

A Positive-Coefficient Counterexample to the Weighted Forsgård–Shapiro Parity Bound

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Abstract

Forsgård and Shapiro proposed bounding the number of real zeros of a positive-coefficient polynomial by the parity changes among all indices at which $(k+1)a_k^2 - ka_{k-1}a_{k+1}$ is positive. We give a counterexample with positive rational coefficients, parity count one, and at least three distinct negative real zeros. An even-degree shift of a degree-77 reciprocal-block seed makes the undesired weighted local quantities negative. A sufficiently small, strictly log-convex factorial prefix fills every missing coefficient without introducing a parity change or destroying the three roots. The explicit witness has degree 1,000,077; no optimal-degree claim is made. Its coefficients have a compact exact formula.

1 The proposed bound

For $H(z) = \sum_{k=0}^N a_k z^k$ with every $a_k > 0$, put

$$\tilde{c}_k = (k+1)a_k^2 - ka_{k-1}a_{k+1}, \quad a_{-1} = a_{N+1} = 0. \quad (1)$$

List *all* indices with $\tilde{c}_k > 0$ in increasing order, and let $v(H)$ count parity changes between consecutive listed indices. Conjecture 9 in Shapiro’s problem collection [1, §7], attributed there to Forsgård and Shapiro, asserts that the number of real zeros is at most $v(H)$. Its strict cutoff and weighted expression differ from the neighboring unweighted Conjecture 10. The latter has a degree-77 counterexample [2]; we transfer its seed to the weighted setting, rather than claiming that an unchanged seed settles both statements.

Related failures of globally weighted tropical root bounds occur in Forsgård–Novikov–Shapiro [3, Theorem 11]. Global tropical dominance indices are not the local quantities (1). The argument below checks the local signs directly.

2 The seed and the positive prefix

We restate a compact formula for the seed. Set

$$r = 19, \quad \eta = 10^{-6}, \quad \theta = 1 - \eta, \quad w_i = \frac{4i+3}{2}, \quad S_j = \sum_{i=0}^{j-1} \frac{1}{w_i}, \quad S = S_{19},$$

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and $x_j = 1 + \eta - S_j/S$ for $0 \leq j \leq 19$. The numbers x_j decrease strictly from $1 + \eta$ to η . Define

$$\begin{aligned} A(z) &= \sum_{j=0}^{19} x_j^2 z^{2j} + \theta \sum_{j=0}^{18} x_j x_{j+1} z^{2j+1}, \\ F(z) &= z^{38} A(1/z) + z^{39} A(z) = \sum_{j=0}^{77} b_j z^j. \end{aligned} \quad (2)$$

The reversed expression is an ordinary polynomial. All b_j are positive rationals and $b_0 = \eta^2 = 10^{-12}$. For $g(t) = F(-t)$, exact rational evaluation gives

$$g(0) > 0, \quad g(999/1000) < -b_0, \quad g(1001/1000) > b_0, \quad g(2) < -b_0. \quad (3)$$

The short exact-arithmetic reproduction in the appendix checks these finite inequalities directly from (2), without a root solver.

Choose the even integer $M = 10^6$ and put

$$\delta = \frac{b_0}{16M 2^{2M}}, \quad a_k = \delta \frac{k!}{(M-1)!} 2^{k^2 - (M-1)^2} \quad (0 \leq k < M). \quad (4)$$

Our polynomial is

$$H(z) = \sum_{k=0}^{M-1} a_k z^k + z^M F(z). \quad (5)$$

Thus the remaining coefficients are $a_{M+j} = b_j$, $0 \leq j \leq 77$. Every coefficient is strictly positive and rational. The tiny rational prefix is not replaced by numerical zeros.

Theorem 1. *The polynomial (5) has degree 1,000,077, satisfies $v(H) = 1$, and has at least three distinct negative real zeros. Consequently the weighted parity bound [1, Conjecture 9] is false.*

3 The exact parity count

For $1 \leq k \leq M-2$, formula (4) gives

$$a_{k-1} a_{k+1} = 4 \frac{k+1}{k} a_k^2, \quad \tilde{c}_k = -3(k+1) a_k^2 < 0.$$

At the transition, $a_{M-1} = \delta$ and $a_{M-2} = \delta 2^{3-2M}/(M-1)$, so

$$\tilde{c}_{M-1} = \delta(M\delta - b_0 2^{3-2M}) < 0,$$

because $M\delta = b_0 2^{-2M}/16$.

For $1 \leq j \leq 76$, write $c_j = b_j^2 - b_{j-1} b_{j+1}$. Then

$$\tilde{c}_{M+j} = b_j^2 + (M+j) c_j. \quad (6)$$

In the reversed left block, at each odd $j = 1, 3, \dots, 37$ there are adjacent positive numbers x_u, x_{u+1} with

$$b_j^2 = \theta^2 x_u^2 x_{u+1}^2, \quad c_j = -(1 - \theta^2) x_u^2 x_{u+1}^2.$$

The same formulas hold at each even $j = 40, 42, \dots, 76$ in the right block. Since

$$M(1 - \theta^2) = 1.999999 > 0.999998000001 = \theta^2,$$

all these undesired weighted quantities are strictly negative. The central seed quantities are strictly positive:

$$c_{38} = c_{39} = x_0^3(x_0 - \theta x_1) > 0,$$

so (6) selects both $M + 38$ and $M + 39$.

The endpoint quantities are also positive: $\tilde{c}_0 = a_0^2 > 0$ and $\tilde{c}_{M+77} = (M + 78)b_{77}^2 > 0$. The sign at M is immaterial, since M is even. Every selected index at or below $M + 38$ is therefore even; every selected index at or above $M + 39$ is odd. Both parts are nonempty, giving exactly one parity change. No local quantity of forbidden parity is zero, so changing the cutoff to nonnegative would not rescue the proposed inequality.

4 Three roots survive the prefix

Let $E(t) = \sum_{k=0}^{M-1} a_k(-t)^k$. For $0 \leq k < M$, formula (4) gives $0 < a_k \leq \delta$. Consequently, for $0 \leq t \leq 2$,

$$|E(t)| \leq \delta \sum_{k=0}^{M-1} 2^k < \delta 2^M = \frac{b_0}{16M} 2^{-M}. \quad (7)$$

Since M is even, $H(-t) = t^M g(t) + E(t)$. Each of the three nonzero arguments in (3) is at least $1/2$, so

$$|t^M g(t)| > b_0 2^{-M}.$$

This is larger than (7). Thus $H(-t)$ has signs $+, -, +, -$ at $0, 999/1000, 1001/1000, 2$, respectively; the value at zero is $a_0 > 0$. The intermediate value theorem gives three distinct roots in the three disjoint intervening intervals for t , hence three distinct negative roots of H . This proves Theorem 1, independently of the exact total root count.

The degree and coefficients were chosen for transparent strict margins, not optimized. Multiplying H by a common denominator would give an integer-coefficient witness without changing roots or local signs, but the compact rational formula avoids an unnecessary enormous expansion. No independent human refereeing, formal proof-assistant certificate, minimal degree or absolute historical priority is claimed. AI-assisted tools were used for exploration, source comparison, exact-arithmetic checks and manuscript preparation; the author is responsible for the argument.

A Exact finite seed verification

The following Python code uses rational arithmetic only. The remaining prefix signs and root-preservation estimate are the all-size identities in the body, so no million-coefficient array or floating-point underflow is needed. The source package contains more extensive checks.

```
from fractions import Fraction as Q
r, eta, M = 19, Q(1, 10**6), 10**6
w = [Q(4*i+3, 2) for i in range(r)]
S = sum((1/u for u in w), Q(0))
x, partial = [], Q(0)
for j in range(r+1):
    x.append(1+eta-partial/S)
    if j < r: partial += 1/w[j]
A = []
for j in range(r):
    A.extend([x[j]**2, (1-eta)*x[j]*x[j+1]])
A.append(x[r]**2)
b = A[::-1]+A
assert len(b) == 78 and all(u > 0 for u in b)
assert b[0] == Q(1, 10**12)
def g(t):
    value = Q(0)
    for u in b[::-1]: value = -t*value+u
    return value
assert g(Q(999,1000)) < -b[0]
assert g(Q(1001,1000)) > b[0]
assert g(Q(2)) < -b[0]
bad = list(range(1,38,2))+list(range(40,77,2))
for j in bad:
    c = b[j]**2-b[j-1]*b[j+1]
    assert b[j]**2+(M+j)*c < 0
for j in [38,39]:
    assert b[j]**2-b[j-1]*b[j+1] > 0
assert M*(1-(1-eta)**2) > (1-eta)**2
print("Exact seed signs and weighted parity checks pass")
```

References

- [1] B. Shapiro, *Problems Around Polynomials: The Good, The Bad and The Ugly...*, Arnold Mathematical Journal **1** (2015), 91–104. [doi:10.1007/s40598-015-0008-4](https://doi.org/10.1007/s40598-015-0008-4).
- [2] A. Ferudun, *A Reciprocal-Block Counterexample to the Forsgård–Shapiro Coefficient-Parity Bound*, unrefered preprint (2026). [doi:10.5281/zenodo.23050917](https://doi.org/10.5281/zenodo.23050917).
- [3] J. Forsgård, D. Novikov, and B. Shapiro, *A tropical analog of Descartes’ rule of signs*, [arXiv:1510.03257](https://arxiv.org/abs/1510.03257) (2015), Theorem 11.