

A Finite Moment-Cone Criterion for Derivative-Root Configurations

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Abstract

We give a necessary and sufficient finite criterion for prescribed numerical positions of all derivative roots of a real-rooted polynomial-like function. Polynomial-like degree n means that the n th derivative is nowhere zero; the function need not be an ordinary polynomial of degree n . Eliminating integration constants produces $m = n(n-1)/2$ explicit piecewise polynomial Peano kernels. Realizability is equivalent to zero being interior to their essential-trace convex hull, and to representing an explicit baseline vector by at most m nonnegative trace terms. This gives a finite semialgebraic criterion and a terminating decision procedure for algebraic coordinates. Permitted cross-order coincidences are included. Every feasible array also has an ordinary-polynomial realization of unspecified larger degree. The argument applies classical positive-moment methods to the exact coordinate question in Shapiro’s Problem 6; it makes no absolute-priority or practical-complexity claim.

1 The coordinate problem and classical ingredients

A smooth function f on an interval is polynomial-like of degree n if $f^{(n)}$ is nowhere zero there. Its real-rooted case has n distinct simple zeros. Repeated Rolle implies that $f^{(i)}$ has exactly $n-i$ simple zeros for $0 \leq i < n$. Shapiro’s Problem 6 [1, §8] asks which prescribed real coordinates can occur, beyond the usual Rolle restrictions. This is a question about positions, not only the merged order of the roots. Exact degree-three inequalities appear in Shapiro–Shapiro [2]; Kostov [3, Remark 14] distinguishes them from higher-degree arrangement results. The same distinction is recalled in Gati–Kostov [4, §2].

Our tools are classical. Takeya’s finite-moment treatment of nonnegative densities [5, Theorem 1] and his reduction of two-endpoint differential inequalities to moments [6] date to 1915. Interior moment-cone conditions and finite representations also occur in the value-interpolation setting of Pinkus–Wulbert [9, §6]. Interior moment representations by smooth mixtures and subtraction of a prescribed component are treated in di Dio [7, Theorems 17 and 25]; smooth-density extensions and isolated-point warnings appear in [8]. We give the root-specific kernels and a self-contained strict-density argument, with separate traces at discontinuities. These generic moment methods are not new theorems of this paper. A bounded primary-source comparison located no matching all-degree fixed-coordinate statement in the inspected sources; this is not an exhaustive priority claim. Whether the application is an already-known specialization remains uncertain; inaccessible Hermite–Birkhoff/monotone-spline literature is an unresolved overlap lead.

Fix $n \geq 2$. Prescribe rows

$$x_1^{(i)} < \cdots < x_{n-i}^{(i)}, \quad 0 \leq i < n,$$

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and assume every standard strict Rolle inequality

$$x_l^{(i)} < x_l^{(j)} < x_{l+j-i}^{(i)} \quad (i < j). \quad (1)$$

Coordinates in different rows may coincide where (1) permits it. Write $x_j = x_j^{(0)}$, $a = x_1$, $b = x_n$, and $m = n(n-1)/2$. Let ℓ_j be the degree- $n-1$ Lagrange basis at $x_1, \dots, x_n$. Index vectors in $\mathbb{R}^m$ by pairs (i, z) with $1 \leq i < n$ and $z = x_l^{(i)}$. Define

$$K_{i,z}(t) = \frac{(z-t)_+^{n-1-i}}{(n-1-i)!} - \sum_{j=1}^n \ell_j^{(i)}(z) \frac{(x_j-t)_+^{n-1}}{(n-1)!}. \quad (2)$$

Exponent zero means the indicator $\mathbf{1}_{\{t < z\}}$, not a constant.

Sort the distinct coordinates of all rows into knots. On each positive-length open knot interval (u_h, v_h) , the vector K is a polynomial P_h . Define

$$S = \bigcup_h P_h([u_h, v_h]), \quad C = \text{conv}(S), \quad Q(x) = \frac{1}{n!} \prod_{j=1}^n (x - x_j), \quad (M_0)_{i,z} = Q^{(i)}(z). \quad (3)$$

Each closed-piece endpoint uses that piece's polynomial formula. At a jump both one-sided traces are kept; arbitrary isolated point values are not. Zero-length pieces are excluded.

Theorem 1. *For the prescribed rows, the following are equivalent:*

1. *They are exactly the derivative-root rows of a smooth real-rooted polynomial-like function of degree n on an interval.*
2. *Such a realization exists on all $\mathbb{R}$, with $f^{(n)} > 0$.*
3. *$0 \in \text{int } C$, with ordinary interior in $\mathbb{R}^m$.*
4. *No nonzero $\lambda \in \mathbb{R}^m$ satisfies $\lambda \cdot P_h(t) \geq 0$ on every closed piece.*
5. *For some $r \leq m$, piece indices h_s , points $t_s \in [u_{h_s}, v_{h_s}]$, and $\alpha_s \geq 0$,*

$$-M_0 = \sum_{s=1}^r \alpha_s P_{h_s}(t_s). \quad (4)$$

The empty sum is permitted. For $n = 1$, every input is realized by $x - x_1$.

2 Elimination and full span

Lemma 2. *For smooth w on $[a, b]$, set*

$$G(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} w(t) dt, \quad F(x) = G(x) - \sum_{j=1}^n G(x_j) \ell_j(x).$$

Then $F(x_j) = 0$, $F^{(n)} = w$, and

$$F^{(i)}(z) = \int_a^b K_{i,z}(t) w(t) dt. \quad (5)$$

Conversely every smooth f vanishing at the x_j has this form with $w = f^{(n)}$. Also $\int_a^b K(t) dt = M_0$, and both outer endpoint traces of K are zero.

Proof. Differentiating G gives the first term in (2), including the indicator at derivative order $n - 1$. Subtracting the interpolant gives (5). Conversely $f - G$ has zero n th derivative, hence is a polynomial of degree at most $n - 1$, uniquely determined by its values $-G(x_j)$. For $w = 1$, $G = (x - a)^n/n!$; its interpolation residual is Q . Thus its derivative values give M_0 . At $t = a$, exact Lagrange reproduction of $(x - a)^{n-1}/(n - 1)!$ cancels every component. At $t = b$, all truncated terms vanish. The indicator's assigned value at its knot does not affect any integral. $\square$

Lemma 3. S spans $\mathbb{R}^m$, contains zero, and C is compact and full-dimensional.

Proof. Prescribe arbitrary values for the m higher-order derivatives, and value zero at every zeroth-row node. All higher nodes lie in (a, b) by (1). Around each distinct interior node choose a compactly supported smooth bump equal to one near that node, with disjoint supports for distinct nodes. Multiply each bump by the Taylor polynomial of order $n - 1$ specifying all its requested derivatives; set unspecified entries to zero. At an interior zeroth-row node the value entry is also zero. Their sum h vanishes at every x_j , is supported inside (a, b) , and has the prescribed jets. Different derivative orders at one coincident node are different entries of the same jet, not incompatible assignments. Lemma 2 gives $\int K h^{(n)}$ equal to the arbitrary vector of requested derivative values. A linear functional annihilating S would annihilate all these vectors, so must be zero. The outer zero trace gives $0 \in S$. Compactness follows from the finite union of closed polynomial images and its finite-dimensional convex hull. $\square$

This argument uses smooth jets, not a genericity assumption or a lacunary polynomial interpolation bound. For example $p(-1) = p(1) = 0$, $p'(0) = 1$ admit no quadratic, although $x - x^3$ satisfies these three conditions.

3 Positive densities and exact realization

Proof of Theorem 1, conditions 1–4. For an interval realization negate f if necessary, so that $w = f^{(n)} > 0$. Lemma 2 gives $\int K w = 0$. If zero is not interior to C , a supporting hyperplane through zero, using Lemma 3, supplies nonzero λ with $\lambda \cdot P_h \geq 0$ on every piece. Full span ensures that one such polynomial is not identically zero. It is then positive on an open subinterval, whence $\int (\lambda \cdot K) w > 0$, a contradiction. This proves 1 implies 3. The same supporting-hyperplane argument makes 3 and 4 equivalent.

Assume 3. There are finitely many regular points, away from knots, whose kernel values v_j surround zero. Indeed, choose a small cross-polytope about zero in C , express each of its finitely many vertices using finite convex combinations of traces, and approximate those traces by values from their own open pieces. A smaller ball remains contained in the resulting hull. Containment is stable because the positive support margin $\min_{\|\lambda\|=1} \max_j \lambda \cdot v_j$ persists under small perturbations.

At each selected point choose a normalized nonnegative smooth bump ϕ_j supported inside its piece. For sufficiently narrow supports, the moment vectors $b_j = \int K \phi_j$ are sufficiently close to v_j that they also surround zero. Thus their positive cone is all $\mathbb{R}^m$. Choose $\gamma_j \geq 0$ with $\sum_j \gamma_j b_j = -M_0$ and put

$$w = 1 + \sum_j \gamma_j \phi_j \geq 1. \quad (6)$$

Then $\int K w = 0$ exactly. This is exact moment correction, not an approximate smoothing of atoms. All bumps are supported inside (a, b) , so extend w by one to a smooth positive function on $\mathbb{R}$. Define its global n th primitive and subtract the zeroth-row interpolant as in Lemma 2. The resulting F has $F^{(n)} > 0$ and every prescribed derivative zero.

There are no additional real zeros of $F^{(i)}$: an extra distinct zero would give a zero of $F^{(n)}$ by repeated Rolle. A multiple prescribed zero would be a zero of $F^{(i+1)}$ distinct from the $n - i - 1$ Rolle

zeros inside consecutive gaps. These $n - i$ distinct zeros again force a zero of $F^{(n)}$. For $i = n - 1$, simplicity follows directly from $F^{(n)} > 0$. This excludes roots on both tails as well. Hence 3 implies 2; restricting the global realization gives 1. $\square$

Proof of Theorem 1, condition 5. If 3 holds, $\text{cone}(S) = \mathbb{R}^m$, so $-M_0$ is a finite nonnegative combination of traces. Conic Caratheodory reduction leaves at most m terms. Explicitly, when more positive terms occur their vectors are linearly dependent; subtract a suitable multiple of a dependence until a coefficient becomes zero, keeping all coefficients nonnegative, and repeat.

Conversely suppose 5 holds but 3 fails. The supporting functional above has $\lambda \cdot M_0 = \int \lambda \cdot K > 0$, since one nonnegative piece is nonzero. But (4) implies $-\lambda \cdot M_0 = \sum_s \alpha_s \lambda \cdot P_{h_s}(t_s) \geq 0$, a contradiction. $\square$

Remark 4. *The baseline $-M_0$ is essential. An atomic representation of zero alone would always exist from the outer zero trace, even for infeasible input. Likewise the scalar kernel $t(1 - t)$ on $[0, 1]$ has zero endpoint atoms but integrates strictly positively against every positive density.*

4 Finite decision and polynomial witnesses

Corollary 5. *For fixed n , realizability of the numerical arrays is a semialgebraic condition. It is decidable for coordinates encoded as real algebraic numbers, including rationals.*

Proof. Fix an ordering/equality stratum of all coordinates and remove zero-length pieces. The coefficients of every P_h are rational functions of the coordinates; denominators involve only factorials and nonzero differences of distinct zeroth-row roots. The entries of M_0 are polynomial functions. Condition 5 is a finite disjunction over at most m piece choices, with existential variables t_s, α_s , piece bounds, coefficient nonnegativity and m equations. Clear nonzero denominators in equations; use their positive squares if clearing inequalities. There are finitely many strata for fixed n . Real closed field quantifier elimination therefore decides the criterion for encoded real algebraic inputs. $\square$

This does not provide a computable encoding for arbitrary noncomputable reals, a practical complexity bound, or a bounded-degree degree- n ordinary-polynomial witness.

Corollary 6. *Every feasible array has an ordinary-polynomial realization on $\mathbb{R}$, possibly of larger algebraic degree, with positive n th derivative. For algebraic coordinates such a realization can have algebraic coefficients.*

Proof. In the bump construction approximate each continuous $\sqrt{\phi_j}$ uniformly on $[a, b]$ by a polynomial r_j . Their squares approximate ϕ_j , so the vectors $\int K r_j^2$ approach the b_j . Choose approximations close enough to retain the interior support margin. Their positive cone remains all $\mathbb{R}^m$; choose $\beta_j \geq 0$ with $\sum_j \beta_j \int K r_j^2 = -M_0$. The polynomial $w = 1 + \sum_j \beta_j r_j^2$ is positive on all $\mathbb{R}$ and has zero required moments. Its n th primitive minus the zeroth-row interpolant is a polynomial realization; the Rolle argument above proves its exact global roots and simplicity. For algebraic input rational-coefficient polynomial approximants suffice. The moments are algebraic, and the resulting feasible nonnegative linear system has an algebraic solution, giving algebraic coefficients. $\square$

5 Exact controls and limitations

For $n = 2$, take zeroth-row roots 0, 1 and derivative root 3/4. Then $K = t$ before 3/4 and $K = t - 1$ after it, $M_0 = 1/4$, and

$$\int_0^1 K(t) t^8 dt = -\frac{32657}{11796480}.$$

With $\alpha = 2949120/32657$, the polynomial

$$F(x) = \frac{x^2}{2} + \frac{\alpha x^{10}}{90} - \left(\frac{1}{2} + \frac{\alpha}{90}\right)x$$

satisfies $F(0) = F(1) = F'(3/4) = 0$ and $F'' = 1 + \alpha x^8 > 0$ globally. Its ordinary degree is 10 while its polynomial-like degree is 2.

For $n = 3$, rows $(0, 1, 2)$, $(3/4, 3/2)$ and $(11/10)$ obey all strict Rolle restrictions, but the kernel for $(i, z) = (1, 3/4)$ is

$$-\frac{3t^2}{8}, \quad -\frac{(1-t)^2}{4} - \frac{(2-t)^2}{8}, \quad -\frac{(2-t)^2}{8}$$

on the respective pieces $(0, 3/4)$, $(3/4, 1)$ and $(1, 2)$. It is strictly negative on $(0, 2)$, giving an exact dual obstruction. These degree-two and degree-three examples are controls, not new low-degree classifications.

The accompanying exact-arithmetic checks compare two separately implemented Python-rational and JavaScript-BigInt kernel builders on 49 arrays of degrees 2–8, including permitted coincidences. All 570 baseline identities and 13,035 polynomial-piece vectors agree. These finite controls verify implementation formulas, not the all-degree theorem, whose analytic proof is given above. No sampled sign test is used in that proof.

This preprint is unrefereed. Its proof received a detailed originating researcher self-audit, not independent human review or formal proof-assistant certification. AI assistance was used for research, algebraic checking, source comparison and drafting; mathematical responsibility remains with the author. Minimal witness degree, practical running time and absolute historical priority are not claimed.

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