

Iterated-Logarithmic Size of Gross Exceptional Sets

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Abstract

For each $q > 1$, we construct a transcendental entire function and one regular inverse germ whose exceptional directions in Gross's star theorem contain a compact set of positive finite Hausdorff measure for the gauge $h_q(t) = t[\log \log(e^e/t)]^q$. Along every direction of this compact set the same inverse branch escapes to infinity at unit distance from its center. This exceptional set has Hausdorff dimension one and has infinite measure for every gauge $t[\log(e/t)]^\varepsilon$, $\varepsilon > 0$, while Gross's theorem still gives length zero. The construction refines Nadi's positive-capacity example. The additional estimates use slowly varying corridor widths, a nonatomic Bernoulli measure summed over ancestral shells, and an exponential holomorphic weight for entire approximation. We make no priority claim for dimension one, already attributed to Nadi on Eremenko's problem page.

1 Statement and earlier work

For a meromorphic f on $\mathbb{C}$ and a regular inverse germ ϕ at a finite value w_0 , let $E(f, \phi) \subset \mathbb{T}$ consist of the directions in which ϕ fails to continue holomorphically along the whole ray $w_0 + t\xi$, $t \geq 0$. Gross's star theorem gives length zero. Stronger size restrictions are asked for in Hayman–Lingham, Problem 1.32 [2], and Eremenko's note [3].

Nadi [1] gives the positive-capacity counterexample. For each $0 < s < 1$, his Theorem 1.1 provides a function and germ with a compact exceptional subset of positive $\mathcal{H}^s$ measure. We use his construction architecture: thin starlike domains, conformal escape, and two applications of Arakelian approximation. The geometric inputs from his paper are identified below. Eremenko's current problem page [4] already attributes a dimension-one example to Nadi. Our contribution is the more precise iterated-logarithmic gauge statement, not priority for dimension one or for the original capacity counterexample.

Gauges are specified near zero and extended monotonically elsewhere. Hausdorff measures use diameters in their covering definition. All logarithms are natural.

Theorem 1. *For every $q > 1$, put $h_q(t) = t[\log \log(e^e/t)]^q$ for sufficiently small $t > 0$. There exist a transcendental entire function f , a simple zero z_0 , and a compact $K \subset (-1/8, 1/8)$ such that, writing $\widehat{K} = \{e^{i\theta} : \theta \in K\}$,*

$$0 < \mathcal{H}^{h_q}(\widehat{K}) < \infty, \quad |\widehat{K}| = 0.$$

The inverse germ ϕ at zero with $\phi(0) = z_0$ continues along $[0, e^{i\theta})$ for every $\theta \in K$, and

$$|\phi(te^{i\theta})| \longrightarrow \infty \quad (t \uparrow 1).$$

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Thus $\widehat{K} \subset E(f, \phi)$, $\dim_H E(f, \phi) = 1$, and

$$\mathcal{H}^{g_\varepsilon}(\widehat{K}) = \infty \quad \text{for every } \varepsilon > 0, \quad g_\varepsilon(t) = t[\log(e/t)]^\varepsilon.$$

No equality $E(f, \phi) = \widehat{K}$, endpoint $q = 1$ result, or classification of all exceptional sets is asserted. Constants below may depend on the fixed q .

2 Angular and radial scales

Let $\gamma = (q - 1)/2 > 0$, $\ell_0 = 1/8$, and

$$\ell_n = \ell_0 2^{-n} \left(\frac{\log 3}{\log(n+3)} \right)^q, \quad r_n = 1 - \frac{1}{2} \left(\frac{\log 3}{\log(n+3)} \right)^\gamma, \quad h_n = r_{n+1} - r_n. \quad (1)$$

Here $r_0 = 1/2$, $r_n \uparrow 1$, and the mean value theorem gives

$$h_n \asymp \frac{1}{(n+3)[\log(n+3)]^{\gamma+1}}, \quad \frac{h_n}{\ell_n} \asymp \frac{2^n [\log(n+3)]^\gamma}{n+3}. \quad (2)$$

The ratios $a_n = \ell_{n+1}/\ell_n$ satisfy $0 < \inf_n a_n \leq a_n < 1/2$.

Start with a centered interval $J_\emptyset$ of length ℓ_0 . Each interval J_v of length ℓ_n and center c_v has two children of length ℓ_{n+1} , centered at $c_v \pm \ell_n/4$. Let $|v| = n$ denote its level. Set

$$\tau_0 = 5/8, \quad \tau_n = \frac{1}{2} + (1 - 2a_{n-1})/8 \quad (n \geq 1), \quad I_v = (c_v - \tau_n \ell_n, c_v + \tau_n \ell_n).$$

For $0 < a < 1/2$,

$$\frac{1}{4} - a \left(\frac{1}{2} + \frac{1-2a}{8} \right) = \frac{(1-2a)(2-a)}{8} > 0. \quad (3)$$

Hence each closed child I lies strictly inside its parent J , and the enlarged intervals at each level have disjoint closures. Put

$$U_n = \bigcup_{|v|=n} I_v, \quad K = \bigcap_{n \geq 0} \bigcup_{|v|=n} J_v.$$

The Bernoulli probability ν on K assigns mass 2^{-n} to every level- n interval.

Lemma 2. *The sets K and $\widehat{K}$ have length zero and positive finite h_q -measure.*

Proof. The total length $2^n \ell_n$ tends to zero, and

$$\lim_{n \rightarrow \infty} 2^n h_q(\ell_n) = \ell_0 (\log 3)^q > 0.$$

The level covers give finite h_q -measure. If $\ell_n \leq t < \ell_{n-1}$, an interval of length t meets at most $t/\ell_n + 2$ disjoint level- n intervals. This is bounded uniformly because $\inf a_n > 0$. Thus $\nu(L) \leq C 2^{-n} \leq C' h_q(t)$. Every real set lies in an interval of its diameter, so summing over covers proves positive h_q -measure. Exponentiation is bi-Lipschitz on $J_\emptyset$, and $h_q(ct)/h_q(t) \rightarrow c$ for fixed $c > 0$, proving the circle assertion. $\square$

Consider the bounded starlike domain

$$D = \Delta(0, 1/2) \cup \bigcup_{n \geq 0} \{r e^{i\theta} : 0 < r < r_{n+1}, \theta \in U_n\}.$$

At $r_n \leq r < r_{n+1}$ its angular section is exactly U_n . For $|v| = n$, put

$$x_n = r_n + h_n/4, \quad y_n = r_n + 3h_n/4, \quad \Gamma_v = \{y_n e^{i\theta} : \theta \in I_v\}.$$

This crosscut separates an outward component D_v^+ from zero.

3 Boundary shadows and the full shell measure

We use Nadi's crosscut estimate [1, Lemma 2.2]. For a disk crosscut separating a component V from a fixed central disk, its extremal distance λ from that disk satisfies

$$\text{diam } V \leq C e^{-c\lambda}. \quad (4)$$

It follows from the planar continuum-modulus bound: a compact arc in V of diameter comparable to $\text{diam } V$ yields $\lambda \leq C \log(C/\text{diam } V)$.

Let $\chi : \mathbb{D} \rightarrow D$ be conformal, $\chi(0) = 0$. A small fixed central disk maps inside $\Delta(0, 1/4)$. Every curve from it to Γ_v crosses the annular sector $x_n < |z| < y_n$, $\arg z \in I_v$. The logarithmic rectangle formula gives

$$\lambda \geq \frac{\log(y_n/x_n)}{2\tau_n \ell_n} \geq c h_n / \ell_n.$$

The crosscut endpoints are in straight local pieces of ∂D . Reflection extends χ^{-1} there, so its preimage is a disk crosscut. For its outward component V_v , (2) and (4) give

$$\text{diam } \overline{V_v} \leq \delta_n, \quad \delta_n = C \exp\left(-c \frac{2^n [\log(n+3)]^\gamma}{n+3}\right). \quad (5)$$

After a fixed starting level, δ_n decreases and is less than one.

For $\theta \in K$, let $v_n(\theta)$ be its level- n word. The nested $\overline{V_{v_n(\theta)}}$ shrink to one point $\beta(\theta) \in \mathbb{T}$. An interior limit would have image modulus at least y_n for every n , contrary to $D \subset \mathbb{D}$. Moreover

$$\text{diam } D_v^+ \leq 1 - y_n + 2\tau_n \ell_n \rightarrow 0.$$

These crosscuts represent a prime end with impression $\{e^{i\theta}\}$. Distinct angles give distinct prime ends, hence distinct $\beta(\theta)$. The shadow estimate makes β continuous. The compact sets

$$B = \beta(K), \quad B_v = \beta(K \cap J_v)$$

therefore have disjoint level partitions, with $V_v \cup B_v \subset \overline{V_v}$ of diameter at most δ_n .

Use the nonatomic measure $\mu = \beta_* \nu$, so $\mu(B_v) = 2^{-n}$, and define

$$A(\zeta) = - \int_B \text{Log}(1 - \bar{\xi}\zeta) d\mu(\xi) \quad (\zeta \in \mathbb{D}), \quad (6)$$

using the branch at 1. This is holomorphic by local uniform convergence. Since $1 - \bar{\xi}\zeta$ is in the right half-plane,

$$\text{Re } A \geq -\log 2, \quad |\text{Im } A| < \pi/2. \quad (7)$$

Lemma 3. *For some $c, C > 0$ and all large n ,*

$$\text{Re } A(\zeta) \geq c[\log(n+3)]^{\gamma+1} - C$$

whenever $\zeta \in V_v$, $|v| = n$. The same form of bound holds when $\text{dist}(\zeta, B) < \delta_n$.

Proof. For ancestors v_j of v , the disjoint shells $B_{v_j} \setminus B_{v_{j+1}}$ have mass 2^{-j-1} , and every point in the j -th shell is within δ_j of $\zeta \in V_v$. All unused contributions are bounded below by $-\log 2$. For a

fixed sufficiently large j_0 ,

$$\begin{aligned} \operatorname{Re} A(\zeta) &\geq \sum_{j=j_0}^{n-1} 2^{-j-1} \log(1/\delta_j) - C \\ &\geq c \sum_{j=j_0}^{n-1} \frac{[\log(j+3)]^\gamma}{j+3} - C \\ &\geq c' [\log(n+3)]^{\gamma+1} - C'. \end{aligned}$$

The last step compares with the integral of $(\log x)^\gamma/x$. For a point within δ_n of B , choose a nearby $\beta \in B$ and its ancestral shells. Their distances are at most $\delta_n + \delta_j \leq 2\delta_j$. The extra logarithmic constant is absorbed in C . $\square$

Thus $\operatorname{Re} A \rightarrow \infty$ uniformly as $\operatorname{dist}(\zeta, B) \rightarrow 0$ inside the disk. Set $Q(\zeta) = \zeta e^{2A(\zeta)}$. Because μ is a probability,

$$\frac{\zeta Q'(\zeta)}{Q(\zeta)} = \int_B \frac{1 + \bar{\xi}\zeta}{1 - \bar{\xi}\zeta} d\mu(\xi)$$

has positive real part. The analytic starlikeness criterion makes Q conformal onto a starlike domain Ω . It is proper, since otherwise its inverse would be bounded and entire. Furthermore $|Q(\zeta)| \rightarrow \infty$ uniformly near B . Write $\Phi = Q \circ \chi^{-1} : D \rightarrow \Omega$ and $g = \Phi^{-1}$.

4 Amplifying the approximation weight

Let $d_D(z) = \operatorname{dist}(z, \partial D)$, $\epsilon = 1/8$, and

$$\rho_\theta(t) = \epsilon d_D(te^{i\theta}), \quad T_{\theta,t} = \overline{\Delta(te^{i\theta}, \rho_\theta(t))} \quad (\theta \in K, 0 \leq t < 1).$$

These disks are compactly contained in D , and all coincide at $t = 0$.

Lemma 4. *For $r_n \leq t < r_{n+1}$, uniformly in $\theta \in K$ and all large n ,*

$$\frac{c\ell_{n+1}}{(n+4)\log(n+4)} \leq d_D(te^{i\theta}) \leq C\ell_n.$$

Hence $\log(2/\rho_\theta(t)) \leq C_q(n+1)$. Also $\chi^{-1}(T_{\theta,t}) \subset V_{v_{n-1}(\theta)}$ after increasing the starting level.

Proof. The mean value theorem gives $\tau_n - 1/2 \asymp 1/[(n+3)\log(n+3)]$. Thus θ has at least the asserted lower-scale angular margin in its level- $(n+1)$ enlarged interval. A small fixed multiple of that radius changes argument by less than half the margin, since $t \geq 1/2$. It crosses at most one radial level: adjacent radial gaps are polynomially small by (2), whereas this radius is exponentially small. Nesting then puts the entire disk in D . This proves the lower bound. An angular endpoint of $I_{v_n(\theta)}$ at radius t is a boundary point within $C\ell_n$, giving the upper bound.

Taking logarithms in (1) proves the precision estimate. Finally $r_n - y_{n-1} = h_{n-1}/4$ and $\ell_n/h_{n-1} \rightarrow 0$. The disk therefore misses the previous crosscut and lies in its outward component for all large n . $\square$

On these disks, Lemmas 3 and 4 imply

$$\operatorname{Re} A(\chi^{-1}(\xi)) \geq c[\log(n+2)]^{\gamma+1} - C.$$

They also imply $\chi^{-1}(T_{\theta,t})$ approaches B uniformly. Consequently

$$\inf_{\theta \in K} \inf_{\xi \in T_{\theta,t}} |\Phi(\xi)| \longrightarrow \infty \quad (t \uparrow 1). \quad (8)$$

The preceding lower bound is only polylogarithmic in n . To dominate the $O(n)$ precision requirement, define on Ω

$$P_0(w) = \exp(A(Q^{-1}(w))/2).$$

The strip bound (7) yields

$$\operatorname{Re} P_0(w) \geq 2^{-1/2} \exp(\operatorname{Re} A(Q^{-1}(w))/2).$$

Since $\gamma + 1 > 1$, the resulting lower bound $c' \exp(c''[\log n]^{\gamma+1})$ dominates $C_q(n+1)$. For $t \leq t_0 < 1$, the disks form a compact subset of D and their radii have a positive minimum. Thus one real constant C_* makes $P = P_0 + C_*$ satisfy

$$e^{-\operatorname{Re} P(w)} \leq \frac{1}{2} \rho_\theta(t) \quad \text{on } \Phi(T_{\theta,t}) \quad (9)$$

simultaneously for every $\theta \in K$, $0 \leq t < 1$.

5 Entire approximation and one inverse germ

We use Nadi's starlike-envelope lemma [1, Lemma 3.2]: a set S closed in $\mathbb{C}$ and contained in a proper starlike domain Ω is contained in a closed Arakelian set $F \subset \Omega$. Arakelian's criterion [5] requires the spherical complement of F to be connected and locally connected at infinity; it then allows uniform entire approximation of continuous functions on F holomorphic in its interior. In the cited lemma, tapered open tubes about the complementary radial rays avoid S ; every point in a tube escapes with nondecreasing modulus. The union of these tubes, together with infinity, is connected, as is its intersection with each exterior neighborhood of infinity. Its closed complement gives the required set F .

The set $S = \bigcup_{\theta,t} \Phi(T_{\theta,t})$ is closed in $\mathbb{C}$. A bounded convergent sequence in it cannot have $t \uparrow 1$, by (8). Otherwise compactness of K and continuity of the centers and radii give a limit in one of the same disks. Choose $S \subset F \subset \Omega$ as above.

Approximate $-P$ on F by an entire H with $|H + P| < \log 2$, and ge^{-H} by an entire G with $|G - ge^{-H}| < 1/8$. Then $f = e^H G$ is entire and

$$|f - g| < \frac{1}{4} e^{-\operatorname{Re} P} \leq \frac{1}{8} \rho_\theta(t) \quad \text{on } \Phi(T_{\theta,t}). \quad (10)$$

The image of the interior of $T_{\theta,t}$ is a bounded Jordan domain. On its boundary, $|g(w) - te^{i\theta}| = \rho_\theta(t)$. Rouché's theorem gives exactly one zero, counted with multiplicity, of $f(w) - te^{i\theta}$ in that domain. Denote it by $z_\theta(t)$. It is simple. At $t = 0$ the domain is independent of θ , so all these zeros equal one simple zero z_0 .

For fixed θ , the zero varies continuously with $t < 1$. Indeed, near $t_0 < 1$ all disks lie in one compact subset of D ; subsequential limits of zeros are zeros in the limiting closed domain. The strict boundary inequality excludes its boundary, and uniqueness identifies the limit with $z_\theta(t_0)$. The inverse function theorem therefore patches these local zeros into the continuation of the same germ at z_0 . By (8), $|z_\theta(t)| \rightarrow \infty$, preventing continuation through $e^{i\theta}$. Also f is not a polynomial, since it stays bounded along this escaping curve.

Lemma 2 proves the gauge statement. The limits $h_q/g_\varepsilon \rightarrow 0$ and $h_q(t)/t^s \rightarrow 0$ for $s < 1$ give the remaining measure and dimension conclusions. Gross's theorem still gives length zero for all of $E(f, \phi)$. This proves Theorem 1.

6 Scope and reproducibility

The variable scales, full ancestral shell masses, and sector-preserving exponential weight are used together. A single-level estimate loses the accumulated potential. A linear weight does not establish (9) at these scales. At $q = 1$, our formulas give $\gamma = 0$ and $r_n = 1/2$ for every n , so no endpoint assertion follows.

A bounded primary-source search through 28 October 2026 did not locate the specific h_q statement; it does not certify absolute priority. The earlier capacity result and dimension-one attribution are credited above. The original open-ended question is not treated as a new complete classification.

The accompanying program performs 16,878 checks of scale identities, exact dyadic shell masses, finite interval geometry and false parameter or weight choices. Floating-point checks use 100 decimal digits and are regression tests, not interval certificates. The universal argument is the written proof with its attributed analytic inputs. This manuscript was developed with AI assistance and self-audited. It is unrefereed and has not been independently reviewed or formally verified.

References

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