

Global Symplectic Equivalence Does Not Determine a Lagrangian Torus Fibration

Alper Ferudun*

12 October 2026

Abstract

Consider the projection of the four-torus onto its last two coordinates, with symplectic form $\omega_c = d\theta_1 \wedge dx_1 + d\theta_2 \wedge dx_2 + c dx_1 \wedge dx_2$. We compare global symplectic equivalence with equivalence preserving this Lagrangian fibration. The latter identifies exactly the parameters $c = \pm d \pmod{\mathbb{Z}}$. The total spaces are globally symplectomorphic exactly when their period subgroups $\mathbb{Z} + c\mathbb{Z}$ and $\mathbb{Z} + d\mathbb{Z}$ agree. For rational parameters this means equality of reduced denominators; for irrational parameters the two equivalence relations coincide. In particular, parameters $1/5$ and $2/5$ give globally symplectomorphic total spaces but no fibre-preserving symplectomorphism, even allowing an arbitrary diffeomorphism of the base. We display an integral matrix realizing the global equivalence. This disproves the unrestricted global-versus-fibre-preserving implication recorded by Leung–Symington for Lagrangian torus bundles over a torus. The construction and the twisting invariant are classical; the comparison is made explicitly within this one-parameter family.

1 The equivalence relations and the family

Leung–Symington [3, Section 7.2, p. 185] record a conjecture, attributed there to Mishachev, that two Lagrangian torus bundles over a torus are globally symplectomorphic if and only if they are fibre-preservingly symplectomorphic. We give an explicit counterexample to that formulation. Our attribution is to the published formulation in [3]; we do not assert that it reproduces every qualification of Mishachev’s original 1996 article, whose full text was not available for comparison.

Magnetic twists and the distinction between the affine lattice, the ordinary bundle obstruction, and the remaining twisting class belong to established classification theory. In particular, Kozlov [2, Theorem 2, Section 3.3 and Table 2] treats these invariants explicitly, following Duistermaat and Mishachev. Our proof below is self-contained and does not assume the earlier automorphism-orbit classification discussed critically in [2, Remark 12].

Write $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$. On $\mathbb{T}^4$, use coordinates $(\theta_1, \theta_2, x_1, x_2)$, and for $c \in \mathbb{R}$ put

$$\omega_c = d\theta_1 \wedge dx_1 + d\theta_2 \wedge dx_2 + c dx_1 \wedge dx_2, \quad \pi(\theta, x) = x. \quad (1)$$

Here the coordinate differentials, unlike the real-valued coordinates, are globally defined. In this order the alternating matrix is

$$W(c) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & c \\ 0 & -1 & -c & 0 \end{pmatrix}. \quad (2)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

It has determinant one. Thus ω_c is symplectic and π is a proper regular Lagrangian fibration with connected torus fibres. For a local function $f(x)$, the vector field defined by $\iota_X \omega_c = df$ is $X = \sum_i (\partial f / \partial x_i) \partial_{\theta_i}$. The period lattice is therefore generated by dx_1, dx_2 , independently of c . All bases have the same standard integral affine structure. All ordinary bundle Chern classes are zero: the underlying bundle is the product. This does not force the symplectic twisting class to vanish.

By a *fibre-preserving equivalence* from parameter c to parameter d we mean a symplectomorphism F satisfying $\pi \circ F = \psi \circ \pi$ for some diffeomorphism ψ of $\mathbb{T}^2$. A *global equivalence* imposes no such condition.

2 Fibre-preserving equivalence

Theorem 2.1. *For $c, d \in \mathbb{R}$, a fibre-preserving equivalence exists if and only if*

$$c = \varepsilon d + k, \quad \varepsilon \in \{1, -1\}, \quad k \in \mathbb{Z}. \quad (3)$$

If the base map must be the identity, the condition is $c - d \in \mathbb{Z}$.

Proof. Write $F(\theta, x) = (\Theta(\theta, x), \psi(x))$. Mixed vertical-horizontal coefficients of $F^* \omega_d = \omega_c$ give

$$(D_\theta \Theta)^T D_x \psi = I.$$

For each fixed x , the derivative of the fibre map is constant in θ . A lift to $\mathbb{R}^2$ is consequently affine. Its linear part is the induced map on $H_1(\mathbb{T}^2; \mathbb{Z})$ and belongs to $\text{GL}(2, \mathbb{Z})$. This matrix depends continuously on x , so it is constant. It follows that, for some $A \in \text{GL}(2, \mathbb{Z})$, $b \in \mathbb{T}^2$ and smooth $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$,

$$\psi(x) = Ax + b, \quad \Theta(\theta, x) = A^{-T} \theta + h(x). \quad (4)$$

Thus allowing nonlinear smooth equivalences gives no extra freedom in the base derivative or in the fibre linear part.

Let $H = (H_{ij}) \in M_2(\mathbb{Z})$ be the degree matrix of h . A lift has the form $\tilde{h}(x) = Hx + \tilde{h}_0(x)$ with $\tilde{h}_0$ periodic. Restricting the pullback identity to the section $\theta = 0$ and integrating over the base yields

$$c = \det(A)d + H_{11}A_{12} - H_{12}A_{11} + H_{21}A_{22} - H_{22}A_{21}. \quad (5)$$

The periodic derivative terms integrate to zero. The final four terms are integers and $\det(A) = \pm 1$, proving necessity. When $A = I$, their sum is $H_{21} - H_{12}$.

Conversely, if (3) holds, set

$$F(\theta_1, \theta_2, x_1, x_2) = (\varepsilon \theta_1, \theta_2 + kx_1, \varepsilon x_1, x_2).$$

This is an integral torus automorphism covering $\text{diag}(\varepsilon, 1)$, and direct pullback gives $F^* \omega_d = \omega_{\varepsilon d + k}$. Taking $\varepsilon = 1$ gives the asserted converse over the identity. $\square$

3 Global equivalence

The period subgroup of a closed two-form is its set of integrals over integral two-cycles. Coordinate two-tori generate $H_2(\mathbb{T}^4; \mathbb{Z})$, so the subgroup for (1) is

$$\Gamma(c) = \mathbb{Z} + c\mathbb{Z} \subset \mathbb{R}. \quad (6)$$

Every global symplectomorphism preserves this subgroup.

Theorem 3.1. *The total spaces $(\mathbb{T}^4, \omega_c)$ and $(\mathbb{T}^4, \omega_d)$ are globally symplectomorphic if and only if $\Gamma(c) = \Gamma(d)$. More explicitly:*

1. *For rational parameters, equivalence means equality of their positive reduced denominators.*
2. *For irrational parameters, equivalence means $c = \pm d + k$ with $k \in \mathbb{Z}$.*
3. *A rational parameter is not equivalent to an irrational parameter.*

Every equivalence in this statement has a linear integral representative.

Proof. Necessity follows from (6). If $c = a/b$ is reduced with $b > 0$, then $\Gamma(c) = b^{-1}\mathbb{Z}$. If c is irrational, $\Gamma(c)$ has rank two as an abelian group, whereas a rational parameter gives rank one. If c, d are irrational and their subgroups agree, there are integers m, n, r, s with $c = m + nd$ and $d = r + sc$. Substitution and irrationality imply $ns = 1$. Thus $c = m \pm d$, and Theorem 2.1 already supplies a linear equivalence.

It remains to construct a global equivalence between reduced fractions a/b and a'/b . Let $e_1, \dots, e_4$ be the standard lattice basis. Choose $u, v \in \mathbb{Z}$ with $-bu + av = 1$ and use the four columns

$$p = e_3, \quad q = ue_1 + ve_4, \quad r = e_2 - bve_3, \quad s = ae_1 + be_4$$

to form the integer matrix

$$B(a, b) = \begin{pmatrix} 0 & u & 0 & a \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -bv & 0 \\ 0 & v & 0 & b \end{pmatrix}. \quad (7)$$

Its determinant is $bu - av = -1$. In the alternating pairing $bW(a/b)$, the pairings of p, q and r, s are 1 and b^2 , respectively, and the four cross-pairings vanish. Consequently

$$B(a, b)^T bW(a/b) B(a, b) = \text{diag}(J, b^2 J), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (8)$$

This is a constructive instance of the usual integral normal form of an alternating pairing. In particular

$$P = B(a', b) B(a, b)^{-1} \in \text{SL}(4, \mathbb{Z}), \quad P^T W(a'/b) P = W(a/b).$$

The corresponding linear map of $\mathbb{T}^4$ is the required symplectomorphism. $\square$

4 An explicit counterexample and its scope

Corollary 4.1. *The two fibrations in (1) with $c = 1/5$ and $d = 2/5$ have the same integral affine base and globally symplectomorphic total spaces, but are not fibre-preservingly symplectomorphic over any base diffeomorphism.*

Proof. An explicit global equivalence is induced by

$$P = \begin{pmatrix} 5 & -2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 10 & 5 & 1 & 0 \\ 10 & -5 & 0 & -2 \end{pmatrix}. \quad (9)$$

Direct multiplication gives $\det P = 1$ and $P^T W(2/5) P = W(1/5)$. However, $1/5 - 2/5 = -1/5$ and $1/5 + 2/5 = 3/5$ are both nonintegers. Theorem 2.1 excludes every fibre-preserving equivalence. The common affine base was established after (2). $\square$

More generally, for any $b \geq 3$, the reduced numerators modulo b give exactly $\varphi(b)/2$ fibre-preserving classes within this family, all of the same global symplectic type. Indeed the only additional identification is $a \mapsto -a$; it has no fixed reduced residue when $b \geq 3$. For primes b , this number is $(b-1)/2$, hence is unbounded. It is a count within the specified family, not a classification of all Lagrangian fibrations on a given symplectic four-torus. The smallest denominator exhibiting the separation is 5.

The example disproves the unrestricted implication in [3, Section 7.2], already among regular fibrations with trivial bundle and monodromy. Regular fibrations are almost toric, since they have no forbidden singularities. Nevertheless a closed torus base is not a domain with nonempty boundary. Symington’s uniqueness result [4, Corollary 5.4] concerns the total symplectic space when the almost-toric base is noncompact or has nonempty boundary; the example does not contradict it. Nor do we claim to settle the boundary-domain setting preceding Problem 2.2 of [1]. That survey question and the global-versus-fibre-preserving implication are distinct.

As a separate classical observation, $c = 0$ and $c = 1/2$ have the same affine base but globally inequivalent total spaces, as their period subgroups differ. This is not the counterexample of Corollary 4.1; it illustrates why the affine base alone does not classify arbitrary closed-base fibrations.

The elementary computation is included for reproducibility in the accompanying source archive. Exact matrix and wedge pullbacks agree; 308 reduced-parameter normal forms, 4790 same-denominator maps and 18 signed-integer fibre-preserving controls were checked in both normal and optimized Python. These finite checks support, but do not replace, the proofs above. AI assistance was used for derivation, literature comparison, computation and writing. This preprint is self-audited and unrefereed, without independent review, proof-assistant certification or an absolute-priority claim.

References

- [1] A. V. Bolsinov, V. S. Matveev, E. Miranda and S. Tabachnikov, Open problems, questions, and challenges in finite-dimensional integrable systems, *Philos. Trans. R. Soc. A* **376** (2018), 20170430. [arXiv:1804.03737](#).
- [2] I. K. Kozlov, Classification of Lagrangian fibrations, *Sb. Math.* **201** (2010), 1647–1688. [doi:10.1070/SM2010v201n11ABEH004126](#); [arXiv:2212.14089](#).
- [3] N. C. Leung and M. Symington, Almost toric symplectic four-manifolds, *J. Symplectic Geom.* **8** (2010), 143–187. [arXiv:math/0312165](#).
- [4] M. Symington, Four dimensions from two in symplectic topology, *Proc. Sympos. Pure Math.* **71** (2003), 153–208. [arXiv:math/0210033](#).