

Sharp Finite Zero Bounds for One-Parameter Separated Polynomial Systems

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Abstract

Let $k \geq 2$ be even. We determine the largest finite real solution set of a separated system $f_i(x_i) = g_i(t)$, $1 \leq i \leq m$, with all degrees at most k . The maximum is $k^2/4$ for $m = 1$, and $m(k-1)k^m/4$ for $m \geq 2$. The upper bound follows from a weighted count of isolated feasible points of univariate polynomial inequalities: root multiplicity pays for the loss of choices in a fiber at a vanishing constraint. The proof allows constant right-hand sides, multiple roots, and a positive-dimensional complex zero set. For $m \geq 2$, the matching construction is the previously published even-degree family of DannyExperiments; the contribution here is the upper bound throughout the separated class. The unrestricted extremal problem for isolated real zeros of sums of squares is not resolved.

1 The separated extremal problem

For positive integers k, m , let $\mathcal{S}_{k,m}$ be the class of systems

$$f_i(x_i) = g_i(t), \quad 1 \leq i \leq m, \quad f_i, g_i \in \mathbb{R}[u], \quad \deg f_i, \deg g_i \leq k, \quad (1)$$

in the $m+1$ variables $x_1, \dots, x_m, t$. Constant and zero polynomials are allowed. Write $Z_{\mathbb{R}}$ for the real solution set. We maximize $|Z_{\mathbb{R}}|$ only over systems for which the *entire* real solution set is finite, including the empty set.

Theorem 1.1. *If $k \geq 2$ is even and $m \geq 1$, then*

$$\max_{\substack{S \in \mathcal{S}_{k,m} \\ |Z_{\mathbb{R}}(S)| < \infty}} |Z_{\mathbb{R}}(S)| = B(k, m) := \begin{cases} k^2/4, & m = 1, \\ m(k-1)k^m/4, & m \geq 2. \end{cases} \quad (2)$$

No genericity assumption or finiteness assumption on the complex solution set is needed.

The real solutions of (1) are also the zeros of

$$P(x_1, \dots, x_m, t) = \sum_{i=1}^m (f_i(x_i) - g_i(t))^2, \quad \deg P \leq 2k.$$

This connects the theorem with the question on isolated real zeros of sums of squares in Shapiro's Problem 3 [1, Section 3]. That question allows arbitrary summands, not just the separated form above, and allows isolated zeros in an otherwise infinite real zero set.

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DannyExperiments [2, Theorem 3.1] already constructed systems in $\mathcal{S}_{k,m}$ attaining the second line of (2). That work disproves the proposed general value k^{m+1} in suitable dimensions, without determining the unrestricted maximum. We use its construction for sharpness and prove the matching upper bound for every system in the stated class. Thus the lower bound and the counterexample to the earlier conjecture are prior results, not contributions of this paper.

2 A weighted root count

The following lemma records the cost of making a feasible point isolated.

Lemma 2.1. *Let $A_1, \dots, A_r$ be nonzero real polynomials, each taking both positive and negative values. Suppose that*

$$T = \{t \in \mathbb{R} : A_i(t) \geq 0 \text{ for every } i\}$$

is finite. For $t \in T$, let $z(t) = |\{i : A_i(t) = 0\}|$. Let $D_i = \deg A_i$, and let O_i be the number of distinct real roots of A_i having odd multiplicity. Then

$$4 \sum_{t \in T} 2^{-z(t)} \leq \sum_{i=1}^r \left(D_i - \frac{O_i}{2} \right). \quad (3)$$

Proof. The assertion is immediate if T is empty. Fix $t \in T$. At least one A_i vanishes at t , since otherwise all the inequalities hold strictly on a neighborhood of t , contradicting finiteness.

For each vanishing polynomial, denote its root multiplicity at t by $\nu_i(t)$. Define

$$\mu(t) = \sum_{i: A_i(t)=0} \nu_i(t), \quad o(t) = |\{i : A_i(t) = 0, \nu_i(t) \text{ odd}\}|.$$

If $z(t) = 1$, the other polynomials are strictly positive near t . The sole vanishing polynomial must therefore be negative on both punctured sides of t ; a positive side would give an interval in T . Its multiplicity is even and at least two, so

$$\mu(t) - \frac{o(t)}{2} \geq 2 = 4 \cdot 2^{-z(t)}.$$

If $z(t) \geq 2$, a root of odd multiplicity contributes at least $1/2$ to $\mu(t) - o(t)/2$, while a root of even multiplicity contributes at least 2. Hence

$$\mu(t) - \frac{o(t)}{2} \geq \frac{z(t)}{2} \geq 4 \cdot 2^{-z(t)}.$$

The last inequality holds for every integer $z(t) \geq 2$, with equality at $z(t) = 2$.

Sum these local inequalities. For a fixed A_i , the total charge over all its real roots is the sum of their multiplicities minus $O_i/2$, and is at most $D_i - O_i/2$. Every omitted real root has positive charge, and nonreal roots only increase the unused degree. Restricting the sum to $t \in T$ thus gives (3). $\square$

Corollary 2.2. *Under the hypotheses of Lemma 2.1, if $D_i \leq k$ for an even integer k , then*

$$\sum_{t \in T} 2^{-z(t)} \leq \frac{r(k-1)}{4}. \quad (4)$$

Proof. A sign-changing polynomial of even degree has at least two odd-multiplicity real roots, because their number is positive and even. For even D_i , this gives $D_i - O_i/2 \leq k - 1$. If D_i is odd, then $D_i \leq k - 1$ and $O_i \geq 1$, which gives the same bound. Substitute in (3). $\square$

3 The upper bound

Proof of the upper bound in Theorem 1.1. Let (1) have a finite real solution set. If some f_i is constant, x_i does not occur in any equation. A single solution would then generate a line of solutions, so the finite solution set must be empty. We may assume all f_i are nonconstant.

Every odd-degree f_i is surjective on $\mathbb{R}$. Its equation therefore imposes no restriction on t , and has at most k solutions in x_i at each parameter.

For an even-degree f_i , multiply both sides of its equation by the sign of its leading coefficient. The resulting left-hand side has positive leading coefficient and a global minimum c_i . Subtracting c_i from both sides rewrites the equation as

$$F_i(x_i) = A_i(t), \quad F_i(\mathbb{R}) = [0, \infty), \quad \deg F_i, \deg A_i \leq k. \quad (5)$$

The feasible parameters form exactly

$$T = \{t \in \mathbb{R} : A_i(t) \geq 0 \text{ for all even-degree left-hand sides}\}.$$

Each feasible parameter has a nonempty finite fiber, since every f_i is nonconstant. Thus T is finite. In particular, the case of no even-degree left-hand sides is impossible here: it would give $T = \mathbb{R}$.

A nonnegative nonconstant polynomial F_i has at most $k/2$ distinct real zeros, since each has even multiplicity. At a positive level, its equation has at most k real roots.

First suppose that one of the A_i is nonzero and nonpositive everywhere. Its real zeros number at most $k/2$; a negative constant has none. Feasibility forces t to be such a zero, and the corresponding coordinate then has at most $k/2$ choices. Therefore

$$|Z_{\mathbb{R}}| \leq \frac{k}{2} \frac{k}{2} k^{m-1} = \frac{k^{m+1}}{4}. \quad (6)$$

For $m \geq 2$, the inequality $k \leq m(k-1)$ shows that this is at most $B(k, m)$.

In the remaining case, delete every everywhere nonnegative A_i from the list of parameter inequalities, including those that vanish identically. This does not change T . We ignore any reductions in fiber size from the deleted equations, which only weakens an upper bound. Every retained A_i changes sign. Let their number be $r \leq m$. There is at least one retained polynomial, since otherwise $T = \mathbb{R}$. For a feasible t , let $z(t)$ count the retained polynomials vanishing there. The fiber bound and Corollary 2.2 give

$$|Z_{\mathbb{R}}| \leq k^m \sum_{t \in T} 2^{-z(t)} \leq \frac{r(k-1)}{4} k^m \leq \frac{m(k-1)}{4} k^m. \quad (7)$$

This proves the upper bound for $m \geq 2$.

For $m = 1$, a nonempty finite solution set requires an even-degree left-hand side. The single normalized A_1 cannot be positive anywhere: continuity would make T infinite. It also cannot vanish identically. Thus it is nonzero and everywhere nonpositive, and (6) is $k^2/4$. $\square$

4 Attainment by the known construction

We include the attaining family from [2, Theorem 3.1] to make the exact extremal statement self-contained. This section supplies an exposition of that previously published construction, not a new lower bound.

Put $k = 2d$. For $m = 1$, the elementary system

$$\prod_{a=1}^d (x_1 - a)^2 = - \prod_{b=1}^d (t - b)^2$$

has precisely the d^2 solutions with $x_1, t \in \{1, \dots, d\}$.

Now let $m \geq 2$ and $N = md$. Define

$$\begin{aligned} A_1(t) &= -(t-d)(t-N) \prod_{a=1}^{d-1} (t-a)^2, \\ A_i(t) &= (t-(i-1)d)(t-id) \prod_{a=(i-1)d+1}^{id-1} (t-a)^2, \quad 2 \leq i \leq m. \end{aligned} \tag{8}$$

Empty products equal one. These polynomials have degree $2d = k$. Their common nonnegative locus is

$$T = \{1, 2, \dots, N\}. \tag{9}$$

Indeed, A_1 is negative for $t < d$ except at $1, \dots, d-1$, and is negative for $t > N$. In each interval $((i-1)d, id)$, $i \geq 2$, the polynomial A_i is negative except at its internal integer roots. All the integers in (9) satisfy every inequality.

At the $m(d-1)$ internal block integers, exactly one A_i vanishes. At the m boundary integers $d, 2d, \dots, md$, exactly two vanish. The first and last boundaries involve A_1 , so this statement also covers $m = 2$ and $d = 1$.

Take

$$F_d(x) = \prod_{a=1}^d (x-a)^2, \quad \eta = \min_{\substack{1 \leq a \leq d \\ \sigma \in \{-1, 1\}}} F_d(a + \sigma/3) > 0,$$

and let

$$M = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq N}} A_i(j) \geq 0, \quad \varepsilon = \frac{\eta}{2(1+M)}.$$

Consider $F_d(x_i) = \varepsilon A_i(t)$. The feasible parameters are exactly (9). At level zero, F_d has d distinct roots. If $A_i(j) > 0$, then $c = \varepsilon A_i(j)$ satisfies $0 < c < \eta$. On each interval $(a-1/3, a+1/3)$, the polynomial $F_d - c$ is positive at both endpoints and negative at a . It has at least two distinct roots in that interval. The intervals are disjoint, and its degree is $2d$; hence it has exactly k distinct real roots in total.

Each internal block parameter therefore contributes dk^{m-1} solutions, and each boundary contributes $d^2 k^{m-2}$. The total is

$$m(d-1)dk^{m-1} + md^2 k^{m-2} = \frac{m(k-1)}{4} k^m.$$

The exceptional-looking case $d = 1, m = 2$ has $M = 0$ and only boundary fibers, and is already included. Together with the upper bound, this completes Theorem 1.1.

5 Limits of the statement and exact checks

The assumption of a finite *entire* real solution set is essential to the present argument. It is not a count of just the isolated points of a system with additional real components. The separation into one variable x_i per equation and a single shared parameter t is also essential.

In fact, the numerical expression $B(k, m)$ cannot be an upper bound for arbitrary sums of squares in $m + 1$ variables. The family in [2] has a quartic example in 18 variables with $17 \cdot 2^{15}$ real zeros. Adding two copies in disjoint variables gives a quartic sum of squares in 36 variables with

$$(17 \cdot 2^{15})^2 = 289 \cdot 2^{30} > 280 \cdot 2^{30} = B(2, 35).$$

This is an application of the product construction already noted in [2], not a new counterexample to the original conjecture.

Although Lemma 2.1 has no parity restriction, Corollary 2.2 uses an even degree cap. For example,

$$A_1(t) = -(t - 1)(t - 2)^2, \quad A_2(t) = (t - 1)t^2$$

have common nonnegative locus $\{0, 1, 2\}$. The endpoint weights are $1/2$ each and the middle weight is $1/4$. Their sum $5/4$ exceeds $2(3 - 1)/4 = 1$. Thus merely substituting an odd cap into (4) is invalid; we make no odd-degree extremal claim.

The accompanying source package contains a rational-arithmetic Python checker and a separately implemented integer sign-transition C++ checker. They agree on the feasible-system counts, maximum weights, and numbers of maximizers in eight finite catalogs. The Python run performs 553,421 checks, including 174,313 finite systems and 96 instances of the known attaining family. Degenerate cases and the preceding scope obstructions are retained as checks. These computations are supplementary tests, not the proof of the universal theorem and not formal certification or independent expert review.

AI tools assisted the author with mathematical exploration, drafting, and exact-check implementation. The author is responsible for the claims and their presentation. This is an unrefereed preprint; no absolute priority claim is made.

References

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- [2] DannyExperiments, *Many Isolated Real Zeros of Sums of Squares*, version 1.0.0, 10 August 2026. Zenodo release entitled *A counterexample to the Ottaviani–Shapiro isolated-zero conjecture*. doi:10.5281/zenodo.21875290. Source repository: <https://github.com/DannyExperiments/ottaviani-shapiro-sos-counterexample>.