

# Finite Equilibria for Rational Inverse-Power Fields in Any Dimension

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## Abstract

We extend the compact-curve method of Enciso and Peralta-Salas for Coulomb equilibria in three dimensions. For distinct real sites, nonzero real charges, and any rational exponent  $s > 1$ , the field  $E_s(x) = \sum_i q_i (x - a_i) |x - a_i|^{-s}$  has only finitely many equilibria at which  $S_s(x) = \sum_i q_i |x - a_i|^{-s}$  is nonzero, in every ambient dimension. In particular, all equilibria are finite in number when the charges have one sign. A scalar projection controls poles at isotropic points at infinity without a three-dimensional cross product. A second local argument treats charge sites without introducing a potential. An algebraic lift gives complex isolation and an explicit, nonsharp Bezout bound. The conclusions include same-sign Coulomb fields in arbitrary dimension and Newtonian point-charge potentials in their natural dimensions. The exponent-one endpoint and the mixed-sign exceptional set admit continua of equilibria.

## 1 Statement and relation to earlier work

Fix pairwise distinct sites  $a_1, \dots, a_N \in \mathbb{R}^n$  and charges  $q_1, \dots, q_N \in \mathbb{R} \setminus \{0\}$ . For  $s > 0$  define, away from the sites,

$$E_s(x) = \sum_{i=1}^N q_i \frac{x - a_i}{|x - a_i|^s}, \quad S_s(x) = \sum_{i=1}^N \frac{q_i}{|x - a_i|^s}. \quad (1)$$

Shapiro's same-sign finiteness question is stated in  $\mathbb{R}^n$  [4, Conjecture 2]. His preceding formula uses the Coulomb exponent  $s = 3$ . In the dimension-dependent Newtonian convention of Gabriellov, Novikov and Shapiro [3], the potential in dimension  $n \geq 3$  is  $\sum_i q_i |x - a_i|^{2-n}$ , whose critical-point equation is  $E_n = 0$ . The result below covers both conventions.

Enciso and Peralta-Salas [1] proved finiteness for  $s = 3$  in  $\mathbb{R}^3$ , including mixed signs where  $S_3 \neq 0$ . Their key construction places position and distance functions on a compact complex curve and controls their poles. We use that construction and its projective squared-distance forms. The changes here are a dimension-independent projection in the pole estimate at infinity and a field-pole argument at a real charge site. These permit rational exponents  $s > 1$  as well. The three-dimensional Coulomb theorem is prior work, not a new claim of this note.

**Theorem 1.** *Let  $s = A/B$ , where  $A > B$  are positive integers, and put*

$$Z_s = \{x \in \mathbb{R}^n \setminus \{a_1, \dots, a_N\} : E_s(x) = 0, S_s(x) \neq 0\}.$$

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Then  $Z_s$  is finite. For  $N \geq 2$ , if  $d = \dim \operatorname{aff}_{\mathbb{R}}\{a_1, \dots, a_N\}$ , then

$$\#Z_s \leq (2B)^N (A(N-1) + 1)^d. \quad (2)$$

For  $N = 1$  the set is empty. If the charges have one sign,  $Z_s$  is the entire equilibrium set.

The numerator and denominator may be chosen relatively prime to improve the displayed bound. No genericity or nondegeneracy assumption is imposed. The estimate is a coarse degree count; for the three-dimensional Coulomb case it does not improve the sharper bound of [1].

## 2 Rigidity on a compact curve

All complex dot products below are bilinear extensions of the real Euclidean product, not Hermitian products. Orders of vector functions mean the minimum of their component orders, and  $\operatorname{ord}_p 0 = +\infty$ .

**Lemma 2.** *Let  $C$  be a smooth connected projective complex curve. Suppose that  $x = (x_1, \dots, x_n)$  and  $u_1, \dots, u_N$  are meromorphic on  $C$ , that no  $u_i$  is identically zero, and that*

$$u_i^{2B} = (x - a_i) \cdot (x - a_i), \quad \sum_i q_i \frac{x - a_i}{u_i^A} = 0. \quad (3)$$

*If  $S = \sum_i q_i/u_i^A$  is not identically zero, then  $x$  and every  $u_i$  are constant.*

Write  $T = \sum_i q_i a_i / u_i^A$ , so that

$$Sx = T. \quad (4)$$

Let  $\operatorname{aff}_{\mathbb{R}}\{a_i\} = a_1 + V$ . Taking normal components in (4) and dividing by  $S$  in the function field gives

$$x \in a_1 + V_{\mathbb{C}}. \quad (5)$$

Zeros of  $S$  at individual points do not obstruct this field identity. When  $N = 1$ , (4) gives  $x = a_1$ , contradicting the nonzero distance-root function. Thus the lemma is vacuous in that case, and we assume  $N \geq 2$  in its proof.

### 2.1 A projective distance calculation

Use homogeneous coordinates  $[X_0 : X : Q]$  on  $\mathbb{P}^{n+1}$  and set

$$Y = \{X \cdot X = X_0 Q, \quad X - a_1 X_0 \in V_{\mathbb{C}}\}, \quad (6)$$

$$L_i = Q - 2a_i \cdot X + |a_i|^2 X_0. \quad (7)$$

As in [1, Section 3], this is the projective closure of the paraboloid embedding of the affine span, and squared distances become linear forms.

**Lemma 3.** *The forms  $L_i$  have no common zero on  $Y$ .*

*Proof.* In the chart  $X_0 = 1$ , a common zero has  $X = a_1 + w$ ,  $w \in V_{\mathbb{C}}$ , and satisfies

$$w \cdot w = 0, \quad 2(a_i - a_1) \cdot w = |a_i - a_1|^2 \quad \text{for every } i.$$

Its imaginary part lies in  $V$  and is orthogonal to a spanning set of  $V$ , so it is zero. Then  $w$  is real, and  $w \cdot w = 0$  implies  $w = 0$ , contradicting the second equation for any  $a_i \neq a_1$ . At  $X_0 = 0$ , the common-zero equations give  $X \in V_{\mathbb{C}}$  and  $Q = 2a_i \cdot X$ . Their differences make  $X$  orthogonal to  $V_{\mathbb{C}}$ . The complexified Euclidean form on  $V_{\mathbb{C}}$  is nondegenerate, whence  $X = 0$  and  $Q = 0$ , which is impossible in projective space.  $\square$

The map  $[1 : x : x \cdot x]$  extends holomorphically to  $C \rightarrow Y$ : locally, remove the common lowest Laurent order of its homogeneous coordinates. Choose a holomorphic representative  $H = (X_0, X, Q)$  near  $p \in C$  with no common zero. The distance equations give

$$L_i(H) = X_0 u_i^{2B}. \quad (8)$$

Put  $m_i = \text{ord}_p u_i$ ,  $m = \min_i m_i$  and  $h = \text{ord}_p X_0$ . By Lemma 3, at least one  $L_i(H)$  is a unit, so

$$h = -2Bm, \quad \text{ord}_p L_i(H) = 2B(m_i - m). \quad (9)$$

In particular  $h \geq 0$ . Let

$$I = \{i : m_i = m\}, \quad J = \{i : m_i > m\}, \quad \mu = -Am,$$

and, for  $K = I, J$ , set  $S_K = \sum_{i \in K} q_i / u_i^A$  and  $T_K = \sum_{i \in K} q_i a_i / u_i^A$ . Then

$$\text{ord}_p S_I \geq \mu, \quad \text{ord}_p T_I \geq \mu. \quad (10)$$

## 2.2 The local estimate

We claim that

$$\text{ord}_p S \geq -Am \quad (11)$$

at every point of  $C$ . There are four cases.

*A finite real image.* Here  $h = m = 0$  and  $z = x(p)$  is real. For  $j \in J$ , the distance identity gives  $z = a_j$ , so there is at most one such index. Suppose  $J = \{j\}$  and set  $w = x - a_j$ . This vector is not identically zero, since otherwise  $u_j$  would vanish identically. If  $k = \text{ord}_p w$ , then  $k \geq 1$  and  $2Bm_j = \text{ord}_p(w \cdot w) \geq 2k$ . Consequently

$$\text{ord}_p (q_j w / u_j^A) = k - Am_j \leq (B - A)m_j < 0. \quad (12)$$

Every other term of the field in (3) is holomorphic at  $p$ . It cannot cancel this vector pole. Thus  $J$  is empty and  $S$  is holomorphic. This argument uses  $A > B$ , but does not use a potential.

*A finite nonreal image.* Again  $h = m = 0$ . Write  $x(p) = v + iw$  with  $v, w \in \mathbb{R}^n$ ,  $w \neq 0$ . For  $j \in J$ , the imaginary part of the squared-distance equation gives  $a_j \cdot w = v \cdot w$ . Projecting (4) onto  $w$  yields

$$(w \cdot x - v \cdot w)S_J = w \cdot T_I - (w \cdot x)S_I. \quad (13)$$

The left coefficient has value  $i|w|^2 \neq 0$  at  $p$  and is therefore a unit. The right side is holomorphic by (10). Hence  $S_J$  and  $S$  are holomorphic.

*The image  $[0 : 0 : 1]$ .* Every  $L_i$  is nonzero at this point. Equation (9) gives  $J = \emptyset$ , and (10) proves (11).

*Any other image at infinity.* Write the chosen representative at  $p$  as  $H(p) = (0, \Xi, \Theta)$ , where  $\Xi \neq 0$  and  $\Xi \cdot \Xi = 0$ . If  $\Xi = v + iw$  with real  $v, w$ , then

$$|v|^2 = |w|^2 > 0, \quad v \cdot w = 0, \quad w \cdot \Xi = i|w|^2 \neq 0. \quad (14)$$

Now  $h > 0$ . Since  $x = X/X_0$ , both  $x$  and  $w \cdot x$  have a pole of order  $h$ . For  $j \in J$ , the equation  $L_j(H)(p) = 0$  gives

$$a_j \cdot w = \frac{1}{2} \text{Im } \Theta =: \beta.$$

The scalar projection of the field identity becomes

$$(w \cdot x - \beta)S_J = w \cdot T_I - (w \cdot x)S_I. \quad (15)$$

Its left coefficient has order  $-h$ , and its right side has order at least  $\mu - h$  by (10). Thus  $\text{ord}_p S_J \geq \mu$ , and the same holds for  $S = S_I + S_J$ . This proves the estimate. Unlike a cross-product reduction, (15) works in every dimension.

### 2.3 End of the rigidity proof

For each  $i$ , (11) implies

$$\text{ord}_p(u_i^A S) \geq A(m_i - m) \geq 0 \quad (p \in C).$$

Thus  $u_i^A S$  is a holomorphic function on a compact connected curve and is constant. It is a nonzero constant because both factors are nonzero in the function field. Hence  $(u_i/u_1)^A$  is a nonzero constant, and connectedness implies  $u_i = c_i u_1$  for constants  $c_i \neq 0$ . It follows that

$$S = u_1^{-A} \sigma, \quad T = u_1^{-A} \tau, \quad \sigma = \sum_i q_i c_i^{-A} \neq 0, \quad \tau = \sum_i q_i c_i^{-A} a_i.$$

Equation (4) gives  $x = \tau/\sigma$ . Finally,  $u_i^{2B} = (x - a_i) \cdot (x - a_i)$  is a nonzero constant, so every  $u_i$  is constant. This completes the proof of Lemma 2.

## 3 Algebraic isolation and a uniform bound

*Proof of Theorem 1.* For  $N = 1$ , the field has no zero away from its site. Assume  $N \geq 2$ . The normal component of the physical balance identity shows that every  $x \in Z_s$  lies in the real affine span of the sites. Identify that span isometrically with  $\mathbb{R}^d$ . A physical point has the unique positive lift  $u_i = |x - a_i|^{1/B}$ . It solves the following system in  $d + N$  complex variables:

$$u_i^{2B} - (x - a_i) \cdot (x - a_i) = 0 \quad (1 \leq i \leq N), \quad (16)$$

$$\sum_i q_i (x - a_i) \prod_{j \neq i} u_j^A = 0. \quad (17)$$

At this lift, all  $u_i$  and the scalar numerator  $Q_u = \sum_i q_i \prod_{j \neq i} u_j^A$  are nonzero.

Suppose the lift is not isolated in the complex zero set. A positive-dimensional algebraic component through it contains an irreducible algebraic curve through it. Normalize the projective closure of that curve. Its coordinate functions  $x, u_i$  are meromorphic on a smooth connected projective curve. No  $u_i$  is identically zero, and  $S = Q_u / \prod_i u_i^A$  is not identically zero, since its value at the physical point is nonzero. Dividing (17) in the function field gives (3). Lemma 2 makes every coordinate constant, contradicting that the image is a curve. Thus each physical lift is an isolated complex zero.

There are  $N$  equations of degree  $2B$  in (16) and  $d$  equations of degree at most  $A(N - 1) + 1$  in (17). The isolated-zero form of affine Bezout gives the product bound (2). It remains applicable when the full system has other, positive-dimensional components. Indeed, any finite collection of its isolated zeros can be enclosed in disjoint neighborhoods. A sufficiently small generic perturbation preserves their positive local intersection multiplicities, while projective Bezout bounds the total by the product of the degrees; see [2, Sections 8 and 10] and the analogous conservation-of-number argument in [1, Lemma 5.2]. Distinct physical points have distinct lifts. Finally, same-sign charges make  $S_s$  strictly of that sign everywhere in the physical domain, proving the last assertion.  $\square$

*Remark 4.* This proof establishes complex isolation directly. It does not infer complex isolation from real isolation, nor does it require a three-dimensional harmonic critical-set dimension estimate. Those distinctions matter when the ambient dimension or the exponent changes.

## 4 Consequences and boundaries

**Corollary 5.** *A finite set of nonzero same-sign point charges has only finitely many equilibria for the denominator-three Coulomb field in every  $\mathbb{R}^n$ ,  $n \geq 1$ . The Newtonian potential  $\sum_i q_i |x - a_i|^{2-n}$  has finitely many critical points for every  $n \geq 3$ . The logarithmic potential in dimension two has finitely many critical points as well.*

*Proof.* Use  $s = 3$ ,  $s = n$ , and  $s = 2$ , respectively. Differentiating the potentials only changes the corresponding field by a nonzero scalar factor.  $\square$

For every positive rational  $\alpha$ , the potential  $\sum_i q_i |x - a_i|^{-\alpha}$  is similarly covered by  $s = \alpha + 2$ . No assertion for general irrational exponents follows from the finite algebraic lift used here.

The restriction on mixed-sign fields is essential. Put charges  $+1$  at  $(1, 1, 0)$  and  $(-1, -1, 0)$  and charges  $-1$  at  $(1, -1, 0)$  and  $(-1, 1, 0)$ . All distances on the perpendicular axis are equal, and  $\sum_i q_i = 0 = \sum_i q_i a_i$ . Therefore both  $E_s$  and  $S_s$  vanish on the entire axis for every exponent  $s$ . This classical example is already recorded in [4, 1].

The endpoint  $s = 1$  also cannot be included: equal positive charges at  $-1$  and  $1$  on the real line have

$$E_1(x) = \frac{x+1}{|x+1|} + \frac{x-1}{|x-1|} = 0 \quad (-1 < x < 1),$$

although  $S_1(x) > 0$ . This is a different model from the denominator-three field on a line. It also explains why the strict inequality in (12) is needed.

This work was developed with AI assistance for literature exploration, algebraic checks and drafting. The proof is presented for mathematical scrutiny as an unrefereed preprint. Its use of the Enciso–Peralta–Salas method is explicit; no absolute priority claim is made. The supplementary exact checks test identities and boundary cases, and are not a formal verification of the general proof.

## References

- [1] A. Enciso and D. Peralta-Salas, *The finiteness conjecture for equilibria of electric fields generated by point charges of one sign*, arXiv:2609.03965v1, 2026. <https://arxiv.org/abs/2609.03965v1>.
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