

On a Question of Bolsinov, Matveev, Miranda and Tabachnikov: Quadratic Poisson Brackets without Polynomial Integrable Systems

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Abstract

Bolsinov, Matveev, Miranda and Tabachnikov asked whether each quadratic Poisson bracket on a vector space admits a polynomial integrable system, that is, $\frac{1}{2}(\dim + \text{corank})$ Poisson-commuting polynomials whose differentials are linearly independent almost everywhere (Question 5.9 in the numbering of arXiv:1804.03737v2; record AMR-020-0507 of the UnsolvedMath collection, where it is labelled Question 5.7); Bolsinov, Izosimov and Tsonev had asked the same question in 2017 for polynomial Poisson brackets of any degree. The answer is no. The log-canonical bracket $\{x, y\} = xy$, $\{y, z\} = \sqrt{2}yz$, $\{z, x\} = \sqrt{3}zx$ on $\mathbb{R}^3$ has generic rank two, and there are no two Poisson-commuting polynomials f, g with $df \wedge dg \neq 0$. The proof is elementary: the leading exponents of commuting polynomials are orthogonal for the skew form of the bracket, and a count of dimensions does the rest; a second proof uses residues on the common level curves of f and g . More generally, a log-canonical bracket $\{x_i, x_j\} = \lambda_{ij}x_i x_j$ on $\mathbb{R}^n$ admits a polynomial integrable system if and only if some maximal isotropic subspace of λ is spanned by vectors with nonnegative integer entries, and then monomials form one; in dimension three this becomes an explicit arithmetic criterion. A log-canonical bracket on $\mathbb{R}^4$ with the constants $1, 2^{1/6}, \dots, 2^{5/6}$ is symplectic on a dense open set and has no polynomial integrable system, so Casimir functions are not the source of the obstruction, and multiplying the three-dimensional example by homogeneous polynomials gives counterexamples of every degree at least two. If the polynomial system is also required to contain polynomial Casimir functions, as in the formulation in the introduction of the 2018 survey, the negative answer is elementary; the substance of the result lies in the formulation of its Section 5, which does not ask for Casimir functions. The questions are due to these authors; for linear brackets the answer is positive by Sadetov's proof of the Mishchenko–Fomenko conjecture; log-canonical brackets, the bracket of two monomials and the description of the polynomial Casimir functions are classical. The contribution of this note is the counterexample with its two proofs, the characterisation for log-canonical brackets and the example on $\mathbb{R}^4$. No printed answer to the question was found in the sources searched; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The question

Let M be a Poisson manifold, with bracket $\{f, g\} = \sum_{i,j} A^{ij} \partial_i f \partial_j g$ in local coordinates. In Section 5 of the survey [BMMT18], Bolsinov, Matveev, Miranda and Tabachnikov call a family of functions an *integrable system* if it consists of $\frac{1}{2}(\dim M + \text{corank } A)$ functions which pairwise Poisson-commute and whose differentials are linearly independent at almost every point, the corank being taken at

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a generic point; equivalently, the differentials span a maximal isotropic subspace for A at a generic point ([BMMT18, §5], lines 984–986). For a symplectic manifold this is the usual count of m integrals on a manifold of dimension $2m$ [BMMT18, §1, lines 181–182]. When M is a vector space with a polynomial bracket, the survey is interested in integrable systems “defined on M by a collection of Poisson commuting polynomials” [BMMT18, §5.2, line 1156]. For a linear (Lie–Poisson) bracket, that is, equation (5.3) of [BMMT18], such a system always exists: this is the Mishchenko–Fomenko conjecture, proved by Sadetov [Sad04] (see also [Bol16]). The authors then ask about brackets of higher degree and pose the following question [BMMT18, Question 5.9]: *does every quadratic Poisson bracket on a vector space admit a polynomial integrable system?*

Here a Poisson bracket on $V = \mathbb{R}^n$ is *quadratic* if all A^{ij} are homogeneous quadratic polynomials in linear coordinates, and a *polynomial integrable system* is an integrable system in the above sense which consists of polynomials. For polynomials $F_1, \dots, F_k$ the differentials are linearly independent almost everywhere if and only if the polynomial k -form $dF_1 \wedge \dots \wedge dF_k$ is not identically zero, since a nonzero polynomial vanishes only on a closed set of measure zero. Nothing in the source restricts the class of quadratic brackets.

Numbering. In [BMMT18] all theorem-like environments share one counter per section. In the arXiv version v2, which we use for all section, line and equation numbers, the question is Question 5.9 (TeX source, lines 1159–1161). It is record AMR-020-0507 of the UnsolvedMath collection [ula26], which reproduces its text verbatim and calls it Question 5.7, counting only problems and questions; in v2, Question 5.7 is a different question, on the algebraic types of pencils.

An earlier formulation. The same question for polynomial Poisson brackets of any degree is Problem 1 of the collection of research problems of Bolsinov, Izosimov and Tsonev [BIT17, Section 2]: does a vector space with a polynomial Poisson structure carry polynomial integrable systems? An integrable system is defined there by the same count $\frac{1}{2}(\dim M + \text{corank } A)$ of commuting functions which are independent almost everywhere [BIT17, formula (1)]. The authors recall that Sadetov’s theorem gives the positive answer for linear brackets and that the problem remains open for many other polynomial brackets. [BMMT18] cites [BIT17] only in another context (line 1111), and its Question 5.9 is the quadratic case of this problem. The results below show that the answer to Problem 1 of [BIT17] can be negative, already for quadratic brackets on $\mathbb{R}^3$, and decide it for log-canonical brackets.

Two formulations. The introduction of [BMMT18] describes, for a Poisson structure of rank $2m$ on a manifold of dimension $2m + r$, integrable systems formed by r Casimir functions and m commuting integrals. For polynomial systems this would require r independent *polynomial* Casimir functions, which is stronger than the definition of Section 5 recalled above (Example 4.1(c)). We use the definition of Section 5; the counterexamples below are counterexamples for both formulations.

1.2 Results

Theorem A. *On $\mathbb{R}^3$ with coordinates x, y, z consider the bracket*

$$\{x, y\} = xy, \quad \{y, z\} = \sqrt{2}yz, \quad \{z, x\} = \sqrt{3}zx.$$

It is a quadratic Poisson bracket of generic rank two, so that an integrable system consists of two functions. There are no polynomials $f, g \in \mathbb{C}[x, y, z]$ with $\{f, g\} = 0$ and $df \wedge dg \not\equiv 0$. In particular this bracket admits no polynomial integrable system, and the answer to [BMMT18, Question 5.9] is negative.

On the dense open set $\{xyz \neq 0\}$ the bracket does have smooth integrable systems, for instance $(|x|^{\sqrt{2}}|y|^{\sqrt{3}}|z|, x)$; the obstruction is polynomiality. Brackets of the form $\{x_i, x_j\} = \lambda_{ij}x_i x_j$ with a real skew-symmetric matrix $\lambda = (\lambda_{ij})$ are called *log-canonical*; see [MO17], where other names used in the literature are also listed. For them we decide exactly when a polynomial integrable system exists.

Let $2s = \text{rank } \lambda$. The generic corank of the bracket is $n - 2s$ (Lemma 2.1), so an integrable system consists of $k = n - s$ functions. We write $\mathbb{N} = \{0, 1, 2, \dots\}$, $\lambda(\alpha, \beta) = \sum_{i,j} \lambda_{ij} \alpha_i \beta_j$ for $\alpha, \beta \in \mathbb{R}^n$, and call a subspace $E \subset \mathbb{R}^n$ *isotropic* if $\lambda(E, E) = 0$; the maximal isotropic subspaces are exactly the isotropic subspaces of dimension $n - s$ (Lemma 2.3).

Theorem B (dimension three). *Let $n = 3$, $\lambda \neq 0$, $k_\lambda = (\lambda_{23}, \lambda_{31}, \lambda_{12})$, and let $\rho \in \{1, 2, 3\}$ be the dimension of the $\mathbb{Q}$ -vector space spanned by $\lambda_{12}, \lambda_{23}, \lambda_{31}$. The log-canonical bracket on $\mathbb{R}^3$ admits a polynomial integrable system if and only if $\rho = 1$, or $\rho = 2$ and the primitive integer vector ν generating the group $\{m \in \mathbb{Z}^3 : m \cdot k_\lambda = 0\}$ has two zero entries or has a positive and a negative entry. For $\rho = 3$ there is none. If a system exists, two monomials form one.*

Theorem C (log-canonical brackets). *For the log-canonical bracket on $\mathbb{R}^n$ the following are equivalent:*

- (i) *it admits a polynomial integrable system, with real or with complex coefficients;*
- (ii) *there are linearly independent $\alpha^{(1)}, \dots, \alpha^{(k)} \in \mathbb{N}^n$ with $\lambda(\alpha^{(i)}, \alpha^{(j)}) = 0$ for all i, j ;*
- (iii) *some maximal isotropic subspace of λ is spanned by vectors of $\mathbb{N}^n$.*

If (ii) holds, the monomials $x^{\alpha^{(1)}}, \dots, x^{\alpha^{(k)}}$ form a polynomial integrable system.

Theorem D. *Let $\theta = 2^{1/6}$ and consider on $\mathbb{R}^4$ the log-canonical bracket*

$$\begin{aligned} \{x_1, x_2\} &= x_1 x_2, & \{x_1, x_3\} &= \theta x_1 x_3, & \{x_1, x_4\} &= \theta^2 x_1 x_4, \\ \{x_2, x_3\} &= \theta^3 x_2 x_3, & \{x_2, x_4\} &= \theta^4 x_2 x_4, & \{x_3, x_4\} &= \theta^5 x_3 x_4. \end{aligned}$$

It is symplectic on the dense open set $\{x_1 x_2 x_3 x_4 \neq 0\}$, so its Casimir functions are constant and an integrable system consists of two functions. There are no polynomials f, g with $\{f, g\} = 0$ and $df \wedge dg \neq 0$.

Further results: if the numbers λ_{ij} ($i < j$) are linearly independent over $\mathbb{Q}$ and $n \geq 3$, there is no polynomial integrable system (Corollary 5.1); if they are rational multiples of one real number, monomials always form one (Proposition 4.2), so counterexamples need constants with irrational ratios, as in Theorems A and D; multiplying the bracket of Theorem A by a homogeneous polynomial gives counterexamples of every degree $d \geq 2$ (Proposition 6.2); and Remark 6.1 gives a second proof of Theorem A, by residues.

1.3 What is known and what is new

The question is due to Bolsinov, Izosimov and Tsonev [BIT17, Problem 1], for polynomial brackets, and to Bolsinov, Matveev, Miranda and Tabachnikov [BMMT18], for quadratic brackets; for linear brackets the positive answer is Sadetov's theorem [Sad04]. In the formulation of the introduction of [BMMT18], with polynomial Casimir functions, the negative answer is elementary: a log-canonical bracket on $\mathbb{R}^3$ whose kernel contains no nonzero integer vector has no non-constant polynomial Casimir function (Lemma 2.2), for instance the bracket with $(\lambda_{12}, \lambda_{23}, \lambda_{31}) = (1, 1, \sqrt{2})$. That bracket nevertheless has the polynomial integrable system (xz, y) in the sense of Section 5 (Example 4.1(c)). The substance of the present note is therefore the Section 5 formulation, where one has to exclude pairs of commuting polynomials which are not Casimir functions; this is what the leading-exponent argument of Section 3 does.

Log-canonical brackets and their Casimir functions are classical. The bracket of two monomials (Lemma 2.2(a)) is [MO17, Lemma 3], stated there for Laurent monomials (see also [Ove19, Lemma 1]).

The description of the polynomial Casimir functions (Lemma 2.2(b)) is the polynomial counterpart of the criterion of Fish and Jordan for the Poisson centre of the Laurent polynomial algebra to be trivial [FJ18, Proposition 7.2], which they derive from a lemma of Vancliff. That an independent commuting family has at most $n - s$ members is standard (see [LLL24, Lemma 2.2]); we use it in the elementary form of Lemma 2.3. Leading and lowest terms are a common tool in this area; for instance Li, Li and Lu [LLL24, Lemma 2.12] pass from pairs of functions with log-canonical bracket to their lowest-degree terms, and Machacek and Ovenhouse [MO17] use iterated Laurent series. What is, to our knowledge, not in print is the negative answer to the question, that is, Theorem A with its two proofs, and Theorems B to D and Corollary 5.1; Propositions 4.2 and 6.2 are simple complements. The searches are described in the paragraph “Scope and priority” at the end; no priority is claimed.

2 Log-canonical brackets

Throughout, $\lambda = (\lambda_{ij})$ is a real skew-symmetric $n \times n$ matrix of rank $2s$, $K = \ker \lambda$, $k = n - s$, and $T = \{x \in \mathbb{R}^n : x_1 \cdots x_n \neq 0\}$. For $\alpha \in \mathbb{N}^n$ we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $e_1, \dots, e_n$ is the standard basis.

Lemma 2.1. (a) *The formula $\{f, g\} = \sum_{i,j} \lambda_{ij} x_i x_j \partial_i f \partial_j g$ defines a quadratic Poisson bracket on $C^\infty(\mathbb{R}^n)$ with $\{x_i, x_j\} = \lambda_{ij} x_i x_j$; it preserves $\mathbb{R}[x]$, and its $\mathbb{C}$ -bilinear extension is a Poisson bracket on $\mathbb{C}[x]$. (b) *Its Poisson tensor is $A(x) = D\lambda D$ with $D = \text{diag}(x_1, \dots, x_n)$. It has rank $2s$ on T and rank at most $2s$ elsewhere, so the generic corank is $n - 2s$ and an integrable system consists of $\frac{1}{2}(n + n - 2s) = k$ functions.**

Proof. (a) Bilinearity, skew-symmetry and the Leibniz rule are clear. The Jacobiator $J(f, g, h) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$ is skew-symmetric, and expanding with the Leibniz rule shows that it is a derivation in each argument; hence $J(f, g, h) = \sum_{i,j,l} J(x_i, x_j, x_l) \partial_i f \partial_j g \partial_l h$, and it suffices to treat coordinates. Since $\{x_i, x_j x_l\} = (\lambda_{ij} + \lambda_{il}) x_i x_j x_l$, we get $\{x_i, \{x_j, x_l\}\} = \lambda_{jl}(\lambda_{ij} + \lambda_{il}) x_i x_j x_l$, and

$$\frac{J(x_i, x_j, x_l)}{x_i x_j x_l} = \lambda_{jl} \lambda_{ij} + \lambda_{jl} \lambda_{il} + \lambda_{li} \lambda_{jl} + \lambda_{li} \lambda_{ji} + \lambda_{ij} \lambda_{li} + \lambda_{ij} \lambda_{lj} = 0,$$

because the second and third, the fourth and fifth, and the sixth and first terms cancel by skew-symmetry. The same computation applies in $\mathbb{C}[x]$. (b) $A^{ij} = \lambda_{ij} x_i x_j$, so $A = D\lambda D$; on T the matrix D is invertible, and T is dense. $\square$

Lemma 2.2. (a) $\{x^\alpha, x^\beta\} = \lambda(\alpha, \beta) x^{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{N}^n$. (b) *A polynomial $C = \sum_m c_m x^m \in \mathbb{C}[x]$ is a Casimir function, that is, $\{x_i, C\} = 0$ for all i , if and only if $\lambda m = 0$ for every m with $c_m \neq 0$. In particular, if $K \cap \mathbb{Z}^n = \{0\}$, every polynomial Casimir function is constant.*

Proof. (a) $x_i \partial_i x^\alpha = \alpha_i x^\alpha$, hence $\{x^\alpha, x^\beta\} = \sum_{i,j} \lambda_{ij} (x_i \partial_i x^\alpha)(x_j \partial_j x^\beta) = \lambda(\alpha, \beta) x^{\alpha+\beta}$. (b) By (a), $\{x_i, x^m\} = (\lambda m)_i x_i x^m$, and different m give different monomials $x_i x^m$. Since $\{f, C\} = \sum_i \partial_i f \{x_i, C\}$, the conditions $\{x_i, C\} = 0$ say that C is a Casimir function. $\square$

The computation in (a) also holds on T for real exponents, with $|x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}$ in place of x^α ; for the bracket of Theorem A it shows that $|x|^{\sqrt{2}} |y|^{\sqrt{3}} |z|$ is a Casimir function on T , which gives the smooth system mentioned after Theorem A.

Lemma 2.3. (a) *Every isotropic subspace $E \subset \mathbb{R}^n$ has $\dim E \leq n - s$, and if $\dim E = n - s$ then $K \subset E$. Every isotropic subspace is contained in one of dimension $n - s$; so the maximal isotropic subspaces are the isotropic subspaces of dimension $n - s$. (b) Let $n = 3$ and $\lambda \neq 0$, and put $k_\lambda = (\lambda_{23}, \lambda_{31}, \lambda_{12})$. Then $K = \mathbb{R} k_\lambda$,*

$$\lambda(\alpha, \beta) = k_\lambda \cdot (\alpha \times \beta) \quad (\alpha, \beta \in \mathbb{R}^3),$$

and the maximal isotropic subspaces are the planes which contain k_λ .

Proof. (a) Since $\lambda(u, \kappa) = u^\top \lambda \kappa = 0$ for $\kappa \in K$, λ induces a nondegenerate skew-symmetric form on $W = \mathbb{R}^n/K$, of dimension $2s$. The image $\bar{E}$ of E in W is isotropic, so $\dim \bar{E} \leq s$, and $\dim E = \dim \bar{E} + \dim(E \cap K) \leq s + (n - 2s)$. If $\dim E = n - s$, then $E + K$ is isotropic and contains E , so $E + K = E$ by the bound. Moreover $\bar{E}$ lies in a Lagrangian subspace $\bar{L}$ of W , and the preimage of $\bar{L}$ is isotropic of dimension $n - s$ and contains E . (b) $(\lambda k_\lambda)_1 = \lambda_{12}\lambda_{31} + \lambda_{13}\lambda_{12} = 0$, and similarly for the other two entries; as $\text{rank } \lambda = 2$ and $k_\lambda \neq 0$, $K = \mathbb{R}k_\lambda$. Both sides of the displayed formula are bilinear, and they agree on pairs of basis vectors, e.g. $\lambda(e_1, e_2) = \lambda_{12} = k_\lambda \cdot e_3$. A plane with basis α, β is isotropic if and only if $\lambda(\alpha, \beta) = 0$, that is, if and only if k_λ is orthogonal to the normal vector $\alpha \times \beta$, that is, k_λ lies in the plane; and $n - s = 2$. $\square$

3 Leading exponents

Fix a total order $\prec$ on $\mathbb{N}^n$ which is compatible with addition, that is, $\alpha \prec \beta$ implies $\alpha + \gamma \prec \beta + \gamma$ for all $\gamma \in \mathbb{N}^n$; the graded lexicographic order is an example. For $p \in \mathbb{C}[x] \setminus \{0\}$ let $\text{LE}(p)$ be the largest exponent in the support of p and $\text{lc}(p)$ its coefficient; for a set S of polynomials, $\text{LE}(S) = \{\text{LE}(p) : p \in S \setminus \{0\}\}$.

Lemma 3.1. *Let p and q be nonzero polynomials. The coefficient of the monomial $x^{\text{LE}(p)+\text{LE}(q)}$ in $\{p, q\}$ is $\text{lc}(p) \text{lc}(q) \lambda(\text{LE}(p), \text{LE}(q))$. In particular, if $\{p, q\} = 0$, then $\lambda(\text{LE}(p), \text{LE}(q)) = 0$.*

Proof. Put $\alpha_0 = \text{LE}(p)$ and $\beta_0 = \text{LE}(q)$. By Lemma 2.2(a), $\{p, q\} = \sum_{\alpha, \beta} p_\alpha q_\beta \lambda(\alpha, \beta) x^{\alpha+\beta}$, the sum running over the supports. If $\alpha \preceq \alpha_0$, $\beta \preceq \beta_0$ and $\alpha \prec \alpha_0$, then $\alpha + \beta \prec \alpha_0 + \beta \preceq \alpha_0 + \beta_0$ by compatibility with addition, and similarly if $\beta \prec \beta_0$. So only the pair (α_0, β_0) contributes to the monomial $x^{\alpha_0+\beta_0}$. $\square$

Lemma 3.2. *For every finite-dimensional subspace $W \subset \mathbb{C}[x]$, $|\text{LE}(W)| = \dim W$.*

Proof. Write the elements of W as coefficient vectors indexed by the finitely many exponents which occur in W , in decreasing order. Gaussian elimination gives a basis $w_1, \dots, w_m$ of W in echelon form: the $\text{LE}(w_i)$ are pairwise distinct, and w_j has coefficient 0 at every exponent above $\text{LE}(w_j)$. If $w = \sum_i c_i w_i \neq 0$ and i is the index with $c_i \neq 0$ and $\text{LE}(w_i)$ largest, then $\text{LE}(w) = \text{LE}(w_i)$. Hence $\text{LE}(W) = \{\text{LE}(w_1), \dots, \text{LE}(w_m)\}$. $\square$

Lemma 3.3. *Let $P \subset \mathbb{C}[x]$ be a subalgebra which contains t algebraically independent polynomials, and let $E_P \subset \mathbb{R}^n$ be the real span of $\text{LE}(P)$. Then $\dim E_P \geq t$.*

Proof. Let $r = \dim E_P$ and $P_{\leq d} = \{p \in P : \deg p \leq d\}$. Choose coordinates $i_1 < \dots < i_r$ such that $v \mapsto (v_{i_1}, \dots, v_{i_r})$ is injective on E_P ; this is possible because some $r \times r$ minor of a matrix whose rows form a basis of E_P is nonzero. The elements of $\text{LE}(P_{\leq d})$ lie in $E_P \cap \mathbb{N}^n$ and have entries at most d , so this map sends them injectively into $\{0, \dots, d\}^r$, and Lemma 3.2 gives $\dim P_{\leq d} = |\text{LE}(P_{\leq d})| \leq (d+1)^r$. On the other hand, if $p_1, \dots, p_t \in P$ are algebraically independent of degree at most D , the products $p_1^{a_1} \dots p_t^{a_t}$ with $a_1 + \dots + a_t \leq \lfloor d/D \rfloor$ are linearly independent elements of $P_{\leq d}$, and there are at least $\lfloor d/D \rfloor^t / t!$ of them. Letting $d \rightarrow \infty$ gives $t \leq r$. $\square$

Proposition 3.4. *Let $P \subset \mathbb{C}[x]$ be a subalgebra with $\{p, q\} = 0$ for all $p, q \in P$, and suppose that P contains t algebraically independent polynomials. Then E_P is an isotropic subspace spanned by vectors of $\mathbb{N}^n$, and $t \leq \dim E_P \leq n - s$.*

Proof. By Lemma 3.1, $\lambda(\alpha, \beta) = 0$ for all $\alpha, \beta \in \text{LE}(P)$, hence $\lambda(E_P, E_P) = 0$ by bilinearity. The bounds are Lemma 3.3 and Lemma 2.3(a). $\square$

Proof of Theorem C. (ii) $\Leftrightarrow$ (iii). The span of the vectors in (ii) is isotropic of dimension k , hence maximal by Lemma 2.3(a). Conversely, a maximal isotropic subspace has dimension k , and if it is spanned by vectors of $\mathbb{N}^n$, then k linearly independent ones among them satisfy (ii).

(ii) $\Rightarrow$ (i). The monomials $x^{\alpha^{(i)}}$ commute by Lemma 2.2(a). On T we have $dx^\alpha = x^\alpha \sum_j \alpha_j dx_j/x_j$, so

$$dx^{\alpha^{(1)}} \wedge \cdots \wedge dx^{\alpha^{(k)}} = x^{\alpha^{(1)} + \cdots + \alpha^{(k)}} \sum_J \det(\alpha_J) \bigwedge_{j \in J} \frac{dx_j}{x_j},$$

where J runs over the k -element subsets of $\{1, \dots, n\}$ and $\det(\alpha_J)$ is the minor on the columns J of the $k \times n$ matrix with rows $\alpha^{(1)}, \dots, \alpha^{(k)}$. Some minor is nonzero, so the form is not identically zero, and by Lemma 2.1(b) the monomials form a polynomial integrable system.

(i) $\Rightarrow$ (iii). Let $F_1, \dots, F_k \in \mathbb{C}[x]$ commute pairwise with $dF_1 \wedge \cdots \wedge dF_k \neq 0$. They are algebraically independent: if $\Phi(F_1, \dots, F_k) = 0$ for a nonzero $\Phi \in \mathbb{C}[y_1, \dots, y_k]$ of least degree, then Φ is not constant and, by minimality, $(\partial\Phi/\partial y_i)(F) \neq 0$ for some i ; wedging $\sum_j (\partial\Phi/\partial y_j)(F) dF_j = 0$ with the dF_l , $l \neq i$, gives $(\partial\Phi/\partial y_i)(F) dF_1 \wedge \cdots \wedge dF_k = 0$, which is impossible in the polynomial ring. The subalgebra $P = \mathbb{C}[F_1, \dots, F_k]$ is Poisson-commutative by the Leibniz rule, so by Proposition 3.4 the space E_P is isotropic, spanned by vectors of $\mathbb{N}^n$, and of dimension k , hence maximal. $\square$

Proof of Theorem A. By Lemma 2.1 the bracket is a quadratic Poisson bracket, and since $\lambda \neq 0$ we have $s = 1$ and $k = 2$. Suppose that $f, g \in \mathbb{C}[x, y, z]$ commute and $df \wedge dg \neq 0$. By Theorem C some maximal isotropic plane E is spanned by vectors of $\mathbb{N}^3$; let $\alpha, \beta \in E \cap \mathbb{N}^3$ be linearly independent. Here $k_\lambda = (\sqrt{2}, \sqrt{3}, 1)$, so by Lemma 2.3(b) the vector $m = \alpha \times \beta \in \mathbb{Z}^3 \setminus \{0\}$ satisfies $m_1\sqrt{2} + m_2\sqrt{3} + m_3 = 0$. But $1, \sqrt{2}, \sqrt{3}$ are linearly independent over $\mathbb{Q}$: if $p + q\sqrt{2} + r\sqrt{3} = 0$ with rational p, q, r , then squaring $q\sqrt{2} + r\sqrt{3} = -p$ gives $2qr\sqrt{6} = p^2 - 2q^2 - 3r^2$, so $qr = 0$ because $\sqrt{6}$ is irrational, and then $p = q = r = 0$ because $\sqrt{2}$ and $\sqrt{3}$ are irrational. Hence $m = 0$, a contradiction. $\square$

Concretely, the argument says that all leading exponents of the algebra $\mathbb{C}[f, g]$ would lie on one line, so that $\mathbb{C}[f, g]$ would contain at most $d + 1$ linearly independent elements of degree at most d , whereas the products $f^i g^j$ give of the order of d^2 of them.

4 Dimension three

Proof of Theorem B. By Theorem C and Lemma 2.3(b), a polynomial integrable system exists if and only if some plane which contains k_λ is spanned by vectors of $\mathbb{N}^3$, that is, if and only if there are linearly independent $\alpha, \beta \in \mathbb{N}^3$ with $k_\lambda \cdot (\alpha \times \beta) = 0$; and then two monomials form one. For such α, β the vector $\alpha \times \beta$ is a nonzero element of $L = \{m \in \mathbb{Z}^3 : m \cdot k_\lambda = 0\}$, and the plane is its orthogonal complement. The homomorphism $m \mapsto m \cdot k_\lambda$ from $\mathbb{Z}^3$ to $\mathbb{R}$ has as image the subgroup generated by $\lambda_{12}, \lambda_{23}, \lambda_{31}$, which is free of rank ρ ; so L is free of rank $3 - \rho$.

If $\rho = 3$, then $L = \{0\}$ and there is no system.

If $\rho = 1$, then $k_\lambda = \tau m$ with $\tau \in \mathbb{R} \setminus \{0\}$ and a primitive $m \in \mathbb{Z}^3$. Write $m = m^+ - m^-$ with $m^\pm \in \mathbb{N}^3$ of disjoint supports. If $m^+ \neq 0 \neq m^-$, these two vectors are linearly independent and k_λ lies in their span. Otherwise $\pm m \in \mathbb{N}^3$, and $\alpha = \pm m$, $\beta = e_i$ with $e_i \notin \mathbb{R}m$ will do.

If $\rho = 2$, then $L = \mathbb{Z}\nu$. A plane spanned by linearly independent vectors of $\mathbb{N}^3$ contains k_λ if and only if it is $\nu^\perp$, so a system exists if and only if $\nu^\perp \cap \mathbb{N}^3$ contains two linearly independent vectors. If ν has no zero entry and all its entries have the same sign, then $\nu \cdot v = 0$ with $v \in \mathbb{N}^3$ forces $v = 0$. If ν has exactly one zero entry, say $\nu_3 = 0$, and $\nu_1 \nu_2 > 0$, then $\nu^\perp \cap \mathbb{N}^3 \subset \mathbb{N}e_3$. If $\nu = \pm e_i$, then $\nu^\perp$ contains the two other basis vectors. If $\nu_i > 0 > \nu_j$ for some i, j and l is the third index, then $v = -\nu_j e_i + \nu_i e_j \in \nu^\perp \cap \mathbb{N}^3$, and so is $w = \nu_l e_j - \nu_j e_l$ if $\nu_l \geq 0$, and $w = -\nu_l e_i + \nu_i e_l$ if $\nu_l < 0$; the

l -th entry of w is positive and that of v is zero, so v and w are linearly independent. These cases are exhaustive. $\square$

Example 4.1. Write $(a, b, c) = (\lambda_{12}, \lambda_{23}, \lambda_{31})$, so that $\{x, y\} = axy$, $\{y, z\} = byz$, $\{z, x\} = czx$ and $k_\lambda = (b, c, a)$.

- (a) $(a, b, c) = (1, 2, 3)$: $\rho = 1$, $k_\lambda = (2, 3, 1)$; the Casimir function x^2y^3z and x form a polynomial integrable system.
- (b) $(1, \sqrt{2}, -1 - \sqrt{2})$: $k_\lambda = (\sqrt{2}, -1 - \sqrt{2}, 1)$ and $\nu = (1, 1, 1)$; there is no polynomial integrable system.
- (c) $(1, 1, \sqrt{2})$: $k_\lambda = (1, \sqrt{2}, 1)$ and $\nu = (1, 0, -1)$. The pair (xz, y) is a polynomial integrable system, since $\{xz, y\} = (a - b)xyz = 0$ and the exponents are independent; but $K = \mathbb{R}(1, \sqrt{2}, 1)$ contains no nonzero integer vector, so every polynomial Casimir function is constant (Lemma 2.2(b)). This bracket has a polynomial integrable system in the sense of Section 5 of [BMMT18], but none formed by a polynomial Casimir function and one further integral.
- (d) $(\sqrt{2}, -2, 1)$: $k_\lambda = (-2, 1, \sqrt{2})$ and $\nu = (1, 2, 0)$; there is no polynomial integrable system.

Proposition 4.2. *Suppose that the entries of λ are rational multiples of one real number. (a) The log-canonical bracket on $\mathbb{R}^n$ admits a polynomial integrable system formed by monomials whose exponents have positive entries. (b) If $n = 4$ and $\det \lambda \neq 0$, two monomials which form a polynomial integrable system can be given explicitly: x_i and x_j if $\lambda_{ij} = 0$ for some $i \neq j$, and otherwise x_i and x^β , where row i has entries $\lambda_{ij} > 0 > \lambda_{il}$ and $\beta = c(-\lambda_{il}e_j + \lambda_{ij}e_l) \in \mathbb{N}^4$ for a suitable $c > 0$.*

So counterexamples need constants with irrational ratios, in every dimension, as in Theorems A and D. For $n = 3$, part (a) is also the case $\rho = 1$ of Theorem B.

Proof. (a) Call a subspace of $\mathbb{R}^n$ rational if it is spanned by vectors of $\mathbb{Q}^n$. Since λ is a real multiple of a matrix with rational entries, the space $E^\lambda = \{v : \lambda(E, v) = 0\}$ is rational whenever E is. Start with the isotropic line E spanned by $p = (1, \dots, 1)$. If E is isotropic, rational and of dimension less than $n - s$, then by Lemma 2.3(a) it lies in an isotropic subspace E' of dimension $n - s$, and $E' \subset E^\lambda$; so the rational space E^λ is larger than E and contains a vector $v \in \mathbb{Q}^n \setminus E$, and $E + \mathbb{R}v$ is again isotropic, because $\lambda(v, v) = 0$, and rational. After finitely many steps E is maximal isotropic, rational and contains p . Let $p, b_2, \dots, b_k$ be a basis of E by vectors of $\mathbb{Q}^n$, and let $\varepsilon > 0$ be rational and so small that the vectors $p + \varepsilon b_j$ have positive entries. Multiplying $p, p + \varepsilon b_2, \dots, p + \varepsilon b_k$ by a common denominator gives linearly independent vectors of $\mathbb{N}^n$ with positive entries which span E , and Theorem C applies.

(b) Here $k = 2$. If $\lambda_{ij} = 0$ for some $i \neq j$, then x_i and x_j commute. If some row i has entries $\lambda_{ij} > 0 > \lambda_{il}$, then, the ratio of these two entries being rational, $\beta = c(-\lambda_{il}e_j + \lambda_{ij}e_l)$ lies in $\mathbb{N}^4$ for a suitable $c > 0$; it satisfies $\lambda(e_i, \beta) = c(-\lambda_{ij}\lambda_{il} + \lambda_{il}\lambda_{ij}) = 0$ and is linearly independent of e_i , because $\beta_i = 0$; so x_i and x^β commute. If neither happens, the off-diagonal entries of each row are nonzero and of one sign. Two different rows i, j cannot both be positive, because $\lambda_{ji} = -\lambda_{ij}$, and they cannot both be negative; but there are four rows. In each case Theorem C applies. $\square$

5 Dimension four and generic constants

Proof of Theorem D. The bracket is Poisson by Lemma 2.1. Its Pfaffian is $\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = \theta^5 - \theta^5 + \theta^5 = \theta^5$, so $\det \lambda = \theta^{10} \neq 0$ and $\det A(x) = \theta^{10}(x_1x_2x_3x_4)^2$, which does not vanish on T . So the bracket is symplectic on the dense set T ; a Casimir function has zero differential

on T , hence everywhere, and is constant; and $s = k = 2$. Suppose that f and g commute and $df \wedge dg \neq 0$. By Theorem C there are linearly independent $\alpha, \beta \in \mathbb{N}^4$ with $\lambda(\alpha, \beta) = 0$. With the integers $P_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$, not all zero,

$$\lambda(\alpha, \beta) = \sum_{i < j} \lambda_{ij} P_{ij} = P_{12} + P_{13}\theta + P_{14}\theta^2 + P_{23}\theta^3 + P_{24}\theta^4 + P_{34}\theta^5.$$

The polynomial $X^6 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion at the prime 2, so it is the minimal polynomial of θ , and $1, \theta, \dots, \theta^5$ are linearly independent over $\mathbb{Q}$. Hence $\lambda(\alpha, \beta) \neq 0$, a contradiction. $\square$

The same argument gives the following.

Corollary 5.1. *Let $n \geq 3$, and suppose that the $n(n-1)/2$ numbers λ_{ij} , $i < j$, are linearly independent over $\mathbb{Q}$. Then the log-canonical bracket on $\mathbb{R}^n$ admits no polynomial integrable system; indeed no two linearly independent integer vectors are λ -orthogonal.*

Proof. Since $2s \leq n$, we have $k = n - s \geq n/2 > 1$. By Theorem C, a system would give linearly independent $\alpha, \beta \in \mathbb{N}^n$ with $0 = \lambda(\alpha, \beta) = \sum_{i < j} \lambda_{ij} P_{ij}$, where the integers $P_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$ are not all zero. $\square$

The brackets of Theorems A and D are instances: there $(\lambda_{12}, \lambda_{13}, \lambda_{23}) = (1, -\sqrt{3}, \sqrt{2})$, and $1, \theta, \dots, \theta^5$, respectively. So within the log-canonical family the brackets without polynomial integrable system include the complement of a countable union of hyperplanes of the space of skew-symmetric matrices. For $n = 2$ and $\lambda \neq 0$, on the other hand, $k = 1$ and x_1 alone is a polynomial integrable system.

6 Further remarks

Remark 6.1 (a second proof of Theorem A). The following argument uses the geometry of the common level curves of f and g instead of leading exponents. Write $(a, b, c) = (\lambda_{12}, \lambda_{23}, \lambda_{31})$, assume only that a, b, c are linearly independent over $\mathbb{Q}$, and put $\kappa = b y z dx + c z x dy + a x y dz$. A check on pairs of coordinates gives $\{f, g\} dx \wedge dy \wedge dz = df \wedge dg \wedge \kappa$. Let $f, g \in \mathbb{C}[x, y, z]$ commute with $df \wedge dg \neq 0$.

(1) As in the proof of Theorem C, f and g are algebraically independent, so $F = (f, g): \mathbb{C}^3 \rightarrow \mathbb{C}^2$ is dominant. By standard results of algebraic geometry (Chevalley's theorem on the image of a morphism, generic smoothness in characteristic zero, and the dimension of fibres; see [Har77]), there is a nonempty Zariski-open set $U \subset \mathbb{C}^2$ such that for $(s, t) \in U$ the fibre $F^{-1}(s, t)$ is a nonempty nonsingular curve on which $df \wedge dg$ vanishes nowhere, and none of whose irreducible components lies in a coordinate plane.

(2) Let Γ be such a component and $p \in \Gamma$. From $df \wedge dg \wedge \kappa = 0$ and $df_p \wedge dg_p \neq 0$ we get $\kappa_p \in \text{span}(df_p, dg_p)$, so κ vanishes on $T_p \Gamma = \ker df_p \cap \ker dg_p$. Hence the restriction of $\kappa/(xyz) = b dx/x + c dy/y + a dz/z$ to the dense open subset of Γ where $xyz \neq 0$ is zero, and so is its restriction to Γ as a rational 1-form.

(3) On the nonsingular projective model $\tilde{\Gamma}$ of Γ , a compact connected Riemann surface, the coordinates restrict to nonzero meromorphic functions $\bar{x}, \bar{y}, \bar{z}$. Since $\text{Res}_P(dh/h) = \text{ord}_P(h)$, taking residues in $b d\bar{x}/\bar{x} + c d\bar{y}/\bar{y} + a d\bar{z}/\bar{z} = 0$ gives $b \text{ord}_P \bar{x} + c \text{ord}_P \bar{y} + a \text{ord}_P \bar{z} = 0$ at every point P , so all these orders vanish. A meromorphic function without zeros and poles on a compact connected Riemann surface is constant; so $\bar{x}, \bar{y}, \bar{z}$ are constant and Γ is a point, a contradiction.

Proposition 6.2. *Let $\{\cdot, \cdot\}_0$ be the bracket of Theorem A, let $d \geq 2$, and let $\varphi \in \mathbb{R}[x, y, z]$ be a nonzero homogeneous polynomial of degree $d-2$. Then $\{f, g\}_\varphi = \varphi \{f, g\}_0$ is a Poisson bracket on $\mathbb{R}^3$ whose coefficients are homogeneous polynomials of degree d , and it admits no polynomial integrable system.*

Proof. Expanding with the Leibniz rule, the Jacobiators satisfy $J_\varphi(f, g, h) = \varphi^2 J_0(f, g, h) + \varphi S(f, g, h)$ with

$$S(f, g, h) = \{f, \varphi\}_0 \{g, h\}_0 + \{g, \varphi\}_0 \{h, f\}_0 + \{h, \varphi\}_0 \{f, g\}_0.$$

Here $J_0 = 0$ by Lemma 2.1, and S is skew-symmetric and a derivation in each argument, so on $\mathbb{R}^3$ it is determined by $S(x, y, z)$. With $a_1 = \{y, z\}_0$, $a_2 = \{z, x\}_0$, $a_3 = \{x, y\}_0$ we have $\{x, \varphi\}_0 = a_3 \varphi_y - a_2 \varphi_z$, $\{y, \varphi\}_0 = a_1 \varphi_z - a_3 \varphi_x$ and $\{z, \varphi\}_0 = a_2 \varphi_x - a_1 \varphi_y$, hence $S(x, y, z) = a_1(a_3 \varphi_y - a_2 \varphi_z) + a_2(a_1 \varphi_z - a_3 \varphi_x) + a_3(a_2 \varphi_x - a_1 \varphi_y) = 0$. So $\{\cdot, \cdot\}_\varphi$ is Poisson. Since $\mathbb{C}[x, y, z]$ is an integral domain and $\varphi \neq 0$, two polynomials commute for $\{\cdot, \cdot\}_\varphi$ if and only if they commute for $\{\cdot, \cdot\}_0$; the generic rank is again two, and Theorem A applies. $\square$

Thus for every degree $d \geq 2$ there is a homogeneous polynomial Poisson bracket of degree d on $\mathbb{R}^3$ without a polynomial integrable system, while for linear brackets ($d = 1$) there always is one, by Sadetov's theorem [Sad04]. For homogeneous brackets the dividing line therefore lies between degree one and degree two.

Remark 6.3 (what is not covered). We give no criterion for quadratic brackets which are not log-canonical; the question needs only one counterexample. We do not consider rational, analytic or smooth integrals: smooth integrable systems exist on T (Section 1.2), while both proofs use polynomiality, through the finite supports in Lemma 3.1 and the growth count in Lemma 3.3, and through the algebraic curves in Remark 6.1. Problem 5.8 of [BMMT18], on pencils of a quadratic and a linear bracket, is a separate question and is not treated. Problem 5(c) of [BIT17] remarks that a polynomial Poisson structure of rank two all of whose polynomial Hamiltonians are chaotic would give a Poisson algebra without a complete commutative subalgebra; the bracket of Theorem A gives such an algebra for an arithmetic reason and says nothing about chaos, so it does not answer Problem 5(c).

7 Computations and reproducibility

No proof in this note depends on a computation. The computations check the finite identities and constants used in the proofs, and test the theorems in bounded degree. The reproducibility package has five parts.

The program of this note, `check_note.py` (standard library only; a few seconds; it ends with **TOTAL failures:** 0), checks: (A) that $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ is a $\mathbb{Q}$ -basis of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, by the irreducibility of $t^4 - 10t^2 + 1$; that $X^6 - 2$ is Eisenstein at 2; and, for Theorem D, the Pfaffian θ^5 , $\det \lambda = \theta^{10}$ and $\det A(x) = \theta^{10}(x_1 x_2 x_3 x_4)^2$; (B) the Jacobi identity for symbolic λ ($n = 3, 4, 5$) on coordinate triples and on random polynomial triples, Lemma 2.2(a), $\lambda k_\lambda = 0$ and $\lambda(\alpha, \beta) = k_\lambda \cdot (\alpha \times \beta)$, the identity $\{f, g\} dx \wedge dy \wedge dz = df \wedge dg \wedge \kappa$ of Remark 6.1, the identity $S(x, y, z) = 0$ and the Jacobi identity in Proposition 6.2, and the pairs of Example 4.1; (C) the identity of Lemma 3.1 on random pairs, for the brackets of Theorems A and D under the graded lexicographic, lexicographic and graded reverse lexicographic orders, and, as a control, under an order which is not compatible with addition, where it fails, as it must; (D) the numbers of commuting pairs of monomials of degree at most six with linearly independent exponents: 0 for the bracket of Theorem A and 191,086,0 for the brackets of Example 4.1(a)–(d), and, in degree at most four on $\mathbb{R}^4$, 0 for Theorem D and 156 for $\lambda_{ij} = 1$; (E) Theorem B for all 578 primitive vectors $\nu \in [-4, 4]^3$ (for 402 of them a system exists): the criterion, the vectors constructed in the proof and a search over exponents at most 4 agree; and the case

$\rho = 1$ for all primitive kernel vectors in $[-4, 4]^3$; (F) Proposition 4.2(b) for all 13,312 nondegenerate integer matrices with entries in $[-2, 2]$; (G) as sanity checks, centralizers $\{g : \deg g \leq D, \{f, g\} = 0\}$ computed modulo the prime $10^9 + 7$: for the bracket of Theorem A and five polynomials f with $D = 10$ and one monomial with $D = 18$, and for the bracket of Theorem D and two polynomials with $D = 7$, the centralizer consists of the polynomials in f . This is a certificate, since reduction modulo the prime can only enlarge the kernel; brackets which have polynomial integrable systems serve as controls and give larger centralizers.

The programs with which the results were first checked (original/): `verify_bracket.py` and `verify_dim4.py` (Python with SymPy; exact arithmetic in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\mathbb{Q}(\theta)$; 4 and 28 seconds), with their outputs. Part E of `verify_dim4.py` now tests the identity of Lemma 3.1 on random pairs, with a control order; its first version tested pairs whose leading exponents are parallel, and it could not fail.

Two independent verification runs, each with its own programs and their recorded outputs, in `verification_run_A/` and `verification_run_B/`; see the next paragraph.

The programs of a third independent verification run, which examined the text of this note: four programs, exact or certified modulo a prime, with their outputs, and the comparison of a re-run of the whole package with the recorded outputs, in `independent_run_2/`; see the next paragraph.

Verification. The results were first obtained with proofs and with the two programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs and did not see the report of the other. The first run read the TeX source of [BMMT18] and checked the statement, the definitions and the numbering; it re-derived both proofs of Theorem A, Theorems B to D, Corollary 5.1 and Proposition 6.2 line by line, in particular that in Lemma 3.1 exactly one product reaches the top monomial; and it tested them with programs of its own: the identities for arbitrary functions; centralizers certified modulo a prime, for the bracket of Theorem A in degree at most 10 (13 polynomials and 5 monomials) and for that of Theorem D in degree at most 7; Theorem B for all 578 primitive vectors in $[-4, 4]^3$; the identity of Lemma 3.1 under three monomial orders (2,700 tests); and a search over 1,501 polynomials with at most three terms on $\mathbb{R}^3$ and 196 on $\mathbb{R}^4$ for a commuting partner with independent differential, which found none, while brackets with polynomial integrable systems produced partners at once. The second run re-derived all proofs, wrote its own exact arithmetic in the number fields, reproduced every count and every centralizer dimension, certified the centralizers of about 1,700 further polynomials modulo a prime, tested Lemma 3.1 in 2,450 cases (commuting pairs taken from brackets with integrable systems, under fourteen monomial orders), with a control order that produced violations, checked the bibliography with Crossref and zbMATH, read the closest literature, and proved Proposition 4.2(b), which it tested on all 108,756 nondegenerate integer matrices with entries in $[-3, 3]$. Both runs re-ran the original programs and obtained the recorded outputs. Neither run found a mathematical error. Their required corrections concern the line and equation numbers of the source and the numbering to be used in public texts; the citations of Hartshorne’s book, of Fish and Jordan and of Sadetov’s paper; a program check that could not fail (Part E above); the description of the literature search, including the problem lists [BIT17, BKM26]; and the remark on the Casimir formulation in Section 1.3. All were made; [BIT17] could be read only later (see the third run below); they are listed in `verification_report.md`.

When this note was written, the leading-exponent proof became the main proof of Theorem A and the residue proof a remark with general references; the growth estimate of Lemma 3.3 was given the projection argument above instead of a lattice count; Proposition 6.2 was given the direct computation above instead of a criterion in terms of 1-forms; Lemma 2.3(a) was supplemented by the extension of isotropic subspaces; and Proposition 4.2 was added. The first two runs did not examine these changes.

The text of this note was then examined by a third independent verification run, also AI-assisted. It read the sources again: the TeX source of [BMMT18] (the question, the definitions, the numbering and every line and equation number), the cited statements of [MO17, FJ18, LLL24] and the problem list [BKM26] in their TeX sources, and, as copies became available during the run, [BIT17] in full and the thesis [Ove19]. It re-derived every proof line by line, in particular the four changes listed above. Before re-running the programs of the package it wrote its own: exact symbolic identities (the Jacobi identity for symbolic λ up to $n = 6$, the identity of Remark 6.1 for arbitrary functions, and Proposition 6.2 for an arbitrary function φ , with a four-dimensional control which is not Poisson); the identity of Lemma 3.1 in 2,140 cases under up to 31 orders compatible with addition per bracket, among them the anti-graded order, which is not a well-order, and weight orders with weights of both signs, with three non-compatible control orders which fail; Lemma 2.3(a) on 300 random pairs (λ, E) ; Proposition 4.2(b) on all nondegenerate integer matrices with entries in $[-2, 2]$ and in $[-3, 3]$, and Proposition 4.2(a) on 400 random rational matrices with $n \leq 9$; Theorem B for all 578 primitive vectors in $[-4, 4]^3$ against two independent searches; the counts of Part (D); and 1,102 centralizers certified modulo the prime 2146484161 (brackets of Theorems A and D and of Example 4.1(b), (d)), with controls. It re-ran all fifteen programs of the package from the archive and obtained the recorded outputs up to running times. It found no error in the mathematics. It required nine corrections, all of which were made: the credit to Problem 1 of [BIT17] in the abstract and in Sections 1.1 and 1.3; the citation of the earlier Russian list [BIKO12]; the reading of [Ove19], and the remark on Problem 5(c) of [BIT17] in Remark 6.3; the assignment of the counts in Part (D) and the description of Part (G); the general form of Proposition 4.2(a), whose proof it supplied, in place of a sentence which called the rational case undecided in dimension at least four; the paragraph “Scope and priority”; the wording of the novelty statement in Section 1.3; the record of this run; and the date of the deposit metadata.

Scope and priority. The note answers Question 5.9 of [BMMT18] (record AMR-020-0507 of [ula26]) negatively, for the definition of Section 5 of [BMMT18] and a fortiori for the formulation with polynomial Casimir functions, and it characterises the log-canonical brackets which admit a polynomial integrable system; quadratic brackets which are not log-canonical are not classified (Remark 6.3). The question was asked first, for polynomial brackets of any degree, by Bolsinov, Izosimov and Tsonev [BIT17, Problem 1]; there it is open apart from linear brackets, and no counterexample is given. Question 5.9 of [BMMT18], asked by Bolsinov, Matveev, Miranda and Tabachnikov, is its quadratic case. [BIT17] says that it is based on an earlier list of problems in Russian [BIKO12], which we have not read; the question may already be there. For linear brackets the positive answer is Sadetov’s theorem [Sad04] (see also [Bol16]). Log-canonical brackets, the bracket of two monomials [MO17, Lemma 3] and their polynomial Casimir functions are classical, and the triviality of the Poisson centre of the Laurent polynomial algebra for a matrix without nonzero integer kernel vectors is [FJ18, Proposition 7.2]. What is, to our knowledge, not in print is listed in Section 1.3; the counterexample is elementary, and experts may regard it as known.

What was read: [BMMT18] in the TeX source of arXiv:1804.03737v2, namely the introduction, Section 5 and the bibliography, against which the quotation, the line numbers and the numbering were checked; its bibliographic data with Crossref. [BIT17] in full, in the accepted manuscript of 30 March 2016 held in the repository of Loughborough University, to which our problem numbers refer: Section 2 states Problem 1 with the definition (1) recalled in Section 1.1; the other problems, on invariant functions on homogeneous spaces, chaotic Poisson brackets, bi-Poisson algebra and geometry, the argument shift method and action variables, do not contain an answer. The 2024 problem list of Bolsinov, Konyaev and Matveev [BKM26], in the TeX source of the arXiv version v1: it does not mention quadratic Poisson brackets, polynomial integrable systems or Sadetov’s theorem. Of [MO17] (arXiv v1) Sections 1 to 4, of [FJ18] (arXiv v3) Section 7, and of [LLL24] (arXiv v2) the abstract

and the parts of Sections 1 and 2 on integrable systems and lowest-degree terms; their TeX sources were also searched for the question. None of them mentions it, and [MO17, FJ18] do not consider commuting families. The thesis of Ovenhouse [Ove19]: Chapter 1 (log-canonical brackets, the bracket of two Laurent monomials, the theorem of Goodearl and Launois that no rational functions f, g satisfy $\{f, g\} = 1$, and the main theorem of [MO17]) and Chapter 10 were read, and the other chapters, on pentagram maps and non-commutative integrability, were searched; it does not consider commuting families of polynomials for log-canonical brackets. Sadetov’s paper [Sad04] was not read; it is cited as in [BMMT18], with the pages of the English translation taken from zbMATH. [Har77] is cited only for standard facts.

The searches were made on 17 October 2026: queries to the arXiv (its search pages and its export interface, for quadratic and log-canonical Poisson brackets with integrable systems, commuting polynomials and the Mishchenko–Fomenko conjecture, and the author listings), the list of 62 works citing [BMMT18] in Semantic Scholar (titles read), zbMATH, Crossref, and ten web searches. Many requests to the arXiv export interface and to OpenAlex were refused because of rate limits, so the arXiv topic search rests on a small number of queries and the check of citing works on the Semantic Scholar list. No printed answer to the question was found in the sources searched; a search that finds nothing is not a proof of novelty. No priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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