

A Reciprocal-Block Counterexample to the Forsgård–Shapiro Coefficient-Parity Bound

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Abstract

The coefficient-parity conjecture of Forsgård and Shapiro uses the indices at which $a_k^2 - a_{k-1}a_{k+1}$ is nonnegative to bound the number of real zeros of a polynomial with positive coefficients. We give an explicit degree-77 polynomial with positive rational coefficients for which the selected indices change parity only once, but which has at least three distinct negative real zeros. The construction joins a degree-38 block to its reciprocal. A weighted sum-of-squares identity proves the required root crossing; all coefficients and signs can also be checked by rational arithmetic.

1 The proposed bound and the construction

For a polynomial $f(z) = \sum_{k=0}^n b_k z^k$ with $b_k > 0$, set

$$c_k = b_k^2 - b_{k-1}b_{k+1}, \quad b_{-1} = b_{n+1} = 0. \quad (1)$$

List *all* indices with $c_k \geq 0$ in increasing order as $0 = k_1 < \dots < k_m = n$, and define

$$v(f) = \#\{i : 1 \leq i < m, k_{i+1} \not\equiv k_i \pmod{2}\}.$$

Conjecture 10 in Shapiro’s problem collection [1, §7, Conjecture 10], attributed there to Forsgård and Shapiro, proposes that the number of real zeros of f is at most $v(f)$. Our example violates this inequality even when only distinct real zeros are counted, and therefore also when multiplicities are counted. Positive coefficients exclude positive real zeros.

Here is a compact specification of every coefficient of the example. Put

$$r = 19, \quad \varepsilon = \delta = 10^{-6}, \quad w_j = \frac{4j+3}{2} \quad (0 \leq j < r), \quad (2)$$

and define

$$\begin{aligned} S_j &= \sum_{i=0}^{j-1} \frac{1}{w_i}, & S &= S_r = \frac{178446135735943305326}{88263547578916184475}, \\ x_j &= 1 + \delta - \frac{S_j}{S} \quad (0 \leq j \leq r). \end{aligned} \quad (3)$$

The empty sum is zero. In particular, $x_0 = 1 + \delta > x_1 > \dots > x_r = \delta > 0$, and

$$x_j - x_{j+1} = \frac{1}{S w_j}. \quad (4)$$

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Define the degree-38 polynomial

$$A(z) = \sum_{j=0}^r x_j^2 z^{2j} + (1 - \varepsilon) \sum_{j=0}^{r-1} x_j x_{j+1} z^{2j+1}, \quad (5)$$

and then the degree-77 polynomial

$$f(z) = z^{38} A(1/z) + z^{39} A(z). \quad (6)$$

The first term in (6) denotes the reversed polynomial, so f is an ordinary polynomial, including at $z = 0$. The two coefficient blocks occupy the disjoint ranges $0, \dots, 38$ and $39, \dots, 77$. Thus every coefficient is positive and rational.

Theorem 1. *The polynomial (6) satisfies $v(f) = 1$ and has at least three distinct negative real zeros. Consequently the bound in [1, §7, Conjecture 10] is false.*

Related obstructions for globally weighted tropicalizations are given by Forsgård, Novikov, and Shapiro [2, Theorem 11]. Their tropical indices are defined by global dominance conditions, not by the neighboring-coefficient differences (1). That result is not used as a substitute for the local-index check below. The neighboring Conjecture 9 in [1] instead uses $(k+1)b_k^2 - kb_{k-1}b_{k+1} > 0$ to select its indices; the present argument does not settle that weighted conjecture.

2 The parity count

Write $A(z) = \sum_{k=0}^{38} a_k z^k$. For $0 \leq j < 19$,

$$a_{2j+1}^2 - a_{2j}a_{2j+2} = ((1 - \varepsilon)^2 - 1)x_j^2 x_{j+1}^2 < 0. \quad (7)$$

Reversal in an even-degree block preserves the parity of an index. Therefore $c_k < 0$ for every odd k in $1, 3, \dots, 37$ in the left block of f . The right block is shifted by 39, which reverses parity, so $c_k < 0$ for every even k in $40, 42, \dots, 76$.

The two central coefficients are both $a_0 = x_0^2$; each outside neighbor is $a_1 = (1 - \varepsilon)x_0x_1$. Hence

$$c_{38} = c_{39} = x_0^4 - (1 - \varepsilon)x_0^3x_1 > 0. \quad (8)$$

Also $c_0 = b_0^2 > 0$ and $c_{77} = b_{77}^2 > 0$. Every selected index in the left block is consequently even, every selected index in the right block is odd, and both blocks contain a selected index. Their ordered parity sequence has exactly one change:

$$v(f) = 1. \quad (9)$$

No assertion about the signs at the other indices is needed.

3 The root crossing

Set $P(t) = A(-t)$ and $Q = P'(1) + P(1)/2$.

Lemma 2. *For the block (5),*

$$Q = \frac{1}{2} \sum_{j=0}^{r-1} w_j (x_j - x_{j+1})^2 - \frac{1}{4} x_0^2 + \left(r + \frac{3}{4}\right) x_r^2 + \varepsilon \sum_{j=0}^{r-1} w_j x_j x_{j+1}. \quad (10)$$

Proof. Direct differentiation gives

$$Q = \sum_{j=0}^r \left(2j + \frac{1}{2}\right) x_j^2 - (1 - \varepsilon) \sum_{j=0}^{r-1} w_j x_j x_{j+1}.$$

Expand the squares on the right side of (10). For $1 \leq j < r$, the diagonal coefficient is $(w_{j-1} + w_j)/2 = 2j + 1/2$. At $j = 0$ it is $w_0/2 - 1/4 = 1/2$, and at $j = r$ it is $w_{r-1}/2 + r + 3/4 = 2r + 1/2$. The cross terms agree as well. $\square$

For the parameters above, (4) yields

$$\sum_{j=0}^{18} w_j (x_j - x_{j+1})^2 = \frac{1}{S}, \quad \sum_{j=0}^{18} w_j = \frac{741}{2}.$$

The fraction in (3) satisfies $S > 201/100$. Since $0 < x_j \leq 1 + 10^{-6}$, Lemma 2 gives the entirely rational estimate

$$Q < \frac{50}{201} - \frac{1}{4} + \frac{79}{4 \cdot 10^{12}} + \frac{741}{2 \cdot 10^6} (1 + 10^{-6})^2 < 0. \quad (11)$$

Both numerical comparisons in this display are exact rational inequalities, obtainable by clearing positive denominators.

For $t > 0$, the reciprocal construction gives

$$g(t) := f(-t) = t^{38} P(1/t) - t^{39} P(t). \quad (12)$$

In particular,

$$g(1) = 0, \quad g'(1) = -P(1) - 2P'(1) = -2Q > 0. \quad (13)$$

Thus $t = 1$ is a simple upward-crossing zero of g . There are points $t_- < 1 < t_+$, arbitrarily close to 1, with $g(t_-) < 0 < g(t_+)$. On the other hand $g(0) = x_r^2 > 0$, and the leading coefficient of g is $-x_r^2$, so $g(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. The intermediate value theorem gives one zero in $(0, t_-)$ and another in (t_+, ∞) , in addition to the zero at 1. These yield three distinct negative zeros of f . Together with (9), this proves Theorem 1.

The degree 77 is the degree of this explicit example, not a claim of minimal degree; nor is an exact count of its real zeros asserted. This is an unrefereed manuscript, and no absolute-priority claim is made. AI-assisted tools were used in exploratory algebra, source checking, manuscript preparation, and independent exact-arithmetic checks. The author is responsible for the statements and arguments.

A Exact-arithmetic reproduction

The following standalone Python 3 code uses only its standard library. It reconstructs all coefficients and checks the sign tests without floating-point arithmetic. Multiplication by a common positive denominator, followed by division by the greatest common divisor, gives positive primitive integer coefficients without changing the roots or the signs of (1).

```
from fractions import Fraction as F
from functools import reduce
from math import gcd, lcm

r = 19
eps = delta = F(1, 10**6)
w = [F(4*j+3, 2) for j in range(r)]
```

```

S = sum((1/q for q in w), F(0))
x = [1+delta]
for q in w:
    x.append(x[-1] - 1/(S*q))
assert x[-1] == delta and all(q > 0 for q in x)
a = []
for j in range(r):
    a.extend([x[j]**2, (1-eps)*x[j]*x[j+1]])
a.append(x[r]**2)
b = list(reversed(a)) + a
den = lcm(*(q.denominator for q in b))
B = [int(den*q) for q in b]
d = reduce(gcd, B)
B = [q//d for q in B]
assert len(B) == 78 and all(q > 0 for q in B)
pad = [0] + B + [0]
c = [pad[k+1]**2-pad[k]*pad[k+2] for k in range(78)]
good = [k for k in range(78) if c[k] >= 0]
assert sum((s-t) % 2 for s,t in zip(good[1:], good[:-1])) == 1
Q = sum(((F(k)+F(1,2))*(-1)**k*q
        for k,q in enumerate(a)), F(0))
assert Q < 0
def g(t):
    value = F(0)
    for q in reversed(B):
        value = value*(-t) + q
    return value
points = [F(0), F(999,1000), F(1), F(1001,1000), F(2)]
signs = [(g(t)>0)-(g(t)<0) for t in points]
assert signs == [1, -1, 0, 1, -1]
print("degree:", len(B)-1, "good indices:", good)
print("Q:", Q, "signs:", signs)
# print(B) # Optional full primitive integer coefficient list.

```

The four nonzero signs also certify negative roots in $(-999/1000, 0)$ and $(-2, -1001/1000)$, in addition to the exact root -1 . This is an optional independent certificate, not an assumption in the proof.

References

- [1] B. Shapiro, *Problems Around Polynomials: The Good, The Bad and The Ugly...*, Arnold Mathematical Journal **1** (2015), 91–104. [doi:10.1007/s40598-015-0008-4](https://doi.org/10.1007/s40598-015-0008-4). See §7, Conjectures 9 and 10.
- [2] J. Forsgård, D. Novikov, and B. Shapiro, *A tropical analog of Descartes' rule of signs*, arXiv:1510.03257v1 (2015). arxiv.org/abs/1510.03257. See Theorem 11.