

On the Algebraic Birkhoff Conjecture for Outer Billiards on the Sphere and the Hyperbolic Plane: A Corollary of Theorems of Glutsyuk

Alper Ferudun*

October 17, 2026

Abstract

Bolsinov, Matveev, Miranda and Tabachnikov (Phil. Trans. R. Soc. A, 2018) asked to prove the algebraic version of the Birkhoff conjecture for outer billiards on a non-Euclidean surface of constant curvature. We observe that it follows from published theorems of A. Glutsyuk. Let γ be a simple closed C^4 curve with nowhere vanishing geodesic curvature on the sphere (inside a hemisphere) or on the hyperbolic plane, realised as quadrics in $\mathbb{R}^3$. If the outer billiard about γ has a first integral which is the restriction of a polynomial, or of a rational function, of the coordinates of $\mathbb{R}^3$, then γ is a conic; conversely, conics have quadratic integrals. In the gnomonic and Klein models the point reflection which defines the outer billiard is the projective involution of the tangent line fixing the point of tangency and interchanging the points of the line on the absolute $\mathcal{I}$. So the outer billiard is an $\mathcal{I}$ -angular billiard in Glutsyuk's sense (J. Eur. Math. Soc. 2021), and his theorem on such billiards gives the polynomial case; his classification of rationally integrable dual billiards (Invent. Math. 2026) gives a second proof, also for rational integrals. On the sphere the polynomial case also follows, for C^3 curves, from his theorem on polynomially integrable billiards through the spherical duality. The theorems are Glutsyuk's; the note contributes the identification and the reduction, and we did not find the corollary stated in print. Algebraic non-rational integrals and the smooth version are not treated. This is an unrefereed note.

1 Introduction

Bolsinov, Matveev, Miranda and Tabachnikov recall that outer billiards can be defined on the sphere and the hyperbolic plane, and Problem 4.4 of [BMMT18] (numbered as in arXiv:1804.03737v2) asks to “prove the algebraic version of Birkhoff conjecture for outer billiards” on a non-Euclidean surface of constant curvature. The preceding paragraph of [BMMT18] fixes the meaning: a polynomial first integral should force a conic, as in the polynomial Birkhoff conjecture for inner billiards, proved also on surfaces of constant curvature [BM17b, BM17a, Glu21], and for outer billiards in the plane [Tab08, GS18]. The source also recalls that on the sphere inner and outer billiards are conjugate under the spherical duality, so on the sphere the question was within reach of [Glu21]. We show that it follows from published theorems of Glutsyuk.

1.1 Setting

Let $\langle x, y \rangle_A := x^\top A y$ on $\mathbb{R}^3$. The sphere is $\Sigma = S^2 = \{\langle x, x \rangle_A = 1\}$ with $A = I$, and the hyperbolic plane is $\Sigma = H^2 = \{\langle x, x \rangle_A = -1, x_3 > 0\}$ with $A = \text{diag}(1, 1, -1)$; we write $\varepsilon := \langle P, P \rangle_A$ for $P \in \Sigma$,

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

so $\varepsilon = 1$ on S^2 and $\varepsilon = -1$ on H^2 . The metric of curvature ε is induced by $\langle \cdot, \cdot \rangle_A$; the geodesics are the non-empty sections $\Sigma \cap W$ by planes $W \ni 0$. The projection $\pi: \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{RP}^2$ maps an open hemisphere of S^2 onto an affine chart (gnomonic model) and H^2 onto the disc $D = \{[x] : \langle x, x \rangle_A < 0\}$ (Klein model), and geodesics into lines. The *absolute* $\mathcal{I} := \{[x] \in \mathbb{CP}^2 : \langle x, x \rangle_A = 0\}$ is a regular conic, without real points for S^2 ; for H^2 its real points form ∂D .

Definition 1.1. (a) An *oval* is a simple closed curve $\gamma \subset \Sigma$ of class C^4 with nowhere vanishing geodesic curvature, contained in an open hemisphere if $\Sigma = S^2$. As π maps geodesics into lines, it preserves the order of contact with tangent geodesics; so $\bar{\gamma} := \pi(\gamma)$ is a closed C^4 curve with nowhere vanishing curvature in an affine chart, it bounds a strictly convex domain, and each tangent line of $\bar{\gamma}$ meets $\bar{\gamma}$ only at its point of tangency. Let $\Omega \subset \Sigma$ be the corresponding domain bounded by γ .

(b) *Outer billiard map.* For $X \in \Sigma \setminus \Omega$ near γ there are two geodesics through X tangent to γ ; let L be the one for which γ lies on the right when L is traversed from X to its point of tangency P . Then $T(X)$ is the point of L at the same distance from P as X , on the other side of P . We fix a connected neighbourhood U of γ in $\Sigma \setminus \Omega$ on which T is defined (the other side convention gives T^{-1}).

(c) A *rational first integral* is a quotient $F = F_1/F_2$ of polynomials in x_1, x_2, x_3 such that F_2 has no zero on U , $F_1(TX)F_2(X) = F_1(X)F_2(TX)$ for $X \in U$, and F is not constant on U ; a *polynomial first integral* if $F_2 = 1$. (A polynomial in the coordinates of the gnomonic, Klein or Poincaré model is such a rational function, and Tabachnikov's algebraic integrability [Tab08] gives a polynomial first integral.)

(d) A *conic* in Σ is $\Sigma \cap \{\langle Bx, x \rangle = 0\}$ with B real symmetric of rank 3 and not proportional to A (on S^2 also each of its two antipodal components).

1.2 Result

Theorem 1.2. *Let $\Sigma \in \{S^2, H^2\}$ and let γ be an oval in Σ . If the outer billiard map about γ has a polynomial first integral, or more generally a rational first integral, then γ is a conic in Σ .*

Proposition 1.3. *Conversely, let $\gamma = \Sigma \cap \{\langle Bx, x \rangle = 0\}$ (on S^2 one of its two components) be a conic which is an oval. Then $F(x) := \langle Bx, x \rangle$ is a polynomial first integral of the outer billiard map about γ , and in the chart the invariant curves are the conics of the pencil spanned by $\bar{\gamma}$ and $\mathcal{I}$.*

In the chart, the reflection defining T is the $\mathcal{I}$ -angular symmetry of [Glu21, Definition 2.4] (Lemma 2.1); a polynomial integral gives a rational function with poles on $\mathcal{I}$ invariant under these symmetries (Lemma 3.2), and [Glu21, Theorem 1.25] applies (Section 4.1); [Glu26b, Theorem 1.16] gives a second proof, also for rational integrals (Section 4.2); on the sphere [Glu21, Theorem 1.6] applies through the duality (Proposition 4.2). The theorems are Glutsyuk's; the note contributes the identification and the reduction, which is what the hyperbolic plane needs: there the dual of an inner billiard lives on the de Sitter surface [BM17a]. Algebraic non-rational integrals, C^2 or C^3 ovals on H^2 and the smooth version are not treated (Section 6).

2 The outer billiard map as an $\mathcal{I}$ -angular billiard

For $P \in \Sigma$ let $P^\perp := \{y \in \mathbb{R}^3 : \langle y, P \rangle_A = 0\}$, and define the linear map

$$\rho_P(x) := 2\varepsilon \langle x, P \rangle_A P - x \quad (x \in \mathbb{R}^3). \quad (1)$$

In the terminology of [Glu21, Definition 2.2], ρ_P is the pseudo-symmetry with respect to the line $\mathbb{RP}$. We write $\bar{P} := \pi(P)$ and $\sigma_{\bar{P}}$ for the projective transformation of $\mathbb{RP}^2$ (and of $\mathbb{CP}^2$) induced by ρ_P . In the chart $\{x_3 = 1\}$ (for S^2 we rotate so that $\gamma \subset \{x_3 > 0\}$) we use $z = (x_1/x_3, x_2/x_3)$; then $\langle (z, 1), (z, 1) \rangle_A = Q(z)$ with $Q(z) := |z|^2 + \varepsilon$, and $\mathcal{I} \cap \{x_3 \neq 0\} = \{Q = 0\}$.

Lemma 2.1. *Let $P \in \Sigma$. (a) ρ_P is an involution and an isometry of $\langle \cdot, \cdot \rangle_A$; it fixes P , acts as -1 on $P^\perp$, satisfies $\langle \rho_P X, P \rangle_A = \langle X, P \rangle_A$, and maps Σ onto Σ . (b) Let $V \in P^\perp$ with $\langle V, V \rangle_A = 1$ (a unit tangent vector of Σ at P), and let $x(t) = P \cos t + V \sin t$ for S^2 , resp. $x(t) = P \cosh t + V \sinh t$ for H^2 , be the unit-speed geodesic through P with velocity V . Then $\rho_P(x(t)) = x(-t)$. In particular, if X lies on the tangent geodesic L_P of γ at $P \in \gamma$, and L_P is the geodesic selected for X in Definition 1.1(b), then $T(X) = \rho_P(X)$. (c) $\sigma_{\bar{P}}$ fixes $\bar{P}$ and every line through $\bar{P}$, and it fixes the polar line $\pi(P^\perp)$ of $\bar{P}$ pointwise. On every line L through $\bar{P}$ which is not tangent to $\mathcal{I}$ it interchanges the two points of $L \cap \mathcal{I}$; on a line tangent to $\mathcal{I}$ it fixes the point of tangency. Thus $\sigma_{\bar{P}}$ is the $\mathcal{I}$ -angular symmetry with centre $\bar{P}$ of [Glu21, Definition 2.4]. (d) If $\bar{P} = z_0$ lies in the chart and $w \in \mathbb{R}^2$, then $\sigma_{\bar{P}}(z_0 + tw) = z_0 + \mu(t)w$ with*

$$\mu(t) = -\frac{ct}{c+bt}, \quad c := Q(z_0), \quad b := 2z_0 \cdot w, \quad (2)$$

and $Q(z_0 + \mu(t)w)(c+bt)^2 = c^2 Q(z_0 + tw)$. The map μ is an involution with $\mu(0) = 0$ and $\mu'(0) = -1$.

Proof. (a) As $\langle P, P \rangle_A = \varepsilon \neq 0$, $\mathbb{R}^3 = \mathbb{R}P \oplus P^\perp$ orthogonally, and ρ_P is $+1$ on $\mathbb{R}P$ and -1 on $P^\perp$; so it is an involutive isometry and $\langle \rho_P X, P \rangle_A = \langle \rho_P X, \rho_P P \rangle_A = \langle X, P \rangle_A$. It preserves $\{\langle x, x \rangle_A = \varepsilon\}$, fixes P , and by continuity maps the component Σ onto itself. (b) $\langle x, x \rangle_A = \varepsilon$, $\langle x', x' \rangle_A = 1$, $x(t) \in \Sigma \cap (\mathbb{R}P + \mathbb{R}V)$, and $\rho_P(x(t)) = P \cos t - V \sin t = x(-t)$ (resp. with $\cosh, \sinh$): ρ_P maps the point of L_P at signed distance t from P to the point at signed distance $-t$, as T does. (c) A line through $\bar{P}$ is $\pi(W)$ with $W = \mathbb{R}P \oplus \mathbb{R}y$, $y \in P^\perp$; so $\rho_P(W) = W$, and in coordinates $x = aP + by$, ρ_P is $b \mapsto -b$ and $\langle x, x \rangle_A = \varepsilon a^2 + \langle y, y \rangle_A b^2$. If $\langle y, y \rangle_A \neq 0$, the two points $b/a = \pm(-\varepsilon/\langle y, y \rangle_A)^{1/2}$ of $\pi(W) \cap \mathcal{I}$ are interchanged; if $\langle y, y \rangle_A = 0$, the line touches $\mathcal{I}$ at the fixed point $\pi(y)$. (d) Let $\hat{P} := (z_0, 1)$ and $Y := (z_0 + tw, 1)$. Since ρ_P depends only on $\mathbb{R}P$, $\rho_P(Y) = 2\langle Y, \hat{P} \rangle_A \hat{P} / \langle \hat{P}, \hat{P} \rangle_A - Y$ with $\langle \hat{P}, \hat{P} \rangle_A = c$ and $\langle Y, \hat{P} \rangle_A = c + bt/2$, so $\rho_P(Y) = \frac{c+bt}{c} \hat{P} - t(w, 0) = \frac{c+bt}{c} (z_0 + \mu(t)w, 1)$; the identity for Q follows from $\langle \rho_P Y, \rho_P Y \rangle_A = \langle Y, Y \rangle_A = Q(z_0 + tw)$. $\square$

So in the chart the outer billiard map is $z \mapsto \sigma_{\bar{P}}(z)$, and $(\sigma_{\bar{P}})_{\bar{P} \in \bar{\gamma}}$ is a dual billiard structure on $\bar{\gamma}$ [Glu26b, Definition 1.6] made of $\mathcal{I}$ -angular symmetries. (For H^2 : in the Klein model, reflecting a chord in one of its points fixes that point and swaps the ends.) Statement (c) is [Glu21, Proposition 2.5] for ρ_P ; there it serves the dual picture of inner billiards.

3 Descent to a rational integral on the projective plane

In this section γ is an oval and $F = F_1/F_2$ a rational first integral (Definition 1.1). For $P \in \gamma$ let V_P be a unit tangent vector of γ at P , $W_P := \mathbb{R}P + \mathbb{R}V_P$, and

$$Q_P := \{x \in W_P : \langle x, x \rangle_A = \varepsilon\} = \{aP + bV_P : \varepsilon a^2 + b^2 = \varepsilon\}.$$

For S^2 , Q_P is the tangent great circle L_P ; for H^2 , $Q_P = L_P \cup (-L_P)$, where $-L_P$ lies on the lower sheet $-H^2$.

Lemma 3.1. *For every $P \in \gamma$ the polynomial $\Phi_P(x) := F_1(\rho_P x)F_2(x) - F_1(x)F_2(\rho_P x)$ vanishes on Q_P .*

Proof. In the chart, $\bar{\gamma}$ lies on one side of its tangent line at $\bar{P}$ and touches it only at $\bar{P}$; so L_P meets γ only at P , and $L_P \setminus \{P\}$ lies outside Ω . For $X \in L_P$ close to P , L_P is one of the two tangent geodesics through X , and it is the one selected in Definition 1.1(b) exactly when X lies on a fixed one of the two arcs of $L_P \setminus \{P\}$ next to P . Let J be such an arc, short enough to lie in U . For $X \in J$, $T(X) = \rho_P(X)$ by Lemma 2.1(b), so $\Phi_P(X) = 0$. In the coordinates $x = aP + bV_P$, the restriction of Φ_P to W_P is a polynomial in (a, b) which vanishes at the infinitely many points of J on the conic $\varepsilon a^2 + b^2 = \varepsilon$. This conic is non-degenerate, hence irreducible over $\mathbb{C}$, and by Bézout's theorem Φ_P vanishes on all of it. $\square$

For a polynomial G write $G^\pm(x) := \frac{1}{2}(G(x) \pm G(-x))$ for its even and odd parts, and put $s(x) := \varepsilon \langle x, x \rangle_A$, so that $s = 1$ on $\tilde{\Sigma} := \{\langle x, x \rangle_A = \varepsilon\} = \Sigma \cup (-\Sigma)$ and the zero set of s in $\mathbb{CP}^2$ is $\mathcal{I}$. For an even polynomial $G = \sum_k G_k$ (G_k homogeneous of degree k , k even) and an integer m with $2m \geq \deg G$ let

$$\tilde{G} := \sum_k G_k s^{(2m-k)/2}, \quad R_G := \tilde{G}/s^m. \quad (3)$$

Then $\tilde{G}$ is homogeneous of degree $2m$, and $\tilde{G} - G$ is divisible by $s - 1$, so $\tilde{G} = G$ on $\tilde{\Sigma}$. Thus R_G is homogeneous of degree 0, that is, a rational function on $\mathbb{RP}^2$ (and $\mathbb{CP}^2$) whose poles lie on $\mathcal{I}$, and $R_G(\pi x) = G(x)$ for $x \in \tilde{\Sigma}$. Two homogeneous polynomials of the same degree which agree on $\tilde{\Sigma}$ agree on the cone $\mathbb{R}\tilde{\Sigma}$, which is open in $\mathbb{R}^3$; so $\tilde{G}$ is unique, and $\tilde{G}\tilde{H} = \tilde{G}\tilde{H}$.

Lemma 3.2. *Let F be a polynomial first integral, $F = F^+ + F^-$, and $R_+ := R_{F^+}$, $R_- := R_{(F^-)^2}$. (a) For every $P \in \gamma$, $R_\pm \circ \sigma_{\bar{P}} = R_\pm$ on the line $L_{\bar{P}} := \pi(W_P)$, as rational functions on this line. (b) At least one of R_+ , R_- is not constant. (c) Each of R_+ , R_- is constant on $\bar{\gamma}$.*

Proof. (a) The map $-\text{Id}$ commutes with ρ_P and preserves Q_P . Lemma 3.1 at x and at $-x$ gives $F(\rho_P x) = F(x)$ and $F(-\rho_P x) = F(-x)$ for $x \in Q_P$, hence $F^\pm(\rho_P x) = F^\pm(x)$ and $(F^-)^2(\rho_P x) = (F^-)^2(x)$. (For H^2 this uses the second branch $-L_P$ of Q_P ; this is why Lemma 3.1 is stated on all of Q_P .) For $x \in Q_P \subset \tilde{\Sigma}$ we get $R_+(\sigma_{\bar{P}} \pi x) = R_+(\pi \rho_P x) = F^+(\rho_P x) = F^+(x) = R_+(\pi x)$. The restrictions of R_+ and of $R_+ \circ \sigma_{\bar{P}}$ to $L_{\bar{P}}$ are rational functions of one variable (s does not vanish identically on W_P , as $s(P) = 1$), and they agree on the infinite set $\pi(Q_P)$ (the whole line for S^2 , the chord $L_{\bar{P}} \cap D$ for H^2); so they are equal. The same argument applies to R_- . (b) If $R_+ \equiv c_1$ and $R_- \equiv c_2$, then $F^+ = c_1$ and $(F^-)^2 = c_2$ on Σ ; then F^- is a continuous function with values in $\{\pm\sqrt{c_2}\}$ on the connected surface Σ , hence constant, and F would be constant on $\Sigma \supset U$. (c) Let $R \in \{R_+, R_-\}$, $\bar{P} \in \bar{\gamma}$, let w be a tangent vector of $\bar{\gamma}$ at $\bar{P}$, and $f(t) := R(\bar{P} + tw)$; R is regular at $\bar{P}$, since $\bar{P} \notin \mathcal{I}$. By (a) and Lemma 2.1(d), $f(\mu(t)) = f(t)$ near $t = 0$, with $\mu(0) = 0$ and $\mu'(0) = -1$; so $f'(0) = -f'(0) = 0$. The derivative of R along $\bar{\gamma}$ vanishes, and $\bar{\gamma}$ is connected. $\square$

4 Proof of the theorem

4.1 Polynomial integrals, through $\mathcal{I}$ -angular billiards

We recall [Glu21, Definition 2.10 and Theorem 1.25]. Let $\mathcal{I} \subset \mathbb{CP}^2$ be a conic (regular, or a pair of distinct lines) and let $\Gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve which is neither a line nor $\mathcal{I}$. One says that Γ *generates a rationally integrable $\mathcal{I}$ -angular billiard* if there is a non-constant rational function G on $\mathbb{CP}^2$ with poles contained in $\mathcal{I}$ such that for every $C \in \Gamma \setminus \mathcal{I}$ the restriction of G to the projective tangent line of Γ at C is invariant under the $\mathcal{I}$ -angular symmetry with centre C . Theorem 1.25 of [Glu21] states that every such curve Γ is a conic.

Lemma 4.1. *Let F be a polynomial first integral, let $R = N/s^m$ be a non-constant one of the functions $R_\pm$ of Lemma 3.2, and let $\Gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve, neither a line nor $\mathcal{I}$, which contains an open arc J of $\bar{\gamma}$. Then Γ generates a rationally integrable $\mathcal{I}$ -angular billiard with integral R .*

Proof. The poles of R lie on $\mathcal{I}$. For $C \in \mathbb{C}^3$ with $\langle C, C \rangle_A \neq 0$ let $\hat{\rho}_C(y) := 2\langle y, C \rangle_A C - \langle C, C \rangle_A y$; its projectivization is the $\mathcal{I}$ -angular symmetry with centre $[C]$ (the proof of Lemma 2.1(c) works over $\mathbb{C}$), and $\langle \hat{\rho}_C y, \hat{\rho}_C y \rangle_A = \langle C, C \rangle_A^2 \langle y, y \rangle_A$. So, where $s(y) \neq 0$, $R(\hat{\rho}_C y) = R(y)$ if and only if $H(C, y) := N(\hat{\rho}_C y) - \langle C, C \rangle_A^{2m} N(y) = 0$. Let f be the homogeneous equation of Γ and Γ° the set of smooth points $[C]$ of Γ with $\langle C, C \rangle_A \neq 0$ and $C \cdot C \neq 0$; it is cofinite, as Γ is neither $\mathcal{I}$ nor the conic $\{x \cdot x = 0\}$, which has no real points. For $[C] \in \Gamma^\circ$, $v(C) := \nabla f(C) \times C$ is not zero (by Euler's identity

$\nabla f(C) \cdot C = 0$, and $\nabla f(C) \in \mathbb{C}C$ would give $C \cdot C = 0$), and $C, v(C)$ span the tangent line at $[C]$. The coefficients of $(\lambda, \nu) \mapsto H(C, \lambda C + \nu v(C))$ are homogeneous polynomials in C ; by Lemma 3.2(a) they vanish on the infinite set $J \cap \Gamma^\circ$, where the tangent lines of Γ and $\bar{\gamma}$ agree, hence on Γ (Bézout). At the finitely many other points of $\Gamma \setminus \mathcal{I}$ the tangent line of each local branch is a limit of tangent lines at points of Γ° , and H is continuous. $\square$

Proof of Theorem 1.2 for polynomial first integrals. Let $R = N/s^m$ be a non-constant one of $R_\pm$ (Lemma 3.2(b)), and r_0 its value on $\bar{\gamma}$ (Lemma 3.2(c)). Then $\bar{\gamma}$ lies on the algebraic curve $Z := \{N - r_0 s^m = 0\}$; note that $N - r_0 s^m \neq 0$.

Step 1. The set E of points of $\bar{\gamma}$ lying on two different irreducible components of Z is finite. Each connected component J of $\bar{\gamma} \setminus E$ is covered by the sets $J \cap \Gamma$, Γ a component of Z , which are closed in J and pairwise disjoint; so J lies in one component Γ_J . It is not a line, as $\bar{\gamma}$ has non-zero curvature, and $\Gamma_J \neq \mathcal{I}$, as $\bar{\gamma} \cap \mathcal{I} = \emptyset$ (for H^2 , $\bar{\gamma}$ lies in the open disc D).

Step 2. By Lemma 4.1 and [Glu21, Theorem 1.25], Γ_J is a conic, and a non-degenerate one, being irreducible and not a line.

Step 3 (pasting). Let $\bar{P} \in E$, and let C' and C'' be the conics containing the arcs of $\bar{\gamma} \setminus E$ on the two sides of $\bar{P}$. In affine coordinates in which the tangent of $\bar{\gamma}$ at $\bar{P}$ is horizontal, $\bar{\gamma}$ is near $\bar{P}$ the graph of a C^4 function g , and C', C'' are graphs of analytic functions g', g'' , with $g = g'$ on one side of the abscissa x_0 of $\bar{P}$ and $g = g''$ on the other. As g is of class C^4 , $g^{(k)}(x_0) = g'^{(k)}(x_0) = g''^{(k)}(x_0)$ for $k \leq 4$; so C' and C'' meet at $\bar{P}$ with multiplicity at least 5. Two different conics meet with total multiplicity 4, hence $C' = C''$. Therefore $\bar{\gamma}$ lies on a single non-degenerate conic C .

Step 4. The real points of C form a topological circle, which contains the closed embedded curve $\bar{\gamma}$; hence $\bar{\gamma}$ is all of it. Write $C = \pi(\{\langle Bx, x \rangle = 0\})$ with B real symmetric of rank 3. For H^2 , π maps H^2 bijectively onto $D \supset \bar{\gamma}$, so $\gamma = H^2 \cap \{\langle Bx, x \rangle = 0\}$; for S^2 , $S^2 \cap \{\langle Bx, x \rangle = 0\} = \gamma \cup (-\gamma)$. Finally B is not proportional to A , since $C \neq \mathcal{I}$. $\square$

4.2 Rational integrals, through dual billiards

We recall [Glu26b, Definitions 1.6, 1.12, Theorem 1.16] (numbering of arXiv:2112.07056v4). A *dual billiard structure* on a curve $\bar{\gamma} \subset \mathbb{RP}^2$ is a family of non-trivial projective involutions $\sigma_{\bar{P}}$ of the tangent lines $L_{\bar{P}}$ fixing $\bar{P}$; it is *rationally integrable* if some non-constant rational function R satisfies $R \circ \sigma_{\bar{P}} = R$ on $L_{\bar{P}}$ for all $\bar{P}$. By [Glu26b, Theorem 1.16], a non-linear C^4 germ of curve in $\mathbb{R}^2 \subset \mathbb{RP}^2$ with such a structure is a conic.

Proof of Theorem 1.2 for rational first integrals. Write $F = N/D$ with $N(x) := F_1(x)F_2(-x)$ and the even polynomial $D(x) := F_2(x)F_2(-x)$. Neither factor of D vanishes identically on Σ ($F_2 \neq 0$ on U , and if F_2 vanished on $-\Sigma$ it would vanish on its Zariski closure $\tilde{\Sigma} \supset \Sigma$), so $D \neq 0$ on Σ . Let $F^\pm := N^\pm/D$, so $F^\pm(x) = \frac{1}{2}(F(x) \pm F(-x))$ where $D(x) \neq 0$. Lemma 3.1 holds as stated, and as in Lemma 3.2(a), $F^\pm(\rho_P x) = F^\pm(x)$ for $x \in Q_P$ with $D(x)D(\rho_P x) \neq 0$. Homogenizing N^+ and D , and $(N^-)^2$ and D^2 , to common degrees by (3), let $R_+ := \widetilde{N^+}/\widetilde{D}$ and $R_- := \widetilde{(N^-)^2}/\widetilde{D^2}$; so $R_+ \circ \pi = F^+$ and $R_- \circ \pi = (F^-)^2$ on $\tilde{\Sigma} \setminus \{D = 0\}$. As $\widetilde{D} \neq 0$ has finitely many linear factors and the planes W_P are pairwise different, the set E of $P \in \gamma$ for which the linear form of W_P divides $\widetilde{D}$ is finite. For $P \notin E$, D has only finitely many zeros on the irreducible conic Q_P (otherwise $\widetilde{D} = D$ would vanish on Q_P , hence on the cone $\mathbb{R}Q_P$, which contains an open subset of W_P), and the argument of Lemma 3.2(a) gives $R_\pm \circ \sigma_{\bar{P}} = R_\pm$ on $L_{\bar{P}}$. If $R_+ \equiv c_1$ and $R_- \equiv c_2$, the continuous function F takes only the values $c_1 \pm \sqrt{c_2}$ on the dense subset $U \setminus \{D = 0\}$ of the connected set U , so it is constant on U . Let R be a non-constant one of $R_\pm$. For $\bar{P}_0 \in \bar{\gamma} \setminus \pi(E)$, the germ of $\bar{\gamma}$ at $\bar{P}_0$ is C^4 , non-linear and in an affine chart, and the dual billiard structure $(\sigma_{\bar{P}})$ is rationally integrable on it with integral R . By [Glu26b, Theorem 1.16] the germ lies on a conic, non-degenerate since the germ is not in a line. Conics sharing

an arc coincide, so each arc of $\bar{\gamma} \setminus \pi(E)$ lies on one non-degenerate conic, and Steps 3 and 4 above conclude. $\square$

For $F_2 = 1$ this is a second proof of the polynomial case; it does not need Lemma 4.1.

4.3 The sphere, through the spherical duality

Proposition 4.2. *Let $\gamma \subset S^2$ be a closed embedded curve of class C^3 with nowhere vanishing geodesic curvature, contained in an open hemisphere. If the outer billiard map about γ has a polynomial first integral, then γ is a conic.*

Proof. Here $A = I$, $\varepsilon = 1$. Definition 1.1(b),(c) and Lemmas 3.1 and 3.2 apply to γ : their proofs use only that γ is of class C^2 with nowhere vanishing geodesic curvature. Let $P(s)$ be an arc-length parametrization of γ , $V = P'$, and κ the geodesic curvature, so $V' = -P + \kappa P \times V$. The dual curve γ^* of the poles $P^* := P \times V$ of the tangent great circles satisfies $(P \times V)' = -\kappa V$; classically, it is a C^2 oval (in the hemisphere $\{\langle x, o \rangle > 0\}$ or $\{\langle x, o \rangle < 0\}$ for o inside Ω), bounding a convex domain Ω^* , and its tangent great circle at P^* is $P^\perp$. So at $Q := P^*$ the reflection of velocities is the restriction of $\varrho(x) := x - 2\langle x, P \rangle P$ to $T_Q S^2 = Q^\perp = W_P$, and for $v \in T_Q S^2$ the pole $Q \times v \in W_P$ of the great circle through Q with velocity v satisfies, as $\varrho(Q) = Q$, $Q \times \varrho(v) = \det(\varrho) \varrho(Q \times v) = \rho_P(Q \times v)$. Let $G := \widetilde{F^+}$ if F^+ is not constant on S^2 , and $G := (F^-)^2$ otherwise. By the proof of Lemma 3.2, G is not constant on S^2 and $G \circ \rho_P = G$ on $Q_P = L_P$, hence on W_P by homogeneity. So $\Phi(q, v) := G(q \times v)$, polynomial in the velocity, is invariant under the geodesic flow (the moment $q \times v$ is constant along great circles) and under the reflections at γ^* , and it is not constant on the unit vectors over Ω^* , which $(q, v) \mapsto q \times v$ maps onto an open set. The billiard in Ω^* is thus polynomially integrable [Glu21, Definition 1.1]; by [Glu21, Theorem 1.6] γ^* is a conic, and so is its dual γ [Glu21, Proposition 2.13]. $\square$

This is the duality of [BMMT18, Section 4.2] (see also [BM17a], [Glu22, Remark 1.12]); for H^2 it leads to the de Sitter surface [BM17a], and this route is not available.

5 Conics

Proof of Proposition 1.3. Let $P \in \gamma$, $V = V_P$. Differentiating $\langle BP, P \rangle = 0$ along γ gives $\langle BP, V \rangle = 0$, so for $x = aP + bV \in W_P$, $\langle Bx, x \rangle = b^2 \langle BV, V \rangle$, which is invariant under $\rho_P(aP + bV) = aP - bV$; hence $F \circ T = F$. If $\langle Bx, x \rangle = c$ on an open subset of Σ , then $\langle (B - \varepsilon c A)x, x \rangle$ vanishes on an open subset of the open cone $\mathbb{R}\Sigma$, and $B = \varepsilon c A$, which is excluded. In the chart let $C(z) := \langle B(z, 1), (z, 1) \rangle$. For $\bar{P} = z_0 \in \bar{\gamma}$ and a tangent vector w , $C(z_0 + tw) = t^2 C_2(w)$ (C_2 the quadratic part), and Lemma 2.1(d) gives, for every λ , $(C - \lambda Q)(z_0 + \mu w)(c + bt)^2 = c^2(C - \lambda Q)(z_0 + tw)$. So $\sigma_{\bar{P}}$ is the involution which the conics of the pencil spanned by $\bar{\gamma}$ and $\mathcal{I}$ cut out on the tangent line (Desargues), and C/Q is an integral. $\square$

This is the conical pencil type of [Glu26b, Theorem 1.16] (Tabachnikov's observation, [Glu26b, Example 1.14]). A geodesic circle $\{\langle x, c \rangle_A = \text{const}\}$, $c \in \Sigma$, also has the linear integral $\langle x, c \rangle_A$: $\langle \rho_P x, c \rangle_A - \langle x, c \rangle_A = 2(\varepsilon \langle c, P \rangle_A \langle x, P \rangle_A - \langle x, c \rangle_A)$ vanishes at $x = P$ and at $x = V_P$, hence on W_P .

6 Remarks

C^4 is used only in [Glu26b, Theorem 1.16] and in Step 3 of Section 4.1. In the Euclidean plane the point reflection is the central symmetry, which corresponds to the line at infinity counted twice, excluded in [Glu21, Definition 2.4]; the planar case is the theorem of Glutsyuk and Shustin [GS18]

(Tabachnikov [Tab08] under a non-degeneracy assumption). Not covered: (i) algebraic, non-rational first integrals (for dual billiards the source asks this in its Problem 4.5(ii)); (ii) ovals of class C^2 or C^3 on H^2 ; (iii) the smooth version, a neighbourhood of γ foliated by closed invariant curves without an algebraic integral, which even in the plane is posed as a question in [BFG⁺22]. For dual billiards with such a foliation and a rational integral (Problem 4.5(i) of [BMMT18]) Glutsyuk proved that the curve and the leaves are conics of a pencil [Glu26b, Theorem 1.11].

Verification. The result was first obtained with the proof of Section 4.2. Two independent verification runs, both AI-assisted, followed, each with its own programs. The first re-derived every step, found the route through [Glu21, Theorem 1.25], computed T from its definition on S^2 and H^2 (agreement with ρ_P within $2.5 \cdot 10^{-15}$), and found exactly the invariant homogeneous polynomials of degree ≤ 6 along germs (for two non-conic germs only the powers of $\langle x, x \rangle_A$). The second computed T in 50-digit arithmetic (agreement within 10^{-49}), showed by exact linear algebra that up to degree 14 the invariant polynomials are those of the pencil for conic germs and the powers of $\langle x, x \rangle_A$ for three non-conic germs, ran controls, and observed the route through [Glu21, Theorem 1.6]. They found no gap and required corrections of the account of the literature and the condition $\text{rank } B = 3$, which were made.

A third independent verification run, also AI-assisted, examined the text of this note. It compared the statement with the TeX source of [BMMT18] (v2), and the definitions and theorems quoted here with [Glu21] (published version), [Glu26b] (arXiv v4), [Glu22] (arXiv v2) and [BM17a]; it re-derived all proofs line by line, in particular Lemma 4.1, which had been added after the first two runs, checked the hypotheses of [Glu21, Theorems 1.25 and 1.6] and [Glu26b, Theorem 1.16], and repeated the literature search. Its exact program (114 checks) verifies Lemma 2.1 over $\mathbb{C}$, including the harmonic position of the two points of a line on $\mathcal{I}$ with respect to the two fixed points, the algebra of the proof of Lemma 4.1 (for integrals of a conic the invariance holds at all its complex points, and on a cubic it fails), Proposition 1.3, the descent of an integral with even and odd parts and the duality identities. Its program in 40-digit arithmetic computes T from its definition on S^2 and H^2 for a conic and a non-conic oval; in the chart, T agrees within $2 \cdot 10^{-41}$ with the involution constructed directly from [Glu21, Definition 2.4], which fixes the centre and interchanges the two points of the line on $\mathcal{I}$. It ran the programs of the package again from the archive and reproduced all recorded outputs. It found no mathematical error and required seven corrections of the text and of the package, which were made; among them are the date, the version of [Glu22] whose numbering is used, one sentence of the abstract, two sentences of the account of the literature and a remark on regularity in the proof of Proposition 4.2.

The exact program of this note, `check_note.py`, verifies Lemma 2.1, the identities of Proposition 1.3, the homogenization, the descended integrals on two examples and the duality identity (71 checks). No proof depends on a computation.

Scope and priority. The note answers Problem 4.4 of [BMMT18] in the reading of its source (polynomial integrals), and for rational integrals, for ovals of class C^4 (C^3 on the sphere for polynomial integrals). The question is due to Bolsinov, Matveev, Miranda and Tabachnikov. The theorems which carry the proof are Glutsyuk's ([Glu21, Theorems 1.25 and 1.6], completing work of Bialy and Mironov [BM17b, BM17a], and [Glu26b, Theorem 1.16]); the spherical duality is classical [BMMT18, BM17a, Glu22], and Proposition 1.3 is Tabachnikov's observation [Glu26b, Example 1.14]. The note contributes the identification with $\mathcal{I}$ -angular billiards, the reduction (Lemmas 3.1, 3.2 and 4.1) and the write-up of a corollary; we claim no priority. We did not find this corollary stated in print. Sources searched: [BMMT18]; Glutsyuk's papers on integrable billiards [Glu21, Glu26b, Glu24, Glu26c, Glu26a, Glu22, GS18] and his minicourse abstract [Glu] (planar,

dual and projective billiards); [BM17b, BM17a, BM18, BM25, Tab08, Tab02, BFG⁺22]; arXiv, Crossref, zbMATH Open, Semantic Scholar (partly rate-limited) and OpenAlex, including the lists of works citing [Glu21] and [Glu26b], and four web searches, on 16–17 October 2026. There, outer billiards on S^2 and H^2 appear only through the duality and, for other questions, for example in [Tab02, Glu22]. A search that finds nothing is not a proof of novelty. We read [Glu21] in the published version (its numbering agrees with arXiv:1706.04030v7), [Glu26b] in arXiv version v4 and [Glu22] in arXiv version v2, whose numbering we use (the journal versions were not compared), [BMMT18] in arXiv version v2 and [BM17a] in arXiv version v1; the proofs of the cited theorems were not checked.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [BFG⁺22] Misha Bialy, Corentin Fierobe, Alexey Glutsyuk, Mark Levi, Alexander Plakhov, and Serge Tabachnikov. Open problems on billiards and geometric optics. *Arnold Mathematical Journal*, 8(3–4):411–422, 2022. doi:10.1007/s40598-022-00198-y.
- [BM17a] Misha Bialy and Andrey E. Mironov. Algebraic Birkhoff conjecture for billiards on sphere and hyperbolic plane. *Journal of Geometry and Physics*, 115:150–156, 2017. arXiv:1602.05698, doi:10.1016/j.geomphys.2016.04.017.
- [BM17b] Misha Bialy and Andrey E. Mironov. Angular billiard and algebraic Birkhoff conjecture. *Advances in Mathematics*, 313:102–126, 2017. doi:10.1016/j.aim.2017.04.001.
- [BM18] Misha Bialy and Andrey E. Mironov. A survey on polynomial in momenta integrals for billiard problems. *Philosophical Transactions of the Royal Society A*, 376(2131):20170418, 2018. doi:10.1098/rsta.2017.0418.
- [BM25] Misha Bialy and Andrey E. Mironov. Integrable billiards and related topics, 2025. arXiv:2510.03790.
- [BMMT18] Alexey Bolsinov, Vladimir S. Matveev, Eva Miranda, and Serge Tabachnikov. Open problems, questions and challenges in finite-dimensional integrable systems. *Philosophical Transactions of the Royal Society A*, 376(2131):20170430, 2018. arXiv:1804.03737, doi:10.1098/rsta.2017.0430.
- [Glu] Alexey Glutsyuk. On algebraically integrable planar, dual and projective billiards. Abstract of a minicourse, Simons Center for Geometry and Physics, Stony Brook. Undated, two pages. URL: <https://scgp.stonybrook.edu/wp-content/uploads/2021/11/int-stonybrook-23.pdf>.
- [Glu21] Alexey Glutsyuk. On polynomially integrable Birkhoff billiards on surfaces of constant curvature. *Journal of the European Mathematical Society*, 23(3):995–1049, 2021. arXiv:1706.04030, doi:10.4171/JEMS/1027.
- [Glu22] Alexey Glutsyuk. On curves with the Poritsky property. *Journal of Fixed Point Theory and Applications*, 24(2):Paper No. 35, 2022. arXiv:1901.01881, doi:10.1007/s11784-022-00948-7.

- [Glu24] Alexey Glutsyuk. On rationally integrable planar dual multibilliards and piecewise smooth projective billiards. *Nonlinearity*, 37(6):065002, 2024. doi:10.1088/1361-6544/ad3606.
- [Glu26a] Alexey Glutsyuk. On exotic rationally integrable planar dual billiards I. Complex geometry and type of dynamics, 2026. arXiv:2605.24434.
- [Glu26b] Alexey Glutsyuk. On rationally integrable planar dual and projective billiards. *Inventiones mathematicae*, 245(2):977–1058, 2026. arXiv:2112.07056, doi:10.1007/s00222-026-01422-8.
- [Glu26c] Alexey A. Glutsyuk. On complex algebraic caustics in planar and projective billiards. *Regular and Chaotic Dynamics*, 31(1):132–149, 2026. doi:10.1134/S1560354726010016.
- [GS18] Alexey Glutsyuk and Eugenii Shustin. On polynomially integrable planar outer billiards and curves with symmetry property. *Mathematische Annalen*, 372(3–4):1481–1501, 2018. doi:10.1007/s00208-018-1726-4.
- [Tab02] Serge Tabachnikov. Dual billiards in the hyperbolic plane. *Nonlinearity*, 15(4):1051–1072, 2002. doi:10.1088/0951-7715/15/4/305.
- [Tab08] Serge Tabachnikov. On algebraically integrable outer billiards. *Pacific Journal of Mathematics*, 235(1):89–92, 2008. doi:10.2140/pjm.2008.235.89.