1080/00029890.1980.11994985.