

Vanishing Density of Realizable Rolle Words

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Abstract

Let b_n count the total orders of the zeros of a real-rooted degree- n polynomial and its derivatives that satisfy the interlacing restrictions of Rolle's theorem, and let a_n count those actually realized by polynomials, with all derivative zeros mutually distinct. We prove $a_n \leq (D_n + 1)(2D_n + 1)^n$, where $D_n = n(n^2 - 1)/2$. Together with the classical formula for b_n , this gives

$$\log(a_n/b_n) = -\frac{1}{2}n^2 \log n + \left(\frac{3}{2} \log 2 - \frac{1}{4}\right)n^2 + O(n \log n).$$

In particular, the proportion of realizable words tends to zero. This answers the asymptotic question attributed to Arnold by Shapiro and Shapiro. The proof applies Milnor's bound to the complement of the derivative-resultant hypersurfaces in coefficient space. It does not classify the individual realizable words.

1 Words and the asymptotic question

Fix $n \geq 2$ and put $N = n(n+1)/2$. Write $x_1^{(i)} < \dots < x_{n-i}^{(i)}$ for the zeros of the i th derivative, where $0 \leq i < n$. A *Rolle word* is a total order of these labeled symbols satisfying

$$x_k^{(i)} < x_k^{(i+1)} < x_{k+1}^{(i)} \quad (0 \leq i < n-1, 1 \leq k < n-i). \quad (1)$$

Equivalently, replace $x_k^{(i)}$ by the letter i in the total order. The k th occurrence of a letter recovers its subscript, so this encoding loses no information. All higher-order Rolle inequalities follow transitively from (1).

A word is *realizable* if it comes from a real polynomial of degree n with n distinct real roots and with no zero shared by two different nonconstant derivatives, including the polynomial itself. The latter condition excludes coincidences between derivative levels. The numbers of all Rolle words and of realizable words are denoted by b_n and a_n , respectively. We take $a_1 = b_1 = 1$.

The classical enumeration, recorded in [4, 1] and tracing back to Thrall [5], is

$$b_n = N! \frac{\prod_{j=1}^{n-1} j!}{\prod_{j=1}^n (2j-1)!}. \quad (2)$$

Its first values are 1, 1, 2, 12, 286, 33592. By contrast, the known realizable counts through degree five are 1, 1, 2, 10, 116; see [4] and its references. The two counts are therefore different already in degree four.

Shapiro and Shapiro [4, Section 2, Problem 1] attribute to Arnold the question whether a_n/b_n tends to zero and, if so, how quickly. This is a counting question, distinct from the classification problem posed in [3, Problem 7]. The following result answers the counting question on a logarithmic scale.

Theorem 1. *For $n \geq 2$, let $D_n = n(n^2 - 1)/2$. Then*

$$1 \leq a_n \leq \min\{b_n, (D_n + 1)(2D_n + 1)^n\}. \quad (3)$$

Moreover,

$$\log \frac{a_n}{b_n} = -\frac{1}{2}n^2 \log n + \left(\frac{3}{2} \log 2 - \frac{1}{4}\right)n^2 + O(n \log n). \quad (4)$$

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In particular $a_n/b_n \rightarrow 0$, and

$$\lim_{n \rightarrow \infty} \frac{\log(a_n/b_n)}{n^2 \log n} = -\frac{1}{2}.$$

All logarithms are natural.

The argument uses established ingredients: the resultant description of collisions is already present in [1], the enumeration (2) is classical, and the topological estimate is Milnor's theorem [2]. Their combination gives the explicit bound and the density asymptotic above. No claim of an exhaustive literature search or absolute priority is made.

2 The collision hypersurface

Use the monic coefficient family

$$p_{\mathbf{c}}(x) = x^n + \sum_{k=0}^{n-1} c_k x^k, \quad \mathbf{c} = (c_0, \dots, c_{n-1}) \in \mathbb{R}^n.$$

Multiplication of a polynomial by a nonzero constant does not change any of its zeros or derivative zeros, so monic polynomials suffice. For $0 \leq i < j < n$, define

$$R_{ij}(\mathbf{c}) = \text{Res}_x(p_{\mathbf{c}}^{(i)}, p_{\mathbf{c}}^{(j)}), \quad Q_n(\mathbf{c}) = \prod_{0 \leq i < j < n} R_{ij}(\mathbf{c}).$$

Lemma 2. *The polynomial Q_n is not identically zero and $\deg Q_n \leq D_n$.*

Proof. The degrees in x of the two derivatives are $n-i$ and $n-j$, with fixed nonzero leading coefficients. The Sylvester determinant has size $2n-i-j$, and each entry is constant or linear in $\mathbf{c}$. Hence $\deg R_{ij} \leq 2n-i-j$.

For a fixed pair $i < j$, specialize to $p(x) = x^n + x^i$. Its j th derivative is a nonzero constant times x^{n-j} . Its i th derivative has constant term $i! \neq 0$. These two polynomials have no common complex root, so the specialized resultant is nonzero. Thus every factor R_{ij} is a nonzero polynomial. Since the polynomial ring is an integral domain, so is their product. Finally,

$$\deg Q_n \leq \sum_{i < j} (2n-i-j) = \sum_{1 \leq b < a \leq n} (a+b) = (n-1) \sum_{a=1}^n a = \frac{n(n^2-1)}{2}.$$

□

Put $U_n = \{\mathbf{c} \in \mathbb{R}^n : Q_n(\mathbf{c}) \neq 0\}$. The resultant R_{01} is one of the factors of Q_n , so every polynomial represented in U_n has only simple complex roots.

Lemma 3. *Every connected component of U_n that contains a polynomial with all roots real consists entirely of such polynomials. On each such component the Rolle word is constant. Every realizable word occurs in one of these components.*

Proof. For monic real polynomials with only simple roots, the number of real roots is locally constant: a real simple root persists as a real root under a small real perturbation, and nonreal roots remain away from the real axis under sufficiently small perturbations. The degree and leading coefficient are fixed, so there is no loss of roots at infinity near a coefficient vector. Consequently the number of real roots is constant on a connected component of U_n .

On a component with n real roots, Rolle's theorem gives $n-i$ simple real roots at every derivative level i . The sorted roots at each level vary continuously with the coefficients. No two roots at different levels coincide on U_n , by its definition. Thus all comparisons between these continuous root functions are locally constant, and therefore constant on the component. Conversely, a polynomial realizing a word has only real roots at every derivative level and has no cross-level coincidence. Its derivative pairs have no common complex root either, since all their roots are real. It therefore belongs to U_n . □

There is at least one realizable word for every n . Indeed, the set of monic polynomials with n distinct real roots is nonempty and open in $\mathbb{R}^n$. A nonzero real polynomial cannot vanish on that open set, so Lemma 2 implies that this set meets U_n .

Different components may have the same word; we do not need, and do not assert, the converse of the constancy statement. Lemma 3 gives precisely the inequality

$$a_n \leq b_0(U_n), \quad (5)$$

where b_0 denotes the number of connected components.

3 A bound for the complement

We use the following convenient consequence of Milnor's bound, including a proof of the reduction from complements to zero sets.

Lemma 4. *If $Q \in \mathbb{R}[c_1, \dots, c_m]$ is nonzero and has degree at most D , then*

$$b_0(\{\mathbf{c} : Q(\mathbf{c}) \neq 0\}) \leq (D+1)(2D+1)^m.$$

Proof. The graph

$$V = \{(\mathbf{c}, t) \in \mathbb{R}^{m+1} : tQ(\mathbf{c}) - 1 = 0\}$$

is homeomorphic to the complement in question, by $\mathbf{c} \mapsto (\mathbf{c}, 1/Q(\mathbf{c}))$ and coordinate projection. The defining equation has degree at most $D+1$. Milnor's theorem [2, Theorem 2] bounds the sum of the Betti numbers of a real algebraic set in $\mathbb{R}^r$, defined by polynomials of degree at most d , by $d(2d-1)^{r-1}$. It applies to affine, possibly noncompact sets. Taking $r = m+1$ and $d = D+1$ bounds $b_0(V)$, and hence the desired component count, by $(D+1)(2D+1)^m$. $\square$

Apply Lemma 4 with $Q = Q_n$, $m = n$, and $D = D_n$. Together with (5), existence, and the inclusion of realizable words among Rolle words, this proves (3). In particular,

$$0 \leq \log a_n \leq \log(D_n + 1) + n \log(2D_n + 1) = O(n \log n). \quad (6)$$

For comparison, the two-level realization theorem of Kostov and Shapiro [1, Theorem 2 and Proposition 3] supplies the stronger known lower bound $a_n \geq 2^{n-2}$. For $n \geq 3$, there are 2^{n-2} possible strict arrangements of the roots of p and p'' , and each is realized. Such a strict arrangement persists in a nonempty open subset of the simple hyperbolicity domain. Removing $Q_n = 0$ from that subset leaves a generic realization and hence a full word. Distinct two-level arrangements give distinct full words. The case $n = 2$ is immediate. This lower bound is not needed for (4).

4 The logarithmic density

We spell out the asymptotic calculation to identify the quadratic term as well as the leading term. Stirling's formula, summed with its $O(\log(j+1))$ remainder, gives

$$\log(N!) = n^2 \log n - \left(\frac{1}{2} \log 2 + \frac{1}{2}\right) n^2 + O(n \log n), \quad (7)$$

$$\sum_{j=1}^{n-1} \log(j!) = \frac{1}{2} n^2 \log n - \frac{3}{4} n^2 + O(n \log n), \quad (8)$$

$$\sum_{j=1}^n \log((2j-1)!) = n^2 \log n + \left(\log 2 - \frac{3}{2}\right) n^2 + O(n \log n). \quad (9)$$

For completeness, $\sum_{j=1}^{n-1} j \log j = \frac{1}{2} n^2 \log n - \frac{1}{4} n^2 + O(n \log n)$ by comparison with the integral of $x \log x$. Replacing $2j-1$ by $2j$ changes the analogous sum by at most $O(n \log n)$, yielding (9). Also $N = n^2/2 + O(n)$, which yields (7).

Substituting (7)–(9) into (2) gives

$$\log b_n = \frac{1}{2} n^2 \log n + \left(\frac{1}{4} - \frac{3}{2} \log 2\right) n^2 + O(n \log n).$$

Subtract this identity from (6). Because $1 \leq a_n$, both sides of the claimed asymptotic are controlled; an upper bound alone without existence would not suffice. This proves (4) and completes Theorem 1.

Thus, if a Rolle word is chosen uniformly from all b_n words, its probability of polynomial realizability is

$$n^{-n^2/2} \exp \left[\left(\frac{3}{2} \log 2 - \frac{1}{4} \right) n^2 + O(n \log n) \right].$$

This is a counting probability on words, not a probability model on polynomial coefficients. The error term permits a multiplicative factor $\exp(O(n \log n))$; no asymptotic equivalent for a_n is asserted.

5 Scope and reproducibility

The result concerns ordinary real polynomials and nondegenerate root orders. It does not classify individual words, count arrangements with equalities, or resolve the analogous realization questions for smooth polynomial-like functions. In particular, Problem 7 of [3], indexed as AMR-021-0017 in the frozen UnsolvedMath v1.6.0 worklist, remains a broader classification problem. The complete conclusion here is the asymptotic counting statement of Theorem 1.

The accompanying code independently counts linear extensions of (1) through $n = 10$, compares them with (2), and verifies the degree sum through $n = 100$. A separate exact SymPy check computes all 20 derivative-pair resultants in degrees two through five, checks their degree bounds and nonzero specializations, and certifies disjoint rational root intervals for nine generic examples in degrees two through ten. These finite checks test the encoding and arithmetic; they do not replace the written proof for arbitrary n .

AI tools assisted with research, drafting, and exact checks. The arguments were self-audited. This preprint has not undergone independent peer review or formal proof verification.

References

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