

Isolated Real Zeros of Sums of Squares: Quaternary Forms and Quinary Quartics

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Abstract

For even n let $B'_{n,m}$ be the supremum of the number of real projective zeros of a sum of squares of real forms of degree n in m variables, taken over those with finitely many real zeros. Fröberg, Lundqvist, Oneto and Shapiro conjectured that $B'_{n,m} = (n/2)^{m-1}$; Choi, Lam and Reznick had proved the lower bound and the case $m = 3$. A related conjecture of Ottaviani and Shapiro states that a sum of squares of real polynomials of degree at most k in l variables has at most k^l isolated real zeros; it was known for $l \leq 2$. Both conjectures follow from the statement that the real points of the base locus of a real linear system of forms of degree k on $\mathbb{P}^N$ have at most k^N isolated points. We prove this statement for $N = 3$ and every k , and for $k = 2$, $N = 4$. Consequently $B'_{2k,4} = k^3$ for every k and $B'_{4,5} = 16$, and the Ottaviani–Shapiro bound holds for $l = 3$ and every k and for $(k, l) = (2, 4)$. We also give a short proof of $B'_{4,4} = 8$, a value that Choi, Lam and Reznick expected and that Dressler stated without proof. The main tools are a weighted Bézout inequality, a Jacobian argument along the curve components of the base locus, and an integral closure estimate for the system of partial derivatives. Both conjectures remain open in general, for instance for sextics in five variables and for quartics in six variables. This is an unrefereed note.

1 Introduction

For a real form p in m variables let $\mathcal{Z}(p) \subset \mathbb{RP}^{m-1}$ be its set of real projective zeros. For even n let $\Sigma_{n,m}$ be the cone of real forms of degree n in m variables that are sums of squares of real forms (necessarily of degree $n/2$), and let

$$B'_{n,m} = \sup\{|\mathcal{Z}(p)| : p \in \Sigma_{n,m}, |\mathcal{Z}(p)| < \infty\}.$$

These numbers, and the analogous numbers $B_{n,m}$ for all nonnegative forms, were introduced by Choi, Lam and Reznick [CLR80]. In this note n is always the degree and m the number of variables, as in the conjecture of Fröberg, Lundqvist, Oneto and Shapiro [FLOS18] stated below. The sentence of [FLOS18] that defines their cones speaks of forms of degree m in n variables, but their conjecture (stated for even n , with the value $(n/2)^{m-1}$ and with the case $m = 3$ proved) makes n the degree and m the number of variables, and we follow the conjecture. Choi, Lam and Reznick and also Dressler [Dre21] write the number of variables first. For instance, the number $B'_{3,m}$ of [CLR80, Thm. 4.6] is $B'_{m,3}$ here, and the case $B'_{5,4}$ in the notation of [Dre21] is our $B'_{4,5}$.

Conjecture 1.1 ([FLOS18, subsection “Non-negative forms”]). *For every even n and every $m \geq 2$, $B'_{n,m} = (n/2)^{m-1}$.*

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Shapiro [Sha15, Section III] considers the maximal number $\widetilde{\sharp}(2k, l)$ of isolated real zeros of a nonnegative polynomial of degree $2k$ in l variables that is a sum of squares of real polynomials of degree at most k , and records the following conjecture of Ottaviani and himself.

Conjecture 1.2 (Ottaviani–Shapiro [Sha15]). *For all $k, l \geq 1$, $\widetilde{\sharp}(2k, l) = k^l$.*

Known results. Both lower bounds, $B'_{2k,m} \geq k^{m-1}$ and $\widetilde{\sharp}(2k, l) \geq k^l$, come from products of linear forms placed on a grid ([CLR80, Prop. 4.1], [Sha15]; see Example 2.2). For binary forms and for quadratic forms one has $B'_{n,2} = n/2$ and $B'_{2,m} = 1$ [CLR80, Sect. 4]. Choi, Lam and Reznick proved $B'_{2k,3} = k^2$ [CLR80, Thm. 4.6]; Shapiro credits them also with the affine case $\widetilde{\sharp}(2k, 2) = k^2$ [Sha15]. Their theorem concerns forms with finitely many real zeros; for isolated zeros one also needs a step for fixed components, such as Lemma 3.4 in the proof of Proposition 4.1. For quaternary quartics they proved $8 \leq B'_{4,4} \leq 10$ [CLR80, Prop. 4.13], and they remarked that applying Bézout’s theorem to quadric surfaces in $\mathbb{P}^3$ seemed to give $B'_{4,4} = 8$, without giving the details [CLR80, p. 13]. Dressler states $B'_{4,4} = 8$ in the remark that closes Section 4 of [Dre21] and notes that the numbers $B'_{n,m}$ are not known in general. For all nonnegative quaternary quartics $B_{4,4} = 10$ ([CLR80, Prop. 4.13] and [BHO⁺12]). Shapiro also records the elementary bounds $k^l \leq \widetilde{\sharp}(2k, l) \leq \sharp(2k, l) \leq (2k-1)^l$, where $\sharp(2k, l)$ is the analogous number for all nonnegative polynomials.

Base loci. A *real linear system* of degree k on $\mathbb{P}^N$ is a real vector space L of real forms of degree k in $x_0, \dots, x_N$. Its *base locus* is the complex projective set $Y = V(L) \subset \mathbb{P}^N$ of common zeros, and $Y(\mathbb{R}) = Y \cap \mathbb{R}\mathbb{P}^N$. Let $\Gamma(L)$ be the set of points of $Y(\mathbb{R})$ that are isolated in $Y(\mathbb{R})$ for the Euclidean topology. All our results are instances of the following statement.

Definition 1.3. For integers $k \geq 0$ and $N \geq 1$, $T(k, N)$ is the statement that $|\Gamma(L)| \leq k^N$ for every real linear system L of degree k on $\mathbb{P}^N$.

If $p = \sum q_i^2$ with q_i real forms of degree k , then $\mathcal{Z}(p) = Y(\mathbb{R})$ for the span L of the q_i . Hence $T(k, m-1)$ gives $B'_{2k,m} = k^{m-1}$, and by homogenisation it gives $\widetilde{\sharp}(2k, m-1) = k^{m-1}$; in fact $T(k, l)$ is equivalent to Conjecture 1.2 for (k, l) (Proposition 2.1).

Theorem 1.4. *$T(k, 3)$ holds for every $k \geq 0$.*

Theorem 1.5. *$T(2, 4)$ holds.*

Corollary 1.6. (a) *$B'_{2k,4} = k^3$ for every $k \geq 1$, and $B'_{4,5} = 16$.*

(b) *$\widetilde{\sharp}(2k, 3) = k^3$ for every $k \geq 1$, and $\widetilde{\sharp}(4, 4) = 16$.*

(c) *If $f_1, \dots, f_s \in \mathbb{R}[y_1, y_2, y_3]$ have degree at most k , then the real algebraic set $\{f_1 = \dots = f_s = 0\} \subset \mathbb{R}^3$ has at most k^3 isolated points. If $f_1, \dots, f_s \in \mathbb{R}[y_1, \dots, y_4]$ have degree at most 2, the corresponding bound is 16.*

The case $k = 2$ of Theorem 1.4 is $T(2, 3)$, which gives $B'_{4,4} = 8$. It has a short proof that does not need the machinery of Section 5.

Proposition 1.7. *$T(2, 3)$ holds. In particular $B'_{4,4} = 8$ and $\widetilde{\sharp}(4, 3) = 8$.*

Table 1 summarises what is known. Proposition 8.1 below also proves $T(2, 5)$ for all systems whose base locus has dimension at most one, for all pencils and for all systems with a fixed component.

case	value	source
$B'_{n,2}$ and $B'_{2,m}$	$n/2, 1$	[CLR80, Sect. 4]
$B'_{2k,3}$	k^2	[CLR80, Thm. 4.6]
$B'_{4,4}$	8	Proposition 1.7; indicated in [CLR80, p. 13], stated in [Dre21]
$B'_{2k,4}$, all k	k^3	Corollary 1.6
$B'_{4,5}$	16	Corollary 1.6
$B'_{2k,m}$, $m \geq 5$, $k \geq 3$; $B'_{4,m}$, $m \geq 6$		open
$\widetilde{\#}(2k, 1)$ and $\widetilde{\#}(2, l)$	$k, 1$	elementary
$\widetilde{\#}(2k, 2)$	k^2	[CLR80], see [Sha15]; Proposition 4.1
$\widetilde{\#}(2k, 3)$, all k	k^3	Corollary 1.6
$\widetilde{\#}(4, 4)$	16	Corollary 1.6
$\widetilde{\#}(2k, l)$, $l \geq 4$, $k \geq 3$; $\widetilde{\#}(4, l)$, $l \geq 5$		open

Table 1: Status of Conjectures 1.1 and 1.2 (n is the degree and m the number of variables). The lower bounds hold in all cases (Example 2.2).

Method. A weighted Bézout inequality (Lemma 3.1) gives each component Z of the base locus a budget $k^{\dim Z} m_Z \deg Z$, and the budgets add up to at most k^N . An isolated real point that lies on a component D of positive dimension is a singular point of D if D is real, and lies on $D \cap \overline{D}$ otherwise (Lemma 3.3). So it suffices to bound these points component by component. For components of multiplicity one a Jacobian argument does this (Lemma 3.7). For curves in $\mathbb{P}^3$ of arbitrary multiplicity we use an iterated Jacobian construction and an induction on the order of vanishing of L along the curve, in which the system of partial derivatives is compared with L by an integral closure argument (Proposition 5.1 and Lemma 5.3). For quadrics in $\mathbb{P}^4$ we use a case analysis (Section 7). Fixed components are removed with the gradient system (Lemma 3.4).

Section 2 contains the reformulation and the examples, Section 3 the general lemmas, Section 4 the cases $N = 2$ and $(k, N) = (2, 3)$, Section 5 the curve estimate in $\mathbb{P}^3$, and Sections 6 and 7 the proofs of Theorems 1.4 and 1.5. Section 8 discusses quadrics in higher dimension.

2 Reformulation and lower bounds

Proposition 2.1. *Let $k, N \geq 1$.*

- (a) *If $q_1, \dots, q_s$ are real forms of degree k in $x_0, \dots, x_N$ with span L , then $\mathcal{Z}(\sum_i q_i^2) = Y(\mathbb{R})$. Hence $T(k, N)$ implies $B'_{2k, N+1} \leq k^N$.*
- (b) *Let $q_1, \dots, q_s \in \mathbb{R}[x_1, \dots, x_N]$ have degree at most k , let $Q_i = x_0^k q_i(x_1/x_0, \dots, x_N/x_0)$, and let L be the span of the Q_i . Then $x \mapsto [1 : x]$ maps the isolated zeros of $p = \sum_i q_i^2$ in $\mathbb{R}^N$ injectively into $\Gamma(L)$. Hence $T(k, N)$ implies $\widetilde{\#}(2k, N) \leq k^N$.*
- (c) *Conversely, if $\widetilde{\#}(2k, N) \leq k^N$, then $T(k, N)$ holds.*

Proof. (a) A real point is a zero of $\sum q_i^2$ exactly when every q_i vanishes there. If $Y(\mathbb{R})$ is finite, all its points are isolated.

(b) The map $\iota(x) = [1 : x]$ is a homeomorphism of $\mathbb{R}^N$ onto the open subset $U_0 = \{x_0 \neq 0\}$ of $\mathbb{RP}^N$, and $Q_i(1, x) = q_i(x)$. So ι maps the zero set of p onto $Y(\mathbb{R}) \cap U_0$. A point that is isolated in $Y(\mathbb{R}) \cap U_0$ is isolated in $Y(\mathbb{R})$, because U_0 is open.

(c) Suppose that $|\Gamma(L)| > k^N$ for some real linear system L of degree k ; then $L \neq 0$. Choose $k^N + 1$ points of $\Gamma(L)$ and a real hyperplane that contains none of them and on which some member of L does not vanish identically. After a real linear change of coordinates this hyperplane is $\{x_0 = 0\}$. For a basis $q_1, \dots, q_s$ of L the polynomial $p(x) = \sum_i q_i(1, x)^2$ is a sum of squares of polynomials of degree at most k , it has degree exactly $2k$ (the top-degree parts of the squares cannot cancel), and by the argument of (b) the chosen points are isolated zeros of p . So $\tilde{\sharp}(2k, N) \geq k^N + 1$. $\square$

Part (b) applies to any finite family $q_1, \dots, q_s$ of real polynomials of degree at most k : the isolated points of their common real zero set are the isolated zeros of $\sum_i q_i^2$. With Theorems 1.4 and 1.5 this gives Corollary 1.6(c).

Example 2.2 (grids). Let $F_i = \prod_{c=0}^{k-1} (x_i - cx_0)$ for $i = 1, \dots, N$. If $x_0 = 0$ then $F_i = x_i^k$, so every common zero has $x_0 \neq 0$, and in the chart $x_0 = 1$ the system is $\prod_c (x_i - c) = 0$ for all i . Thus $V(F_1, \dots, F_N)$ consists of the k^N real points $[1 : c_1 : \dots : c_N]$ with $c_i \in \{0, \dots, k-1\}$. The form $\sum_i F_i^2 \in \Sigma_{2k, N+1}$ has exactly k^N real zeros, and the polynomial $\sum_i \prod_c (x_i - c)^2$ has exactly k^N real zeros, all of them isolated. This is the example of [CLR80, Prop. 4.1]. Together with Proposition 2.1, $T(k, N)$ gives $B'_{2k, N+1} = \tilde{\sharp}(2k, N) = k^N$.

3 General lemmas

Conventions. Varieties are over $\mathbb{C}$, points are closed points, and a *component* is an irreducible component. Complex conjugation acts on $\mathbb{P}^N(\mathbb{C})$; a closed subvariety V is *real* if $\bar{V} = V$. For a real linear system L the base locus Y is real, and conjugation permutes its components; a non-real component D comes with the distinct component $\bar{D}$ of the same dimension and degree. We write $L_{\mathbb{C}} = L \otimes \mathbb{C}$; then L is Zariski dense in $L_{\mathbb{C}}$. A member $F \in L$ is called *general* if it avoids a given proper Zariski closed subset of $L_{\mathbb{C}}$; such real members exist, and finitely many conditions of this kind can be imposed at once. Members of a sequence are chosen one after the other, each general with respect to conditions that may depend on the previous ones. Forms are dehomogenised in an affine chart containing the points under consideration; the multiplicity $\text{mult}_y F$ of a form F at a point y is the order of vanishing of its dehomogenisation at y .

Intersections with hypersurfaces. Let $\alpha = \sum_i n_i [V_i]$ be an effective cycle on $\mathbb{P}^N$ of pure dimension $d \geq 1$, and let F be a form that vanishes identically on no V_i . Put $V(F) \cdot \alpha = \sum_i n_i \sum_W i(V_i, V(F); W) [W]$, where W runs over the components of $V_i \cap V(F)$ and $i(V_i, V(F); W)$ is the length of $\mathcal{O}_{V_i, W}/(F)$. Every such W has dimension $d-1$ [Har77, Thm. I.7.2] and a positive coefficient, so the support of $V(F) \cdot \alpha$ is $|\alpha| \cap V(F)$, and $\deg(V(F) \cdot \alpha) = \deg F \cdot \deg \alpha$ [Har77, Thm. I.7.7].

Multiplicities. For an $\mathfrak{m}$ -primary ideal $\mathfrak{q}$ of a Noetherian local ring $(R, \mathfrak{m})$, $e(\mathfrak{q})$ is its Samuel multiplicity. We use: (M1) $\mathfrak{q} \subset \mathfrak{q}'$ implies $e(\mathfrak{q}) \geq e(\mathfrak{q}')$; (M2) if R is Cohen–Macaulay and $\mathfrak{q}$ is generated by a system of parameters, then $\ell(R/\mathfrak{q}) = e(\mathfrak{q})$ [Mat86, Thm. 17.11]; (M3) the associativity formula $e(\mathfrak{q}; M) = \sum_P \ell(M_P) e(\mathfrak{q}; R/P)$, the sum over the primes P with $\dim R/P = \dim M$ [Mat86, Thm. 14.7]; (M4) two $\mathfrak{m}$ -primary ideals with the same integral closure have the same multiplicity [LJT08, Prop. 1.18]. Regular local rings are domains [Mat86, Thm. 14.3], and in a Cohen–Macaulay local ring a part of a system of parameters is a regular sequence [Mat86, Thm. 17.4]. Lengths and multiplicities of $\mathfrak{m}$ -primary ideals of the local ring of a smooth surface at a point do not change if one passes to the ring of convergent power series in local coordinates, since both rings have the same completion.

Lemma 3.1 (weighted Bézout inequality). *Let $L \neq 0$ be a linear system of degree $k \geq 1$ on $\mathbb{P}^N$ with base locus Y , and put $A_0 = [\mathbb{P}^N]$. Let $F_1, \dots, F_N \in L$ be such that for each j the form F_j vanishes identically on no component of A_{j-1} , where*

$$V(F_j) \cdot A_{j-1} = B_j + A_j,$$

B_j being the sum of the terms whose support lies in Y and A_j the sum of the others. Such sequences exist, and if L is real they can be chosen real and general. Then every component Z of Y of dimension $N - j$ occurs in B_j with a coefficient $m_Z \geq 1$, and

$$\sum_Z k^{\dim Z} m_Z \deg Z \leq \sum_{j=1}^N k^{N-j} \deg B_j = k^N - \deg A_N \leq k^N.$$

Proof. If W is a component of A_{j-1} , then $W \not\subset Y$, so the members of $L_{\mathbb{C}}$ vanishing on W form a proper linear subspace; a general real F_j avoids these finitely many subspaces. From $\deg A_j = k \deg A_{j-1} - \deg B_j$ we get $\deg A_N + \sum_j k^{N-j} \deg B_j = k^N$.

We claim $V(F_1, \dots, F_j) = |B_1| \cup \dots \cup |B_j| \cup |A_j|$. This is clear for $j = 0$. Since $|B_i| \subset Y \subset V(F_j)$, the claim for $j - 1$ gives $V(F_1, \dots, F_j) = |B_1| \cup \dots \cup |B_{j-1}| \cup (|A_{j-1}| \cap V(F_j))$, and $|A_{j-1}| \cap V(F_j) = |B_j| \cup |A_j|$.

Let Z be a component of Y with $\dim Z = N - j$; here $j \geq 1$ because $L \neq 0$. As $Z \subset V(F_1, \dots, F_j)$ and Z is irreducible, either Z lies in a component W of some B_i with $i \leq j$, or $Z \subset |A_j|$. In the first case W is an irreducible subset of Y containing the component Z , so $W = Z$, and $\dim W = N - i$ gives $i = j$. In the second case Z would be a component of A_j , because $|A_j|$ has pure dimension $N - j$; but no component of A_j lies in Y . Hence Z occurs in B_j , and the inequality follows. $\square$

This is a weighted form of the refined Bézout inequality; compare [Ful98, Sect. 12.3] and the intersection algorithm of Stückrad and Vogel [Vog84, FOV99]. If $N = 3$ and L has no fixed component, then Y has no surface, $B_1 = 0$ and $A_1 = [V(F_1)]$, and the condition on (F_1, F_2) is that $F_1 \neq 0$ and F_2 has no common factor with F_1 . We then write $m_D(F_1, F_2)$ for the coefficient of a curve component D of Y in $V(F_2) \cdot [V(F_1)]$; this is the number m_D of the lemma.

Lemma 3.2 (multiplicities as lengths). *Let $F_1, \dots, F_c$ be forms on $\mathbb{P}^N$ such that $V(F_1, \dots, F_j)$ has pure codimension j for $j \leq c$, and put $C_0 = [\mathbb{P}^N]$ and $C_j = V(F_j) \cdot C_{j-1}$.*

- (a) *The coefficient of a component W of $V(F_1, \dots, F_c)$ in C_c is $\ell(\mathcal{O}_{\mathbb{P}^N, W}/(F_1, \dots, F_c))$.*
- (b) *Let $c = N - 1$, let W be a curve component, $y \in W$, and let H be a hyperplane through y such that y is an isolated point of $H \cap V(F_1, \dots, F_{N-1})$. Then $\dim_{\mathbb{C}} \mathcal{O}_{H, y}/(F_1|_H, \dots, F_{N-1}|_H)$ is at least the coefficient of W in C_{N-1} , with equality if W is smooth at y , no other component of $V(F_1, \dots, F_{N-1})$ passes through y , and H is transversal to W at y .*

Proof. (a) Induction on c ; the case $c = 1$ is the definition. Let $R = \mathcal{O}_{\mathbb{P}^N, W}$, a regular local ring of dimension c . The F_i form a system of parameters of R , hence a regular sequence, and $A = R/(F_1, \dots, F_{c-1})$ is Cohen–Macaulay of dimension one. Its minimal primes P_V correspond to the components $V \supset W$ of $V(F_1, \dots, F_{c-1})$, and $\ell(A_{P_V})$ is the coefficient of V in C_{c-1} by induction. By (M2), (M3) and (M2) again (the rings $R/P_V = \mathcal{O}_{V, W}$ are one-dimensional domains, hence Cohen–Macaulay),

$$\ell(A/F_c A) = e(F_c; A) = \sum_V \ell(A_{P_V}) \ell(\mathcal{O}_{V, W}/(F_c)),$$

which is the coefficient of W in $C_c = V(F_c) \cdot C_{c-1}$.

(b) Let $A = \mathcal{O}_{\mathbb{P}^N, y}/(F_1, \dots, F_{N-1})$, a Cohen–Macaulay ring of dimension one, and let h be a local equation of H . The hypothesis says that h is a parameter of A , and $A/hA = \mathcal{O}_{H, y}/(F_1|_H, \dots, F_{N-1}|_H)$. By (M2) and (M3), $\ell(A/hA) = \sum_V \ell(A_{P_V})e(h; \mathcal{O}_{V, y})$, the sum over the components V of $V(F_1, \dots, F_{N-1})$ through y . By (a) the term of W is the coefficient of W times $e(h; \mathcal{O}_{W, y}) \geq 1$. Under the extra hypotheses it is the only term, and $e(h; \mathcal{O}_{W, y}) = \ell(\mathcal{O}_{W, y}/(h)) = 1$. $\square$

Lemma 3.3 (location of isolated real points). *Let L be a real linear system, $x \in \Gamma(L)$, and let D be a component of Y with $\dim D \geq 1$ and $x \in D$. If D is real, then $x \in \text{Sing } D$. If D is not real, then $x \in D \cap \overline{D}$. Hence*

$$\Gamma(L) \subset Y_0 \cup \bigcup_{D \text{ real}} \text{Sing } D \cup \bigcup_{D \text{ not real}} (D \cap \overline{D}),$$

where Y_0 is the set of isolated points of Y and D runs over the components of Y of positive dimension.

Proof. Let D be real of dimension $d \geq 1$ and suppose that x is a smooth point of D . The ideal of D is generated by real polynomials, so we can choose real polynomials $g_1, \dots, g_{N-d}$ vanishing on D whose differentials at x are linearly independent. Near x their common zero set is a complex manifold of dimension d containing the germ of D , hence equal to it, and by the real implicit function theorem its real points form a real manifold of dimension d near x . So $D(\mathbb{R})$, and with it $Y(\mathbb{R})$, contains a real manifold of positive dimension through x , and x is not isolated. If D is not real, then $x = \bar{x} \in \overline{D}$. $\square$

Lemma 3.4 (fixed components). *Let $N \geq 2$ and let L be a real linear system of degree k whose members have a greatest common divisor G of degree $f \geq 1$; we may take G real. Let $L' = \{F/G : F \in L\}$, a real system of degree $k - f$ without fixed component, and let L_G be the real span of $\partial G/\partial x_a$, $a = 0, \dots, N$, a system of degree $f - 1$. Then $|\Gamma(L)| \leq |\Gamma(L_G)| + |\Gamma(L')|$.*

Proof. We have $Y(\mathbb{R}) = V_{\mathbb{R}}(G) \cup Y'(\mathbb{R})$ with $Y' = V(L')$. Let $x \in \Gamma(L)$. If $G(x) \neq 0$, then $Y(\mathbb{R}) = Y'(\mathbb{R})$ near x , so $x \in \Gamma(L')$. Let $G(x) = 0$. Since $V_{\mathbb{R}}(G) \subset Y(\mathbb{R})$, the point x is an isolated zero of G in $\mathbb{RP}^N$. In an affine chart around x the dehomogenisation of G has no zero in $B \setminus \{x\}$ for a small ball B , and $B \setminus \{x\}$ is connected because $N \geq 2$; so it has constant sign there, x is a local extremum, and its gradient vanishes at x . With Euler's identity $fG = \sum_a x_a \partial G/\partial x_a$ this gives $\partial G/\partial x_a(x) = 0$ for all a . Euler's identity also shows $V(L_G) \subset V(G)$, so $V_{\mathbb{R}}(L_G) \subset V_{\mathbb{R}}(G)$, in which x is isolated. Hence $x \in \Gamma(L_G)$. $\square$

Lemma 3.5 (pencils of quadrics). *Let Q_1, Q_2 be real quadratic forms in $n + 1$ variables, $n \geq 3$. Then the real set $V_{\mathbb{R}}(Q_1, Q_2) \subset \mathbb{RP}^n$ has at most $2n$ isolated points.*

Proof. Let A, B be the symmetric matrices of Q_1, Q_2 and $M_t = \lambda A + \mu B$ for $t = [\lambda : \mu] \in \mathbb{P}^1$. Let x be an isolated point. If Ax and Bx were linearly independent, then near x the zero set of (Q_1, Q_2) in $\mathbb{R}^{n+1}$ would be a manifold of dimension $n - 1$ and $V_{\mathbb{R}}(Q_1, Q_2)$ a manifold of dimension $n - 2 \geq 1$ near $[x]$. So $M_t x = 0$ for some real t .

Points in one kernel. Let $K = \ker M_t$ for real t , $d = \dim K$, and let M' be a member of the pencil other than M_t . On K the form of M_t vanishes, so $V_{\mathbb{R}}(Q_1, Q_2) \cap \mathbb{P}(K)$ is the real zero set of the real quadratic form $q = M'|_K$ on $\mathbb{P}(K) \cong \mathbb{RP}^{d-1}$. The real zero set of q is empty (q definite), a real projective space $\mathbb{P}(\ker q)$ ($q \neq 0$ semidefinite), all of $\mathbb{P}(K)$ ($q = 0$), or, if q is indefinite, a real quadric of dimension $d - 2$ whose points of $\mathbb{P}(\ker q)$ are limits of its other points. Hence it has at most 1 isolated point if $d \neq 2$ and at most 2 if $d = 2$; in every case at most $\min(d, 2)$.

Regular pencils. If $\det M_t \not\equiv 0$, it is a binary form of degree $n + 1$, and $\dim \ker M_{t_0}$ is at most the multiplicity of t_0 as a root: after a change of basis $M_{t_0} = \text{diag}(0_d, E)$, and d columns of $M_{t_0} + sM'$ are divisible by s . So the number of isolated points is at most $\sum_t \dim \ker M_t \leq n + 1 \leq 2n$.

Singular pencils. Let $\det M_t \equiv 0$, let $d_0 \geq 1$ be the minimal nullity and $T_0 \subset \mathbb{P}^1$ the Zariski open set where it is attained. A nonzero minor of size $n + 1 - d_0$ vanishes at every $t \notin T_0$, so there are at most

n such t . Let U be the span of all $\ker M_t$, $t \in T_0$. If $u \in \ker M_t$ and $w \in \ker M_{t'}$ with $t \neq t'$, write any member as $M_s = \alpha M_t + \beta M_{t'}$; then $u^T M_s w = 0$. If $t = t'$, choose a continuous family $u(\tau) \in \ker M_\tau$ with $u(t) = u$ (the kernels form a vector bundle over T_0), and let $\tau \rightarrow t$. So U is totally isotropic for every member. It is real, since $\overline{\ker M_t} = \ker M_{\bar{t}}$ and T_0 is conjugation invariant. If $\dim U \geq 2$, then $\mathbb{P}(U)(\mathbb{R})$ is a real projective space of positive dimension inside $V_{\mathbb{R}}(Q_1, Q_2)$, so an isolated point lies in $\ker M_t$ for one of the at most n real $t \notin T_0$; this gives at most $2n$ points. If $\dim U = 1$, then $U = \mathbb{C}v$ with v real and $M_t v = 0$ for all $t \in T_0$, hence $Av = Bv = 0$. Then $Q_i(y + cv) = Q_i(y)$, so every real point $y \neq [v]$ of $V_{\mathbb{R}}(Q_1, Q_2)$ lies on a real line inside it, and only $[v]$ can be isolated. $\square$

Lemma 3.6 (singular points of curves). *Let $D \subset \mathbb{P}^N$ be an irreducible curve of degree a whose linear span has dimension w .*

- (a) *If $x \in \text{Sing } D$ and M is a form with $M(x) = 0$ that does not vanish identically on D , then $\ell(\mathcal{O}_{D,x}/(M)) \geq 2$.*
- (b) *If $a < 2w$, then D has at most $a - w$ singular points. In particular lines, conics and twisted cubics are smooth, an irreducible quartic curve spanning a $\mathbb{P}^3$ has at most one singular point, and an irreducible curve of degree $a \leq 7$ spanning $\mathbb{P}^4$ has at most $a - 4$ singular points.*

Proof. (a) $\mathcal{O} = \mathcal{O}_{D,x}$ is a one-dimensional local domain and $M \in \mathfrak{m}_{\mathcal{O}}$. If the length were 1, then $M\mathcal{O} = \mathfrak{m}_{\mathcal{O}}$, the maximal ideal would be principal and $\mathcal{O}$ would be regular, contradicting $x \in \text{Sing } D$.

(b) Suppose D has at least j singular points, $j \leq w$. Choose j of them and $w - j$ further points of D . These w points lie in a hyperplane of the span $\Lambda \cong \mathbb{P}^w$, which does not contain D , and hence in a hyperplane H of $\mathbb{P}^N$ with $D \not\subset H$. By [Har77, Thm. I.7.7] and (a), $a \geq 2j + (w - j) = w + j$. If D had w singular points we would get $a \geq 2w$; so it has fewer than w , and taking j equal to their number gives $j \leq a - w$. $\square$

Lemma 3.7 (Jacobian lemma). *Let L be a real linear system of degree k on $\mathbb{P}^N$, and let $F_1, \dots, F_{N-1} \in L$ be real with $X = V(F_1, \dots, F_{N-1})$ of pure dimension one. Let D be a curve component of Y whose coefficient in $C_{N-1} = V(F_{N-1}) \cdots V(F_1) \cdot [\mathbb{P}^N]$ equals 1, and $a = \deg D$. Then*

$$|\text{Sing } D| \leq \frac{1}{2}(N-1)(k-1)a \quad \text{if } D \text{ is real}, \quad |D \cap \overline{D}| \leq (N-1)(k-1)a \quad \text{if } D \text{ is not real}.$$

Proof. For $j \leq N-1$ every component of $V(F_1, \dots, F_j)$ has dimension at least $N-j$ [Har77, Thm. I.7.2]. A component of larger dimension would meet $V(F_{j+1}, \dots, F_{N-1})$ in a nonempty set whose components have dimension at least 2, and X would have a component of dimension at least 2. So $V(F_1, \dots, F_j)$ has pure codimension j for every $j \leq N-1$, which is the hypothesis of Lemma 3.2(a) used below.

The curves D and $\overline{D}$ are components of X , since $Y \subset X$. Let $\mathcal{J}$ be the $(N-1) \times (N+1)$ matrix $(\partial F_i / \partial x_a)$. In a chart, say $x_0 = 1$, Euler's identity gives $\partial F_i / \partial x_0 = kF_i - \sum_{a \geq 1} x_a \partial F_i / \partial x_a$, so at points of X the column $a = 0$ of $\mathcal{J}$ is a combination of the other columns, which form the affine Jacobian matrix. So at points of X the rank of $\mathcal{J}$ equals the rank of the affine Jacobian matrix, and X is smooth at such a point exactly when this rank is $N-1$: the Zariski tangent space of the scheme X is the kernel of the affine Jacobian matrix.

By Lemma 3.2(a), $\mathcal{O}_{X,D}$ has length one, so X is reduced at the generic point of D . The nilradical of $\mathcal{O}_X$ is a coherent ideal whose support is closed and does not contain the generic point of D . Hence at a general point $y \in D$ the scheme X is reduced, no other component of X passes through y , and D is smooth at y ; so X is smooth at y and some maximal minor of $\mathcal{J}$ does not vanish at y . The combinations of the maximal minors of $\mathcal{J}$ vanishing on D , or on $\overline{D}$, form proper linear subspaces, so a general real combination M , a form of degree $(N-1)(k-1)$, vanishes identically neither on D nor on $\overline{D}$.

Let $x \in \text{Sing } D$, or $x \in D \cap \overline{D}$ if D is not real. If X were smooth at x , then $\mathcal{O}_{X,x}$ would be a regular local ring, hence a domain, so only one component of X would pass through x , and X would coincide near x with that component, which would then be smooth at x . Both cases are impossible. Hence all maximal minors of $\mathcal{J}$ vanish at x , and $M(x) = 0$. By [Har77, Thm. I.7.7] the intersection multiplicities of D and $V(M)$ add up to $a(N-1)(k-1)$. Each point of $\text{Sing } D$ contributes at least 2 (Lemma 3.6(a)) and each point of $D \cap \overline{D}$ at least 1. $\square$

4 The plane, and quadrics in $\mathbb{P}^3$

Proposition 4.1 (Choi–Lam–Reznick). *$T(k, 2)$ holds for every $k \geq 0$.*

Proof. Induction on k . If $k = 0$, then $Y = \emptyset$ or $Y = \mathbb{P}^2$, so $\Gamma(L) = \emptyset$. If $k = 1$, then $Y(\mathbb{R})$ is empty or a real projective space, so $|\Gamma(L)| \leq 1$. Let $k \geq 2$. If L has a fixed component of degree $f \geq 1$, Lemma 3.4 and induction give $|\Gamma(L)| \leq (f-1)^2 + (k-f)^2 \leq k^2$. Otherwise Y is finite and $|\Gamma(L)| \leq |Y| \leq k^2$ by Lemma 3.1. $\square$

This is a variant of the proof of [CLR80, Thm. 4.6], where the fixed part is treated with a bound for the zeros of psd ternary forms instead of the gradient system. With Proposition 2.1 it gives $B'_{2k,3} = \sharp(2k, 2) = k^2$.

Proof of Proposition 1.7. Let L be a real linear system of quadrics on $\mathbb{P}^3$. If L has a fixed component G , then either $\deg G = 2$, $L = \mathbb{R}G$, $\Gamma(L') = \emptyset$ and L_G consists of linear forms, or $\deg G = 1$, L_G consists of constants and L' of linear forms. A real linear system of linear forms has at most one isolated real base point, so $|\Gamma(L)| \leq 1$ by Lemma 3.4.

Otherwise $\dim Y \leq 1$. Choose real $F_1 \neq 0$ and F_2 without common factor with F_1 , and write $m_D = m_D(F_1, F_2)$. Lemma 3.1 gives $2 \sum_D m_D \deg D + |Y_0| \leq 8$, the sum over the curve components. We show that the points of $\Gamma(L)$ assigned to a real D by Lemma 3.3 number at most $2m_D \deg D$, and those assigned to a pair $\{D, \overline{D}\}$ at most $4m_D \deg D$; then $|\Gamma(L)| \leq 8$. Let $a = \deg D$. If $m_D = 1$, Lemma 3.7 with $N = 3$, $k = 2$ gives $|\text{Sing } D| \leq a$ and $|D \cap \overline{D}| \leq 2a$. If $m_D \geq 2$ and D is real, then $4a \leq 8$, so D is a line or a conic and is smooth. If $m_D \geq 2$ and D is not real, then $8a \leq 8$, so D and $\overline{D}$ are lines, which meet in at most one point. The grid shows $B'_{4,4} \geq 8$ and $\sharp(4, 3) \geq 8$. $\square$

5 Curve components in $\mathbb{P}^3$

In this section L is a real linear system of degree k on $\mathbb{P}^3$ without fixed component, D is a curve component of Y , and $a = \deg D$. For a form F let $\text{mult}_D F$ be its multiplicity at a general point of D , and let $\mu(L, D) = \min\{\text{mult}_D F : 0 \neq F \in L_{\mathbb{C}}\} \geq 1$. The members with $\text{mult}_D F > \mu(L, D)$ form a proper linear subspace of $L_{\mathbb{C}}$.

Proposition 5.1. *Let $F_1, F_2 \in L$ be real without common factor, with $\text{mult}_D F_1 = \mu(L, D)$, and let $m = m_D(F_1, F_2)$. Then*

$$|\text{Sing } D| \leq k m a \quad \text{if } D \text{ is real}, \quad |D \cap \overline{D}| \leq 2k m a \quad \text{if } D \text{ is not real}.$$

We first prove two lemmas. For forms f, G let $M_{ab}(f, G) = \partial_a f \partial_b G - \partial_b f \partial_a G$, where $0 \leq a < b \leq 3$ and $\partial_a = \partial/\partial x_a$.

Lemma 5.2 (order drop). *Let f be an irreducible form of degree s and $S = V(f)$, and suppose that the differential of f does not vanish identically on the curve $D \subset S$. Let G be a form of degree d whose restriction to S vanishes along D to order exactly $t \geq 1$, i.e. $G|_S$ has valuation t in the discrete*

valuation ring $\mathcal{O}_{S,D}$. For $c \in \mathbb{C}^6$ let $J_c = \sum_{a < b} c_{ab} M_{ab}(f, G)$, a form of degree $d + s - 2$ or zero. Then $J_c|_S$ vanishes along D to order at least $t - 1$ for every c , and to order exactly $t - 1$ for all c outside a proper linear subspace of $\mathbb{C}^6$.

Proof. S is smooth at the general points of D , so $\mathcal{O}_{S,D}$ is a discrete valuation ring, and the valuation of a function can be read off near any general point of D . Choose a general point $y \in D$ in an affine chart meeting D , say $x_0 = 1$, and write $\hat{F}(x_1, x_2, x_3) = F(1, x_1, x_2, x_3)$ for a form F . Then $u = \hat{f}$ has nonzero differential at y , and there are local holomorphic coordinates (u, v, w) at y with $S = \{u = 0\}$ and $D = \{u = v = 0\}$. Write $\hat{G}(0, v, w) = v^t \psi$ with $\psi(y) \neq 0$. In these coordinates $d\hat{f} \wedge d\hat{G} = \hat{G}_v du \wedge dv + \hat{G}_w du \wedge dw$, and on S

$$\hat{G}_v|_S = v^{t-1}(t\psi + v\psi_v), \quad \hat{G}_w|_S = v^t\psi_w,$$

of orders exactly $t - 1$ (as $t \neq 0$ in $\mathbb{C}$) and at least t . For $1 \leq a < b \leq 3$, $\hat{M}_{ab} = \partial_a \hat{f} \partial_b \hat{G} - \partial_b \hat{f} \partial_a \hat{G}$ are the coefficients of $d\hat{f} \wedge d\hat{G}$ in the basis $dx_a \wedge dx_b$. The change to the basis $du \wedge dv, du \wedge dw, dv \wedge dw$ is invertible with holomorphic coefficients, so $\hat{M}_{ab} = \lambda_{ab} \hat{G}_v + \lambda'_{ab} \hat{G}_w$ with holomorphic $\lambda_{ab}, \lambda'_{ab}$, and the vector $(\lambda_{ab}(y))_{a < b}$ is nonzero. For $a, b \geq 1$ we have $\widehat{M_{ab}(f, G)} = \hat{M}_{ab}$, and Euler's identities for f and G give

$$\widehat{M_{0b}(f, G)} = s\hat{f} \partial_b \hat{G} - d\hat{G} \partial_b \hat{f} - \sum_{a=1}^3 x_a \hat{M}_{ab} \quad (\hat{M}_{aa} = 0, \hat{M}_{ab} = -\hat{M}_{ba}).$$

On S the first term vanishes and the second has order at least t . Hence $\hat{J}_c|_S = \alpha_c \hat{G}_v|_S +$ (terms of order $\geq t$), where α_c is holomorphic and depends linearly on c . So the order is at least $t - 1$, and it equals $t - 1$ unless α_c vanishes on D near y . These c form a linear subspace, and it is proper: for c with $c_{0b} = 0$ for all b we have $\alpha_c(y) = \sum_{a < b} c_{ab} \lambda_{ab}(y)$. $\square$

Lemma 5.3 (the derivative system). *Suppose $\mu = \mu(L, D) \geq 2$, and let $L^{(1)}$ be the real span of the partial derivatives $\partial_a F$, $F \in L$, $a = 0, \dots, 3$, a system of degree $k - 1$. Then:*

- (i) $V(L^{(1)}) \subset Y$, the curve D is a component of $V(L^{(1)})$, $L^{(1)}$ has no fixed component, and $1 \leq \mu(L^{(1)}, D) \leq \mu - 1$;
- (ii) for every pair $F_1, F_2 \in L$ of real forms without common factor there are real $G_1, G_2 \in L^{(1)}$ without common factor, avoiding any given proper Zariski closed subset of $L_{\mathbb{C}}^{(1)} \times L_{\mathbb{C}}^{(1)}$, such that $m_D(G_1, G_2) \leq m_D(F_1, F_2)$.

Proof. (i) Euler's identity $kF = \sum_a x_a \partial_a F$ gives $V(L^{(1)}) \subset Y$, so $V(L^{(1)})$ contains no surface and $L^{(1)}$ has no fixed component. Every $F \in L$ vanishes to order at least 2 along D , so its first partial derivatives vanish on D , and D , a component of Y , is a component of $V(L^{(1)})$. If $\text{mult}_D F = \mu$, some partial derivative of order μ of $\hat{F}$ is nonzero at a general point of D ; it is a derivative of order $\mu - 1$ of some $\partial_a \hat{F} = \widehat{\partial_a F}$, $a \geq 1$, so $\text{mult}_D \partial_a F \leq \mu - 1$.

(ii) Choose a point $y \in D$ that is a smooth point of D , lies on no other component of $V(F_1, F_2)$ and on no other component of Y ; we use an affine chart containing y , say $x_0 = 1$. Let H be a plane through y transversal to D , with affine coordinates $s = (s_1, s_2)$ centred at y , so that $x(s) = y + s_1 v_1 + s_2 v_2$. Let $R = \mathbb{C}\{s_1, s_2\}$ with maximal ideal $\mathfrak{m}$. For a form F put $F^b(s) = F(1, x(s)) \in R$, and let

$$I = (F^b : F \in L)R, \quad K = (G^b : G \in L^{(1)})R.$$

Near y the set $Y \cap H$ is $\{y\}$, and $y \in V(L^{(1)}) \cap H \subset Y \cap H$; so I and K are $\mathfrak{m}$ -primary.

Step 1: $e(I) \leq m_D(F_1, F_2)$. By Lemma 3.2(b), $m_D(F_1, F_2) = \ell(R/(F_1^b, F_2^b))$. This ideal has finite colength, so by (M2) and (M1) the length equals $e((F_1^b, F_2^b)) \geq e(I)$.

Step 2: $I \subset \overline{\mathbf{m}K}$. Let $F \in L$. By the chain rule $\partial F^b / \partial s_j = \sum_{a=1}^3 v_{j,a} (\partial_a F)^b \in K$, and $F^b(0) = F(1, y) = 0$. Let $\gamma = (\gamma_1, \gamma_2)$ be a nonconstant analytic arc with $\gamma(0) = 0$, and write ord for the order in t . If $F^b \circ \gamma \neq 0$, then, since the characteristic is 0 and $F^b(\gamma(0)) = 0$,

$$\text{ord}(F^b \circ \gamma) = 1 + \text{ord}\left(\sum_j (\partial_{s_j} F^b)(\gamma) \gamma'_j\right) \geq \text{ord}(K \circ \gamma) + \text{ord}(\mathbf{m} \circ \gamma) = \text{ord}((\mathbf{m}K) \circ \gamma),$$

using $\text{ord} \gamma'_j \geq \text{ord} \gamma_j - 1 \geq \text{ord}(\mathbf{m} \circ \gamma) - 1$; here $\text{ord}(\mathbf{a} \circ \gamma)$ is the least order of $g \circ \gamma$, $g \in \mathbf{a}$. The inequality is also true if $F^b \circ \gamma = 0$, and trivially for constant arcs. By the valuative criterion of Lejeune-Jalabert and Teissier [LJT08, Thm. 2.1, (iii) $\Rightarrow$ (i)], applied to the $\mathbf{m}$ -primary ideal $\mathbf{m}K$ of $\mathbb{C}\{s_1, s_2\}$, we get $F^b \in \overline{\mathbf{m}K}$.

Step 3: $e(K) \leq e(I)$. By Step 2, (M1) and (M4), $e(I) \geq e(\overline{\mathbf{m}K}) = e(\mathbf{m}K) \geq e(K)$.

Step 4: choice of G_1, G_2 . The fibre cone $\mathcal{F} = \bigoplus_{n \geq 0} K^n / \mathbf{m}K^n$ is a standard graded $\mathbb{C}$ -algebra of dimension 2, because K is $\mathbf{m}$ -primary in a two-dimensional local ring [NR54], [HS06, Ch. 8]. By graded Noether normalisation, the pairs $(\lambda_1, \lambda_2) \in \mathcal{F}_1^2$ for which $\mathcal{F}/(\lambda_1, \lambda_2)$ has finite length form a nonempty Zariski open set. For such a pair, lifted to $g_1, g_2 \in K$, we get $K^{n+1} = (g_1, g_2)K^n + \mathbf{m}K^{n+1}$ for large n , hence $K^{n+1} = (g_1, g_2)K^n$ by Nakayama's lemma; so (g_1, g_2) is a reduction of K [NR54], it has the same integral closure and, by (M4), the same multiplicity. The map $(G_1, G_2) \mapsto (G_1^b, G_2^b) \bmod \mathbf{m}K$ from $L_{\mathbb{C}}^{(1)} \times L_{\mathbb{C}}^{(1)}$ to $\mathcal{F}_1^2$ is linear and surjective. So there is a nonempty Zariski open set U of pairs with (G_1^b, G_2^b) a reduction of K , and then, by (M2), $\ell(R/(G_1^b, G_2^b)) = e(K)$. Let U' be the complement in U of the given closed set, and choose a real pair $(G_1, G_2^0) \in U'$. The G_2 with $(G_1, G_2) \in U'$ form a Zariski open subset of $L_{\mathbb{C}}^{(1)}$ containing G_2^0 , and the G_2 with a common factor with G_1 lie in finitely many proper linear subspaces, because $L^{(1)}$ has no fixed component. So there is a real G_2 with $(G_1, G_2) \in U'$ and without common factor with G_1 . Then $V(G_1, G_2)$ has pure dimension one and contains D as a component. Since (G_1^b, G_2^b) is $\mathbf{m}$ -primary, y is isolated in $H \cap V(G_1, G_2)$, and Lemma 3.2(b) gives $m_D(G_1, G_2) \leq \ell(R/(G_1^b, G_2^b)) = e(K)$. With Steps 1 and 3, $m_D(G_1, G_2) \leq e(K) \leq e(I) \leq m_D(F_1, F_2)$. $\square$

Proof of Proposition 5.1. Induction on $\mu = \mu(L, D)$, for all systems and all degrees at once.

Case $\mu = 1$. Then $\text{mult}_D F_1 = 1$: the differential of F_1 does not vanish at a general point of D . Hence exactly one component $S = V(f)$ of $V(F_1)$ contains D , the irreducible form f (of degree $s \leq k$) divides F_1 exactly once, and the differential of f does not vanish identically on D ; otherwise dF_1 would vanish on D . In particular S is smooth at the general points of D , $\mathcal{O}_{S,D}$ is a discrete valuation ring, and by the definition of $V(F_2) \cdot [V(F_1)]$ we have $m = r$, where $r \geq 1$ is the order of vanishing of $F_2|_S$ along D ($F_2|_S \neq 0$ as $f \nmid F_2$).

If S is not real, then D is not real (if $\overline{D} = D$, then $\bar{S}$ is a component of $V(F_1)$ containing D , so $\bar{S} = S$). The component $\bar{S}$ of $V(F_1)$ contains $\overline{D}$, and $D \not\subset \bar{S}$ by uniqueness of S . Hence $D \cap \overline{D} \subset D \cap \bar{S}$, which has at most $as \leq ak \leq 2kma$ points [Har77, Thm. I.7.7].

Let S be real; take f real. Put $J_0 = F_2$ and, for $i = 1, \dots, r$, let J_i be a real combination $\sum c_{ab}^{(i)} M_{ab}(f, J_{i-1})$ with $c^{(i)}$ outside the proper linear subspaces of Lemma 5.2 for D and, if D is not real, for $\overline{D}$. The orders of a real form along D and along $\overline{D}$ agree, so by Lemma 5.2 the real form J_i has degree $k + i(s - 2)$ and $J_i|_S$ vanishes to order exactly $r - i$ along D and along $\overline{D}$. (For $s = 1$ these degrees stay nonnegative: $F_2|_S$ is a plane curve of degree k divisible by the r -th power of the equation of D , so $ra \leq k$.) Let $X' = V(f, J_{r-1}) = S \cap V(J_{r-1})$. As $J_{r-1}|_S \neq 0$, this is a complete intersection of pure dimension one, and it contains D and $\overline{D}$.

Let $x \in \text{Sing } D$, or $x \in D \cap \overline{D}$ if D is not real. As in the proof of Lemma 3.7, Euler's identities show that at points of X' the homogeneous Jacobian matrix of (f, J_{r-1}) has the rank of the affine one. If this rank were 2 at x , the complete intersection X' would be smooth at x , hence locally irreducible there, and it would coincide near x with D , which would then be smooth at x ; if D is not real, D and

$\overline{D}$ would be two components of X' through x . So all minors $M_{ab}(f, J_{r-1})$ vanish at x , and $J_r(x) = 0$. Since J_r does not vanish identically on D , the intersection multiplicities of D and $V(J_r)$ add up to $a(k + r(s - 2)) \leq a(k + r(k - 2))$. With Lemma 3.6(a),

$$|\text{Sing } D| \leq \frac{1}{2}a(k + r(k - 2)) \leq kra \quad (D \text{ real}), \quad |D \cap \overline{D}| \leq a(k + r(k - 2)) \leq 2kra \quad (D \text{ not real}),$$

because $k + r(k - 2) \leq 2kr$ for $r \geq 1$.

Case $\mu \geq 2$. By Lemma 5.3 there are real $G_1, G_2 \in L^{(1)}$ without common factor, with $\text{mult}_D G_1 = \mu(L^{(1)}, D)$ and $m_D(G_1, G_2) \leq m$. Since $1 \leq \mu(L^{(1)}, D) < \mu$ and D is a curve component of the base locus of the real system $L^{(1)}$ of degree $k - 1$ without fixed component, the induction hypothesis applies to $(L^{(1)}, D, G_1, G_2)$: $|\text{Sing } D| \leq (k - 1)m_D(G_1, G_2)a \leq kma$ if D is real, and $|D \cap \overline{D}| \leq 2(k - 1)m_D(G_1, G_2)a \leq 2kma$ otherwise. $\square$

6 Proof of Theorem 1.4

We use induction on k . If $k = 0$, then $Y = \emptyset$ or $Y = \mathbb{P}^3$, so $\Gamma(L) = \emptyset$. If $k = 1$, then $Y(\mathbb{R})$ is empty or a real projective space, so $|\Gamma(L)| \leq 1$. Let $k \geq 2$.

If L has a fixed component of degree $f \geq 1$, then by Lemma 3.4 and the induction hypothesis $|\Gamma(L)| \leq (f - 1)^3 + (k - f)^3 \leq (k - 1)^3 < k^3$.

Otherwise Y has no surface. Choose a real $F_1 \in L$ with $\text{mult}_D F_1 = \mu(L, D)$ for every curve component D of Y , and a real $F_2 \in L$ without common factor with F_1 . By Lemma 3.1,

$$k \sum_D m_D \deg D + |Y_0| \leq k^3, \quad m_D = m_D(F_1, F_2).$$

Since F_1 and F_2 are real, conjugation preserves the cycle $V(F_2) \cdot [V(F_1)]$, so $m_{\overline{D}} = m_D$. By Lemma 3.3 and Proposition 5.1,

$$|\Gamma(L)| \leq |Y_0| + \sum_{D \text{ real}} k m_D \deg D + \sum_{\{D, \overline{D}\}} 2k m_D \deg D \leq k^3,$$

the last sum over the pairs of conjugate non-real curve components. This proves Theorem 1.4, and Corollary 1.6 for $m = 4$ and $l = 3$ follows from Proposition 2.1 and Example 2.2. $\square$

Remark 6.1. If G is a nonnegative form of degree d in four variables with finitely many real zeros, then every real zero is a critical point, $\mathcal{Z}(G)$ is the real base locus of the gradient system of G , and Theorem 1.4 gives $|\mathcal{Z}(G)| \leq (d - 1)^3$. This bound is not new: it also follows from the standard perturbation and critical point argument behind the bound $\sharp(2k, l) \leq (2k - 1)^l$ recorded in [Sha15]. It is far from sharp for $d = 4$, where $B_{4,4} = 10$ [CLR80, BHO⁺12].

7 Proof of Theorem 1.5

Let L be a real linear system of quadrics on $\mathbb{P}^4$.

Fixed component. As in the proof of Proposition 1.7, Lemma 3.4 gives $|\Gamma(L)| \leq 1$. From now on L has no fixed component, so $\dim Y \leq 2$. We use two facts.

- (F1) If an irreducible curve of degree at least 3 in Y lies in a plane Π , or two distinct conics in Y lie in Π , then $\Pi \subset Y$. If an irreducible surface of degree at least 3 in Y lies in a hyperplane W , then $W \subset Y$. Indeed, a nonzero restriction $Q|_{\Pi}$ is a conic and a nonzero restriction $Q|_W$ is a quadric surface; neither can contain such a curve or surface.

(F2) Let W be a hyperplane with $W \not\subset Y$, and let $D \subset Y$ be an irreducible curve spanning W that is not contained in a two-dimensional component of Y . Then $L|_W$ has no fixed component, and $Y \cap W$ lies in a curve of degree 4, the intersection of two general members of $L|_W$; in particular $\deg D \leq 4$. Indeed, a fixed component T of $L|_W$ would lie in Y , so each of its components would lie in a two-dimensional component of Y . If $\deg T = 2$ then $L|_W = \mathbb{C}T$ and $D \subset T$, which is excluded. If $\deg T = 1$ then $Y \cap W = T \cup V(L'')$ with L'' a nonzero system of linear forms on W , and $D \subset V(L'')$, a proper linear subspace of W , which contradicts the assumption that D spans W .

Case $\dim Y \leq 1$. Choose real $Q_1 \neq 0$, then Q_2 without common factor with Q_1 , then Q_3 vanishing identically on no component of $V(Q_1, Q_2)$ (these are surfaces, not contained in Y). Then $X = V(Q_1, Q_2, Q_3)$ has pure dimension one, and by Lemma 3.1 $2 \sum_D m_D \deg D + |Y_0| \leq 16$, where m_D is the coefficient of D in $V(Q_3) \cdot V(Q_2) \cdot [V(Q_1)]$ and $m_{\bar{D}} = m_D$. By Lemma 3.3 it suffices to show $|\text{Sing } D| \leq 2m_D a$ for real D and $|D \cap \bar{D}| \leq 4m_D a$ for non-real D , where $a = \deg D$.

- $m_D = 1$: Lemma 3.7 with $N = 4$, $k = 2$ gives $|\text{Sing } D| \leq \frac{3}{2}a$ and $|D \cap \bar{D}| \leq 3a$.
- $m_D \geq 2$, D real: $4a \leq 16$, so $a \leq 4$. By (F1) and $\dim Y \leq 1$, D is not a plane curve of degree at least 3. So D is a line or a conic, or it spans a $\mathbb{P}^3$ or $\mathbb{P}^4$ with $a \leq 4$, and Lemma 3.6(b) gives $|\text{Sing } D| \leq 1$.
- $m_D \geq 2$, D not real: $8a \leq 16$, so $a \leq 2$. Two conjugate lines meet in at most one point. If D is a conic in a plane P , then $P \neq \bar{P}$ by (F1), and $D \cap \bar{D} \subset D \cap P \cap \bar{P}$, where $P \cap \bar{P}$ is a point or a line; so $|D \cap \bar{D}| \leq 2$.

Hence $|\Gamma(L)| \leq |Y_0| + \sum_D 2m_D \deg D \leq 16$.

Case $\dim Y = 2$. Choose real $Q_1 \neq 0$ and Q_2 without common factor. In Lemma 3.1 the two-dimensional components Z of Y occur in $B_2 \leq V(Q_2) \cdot [V(Q_1)]$, a cycle of degree 4, and

$$4 \sum_Z m_Z \deg Z + 2 \sum_D m_D \deg D + |Y_0| \leq 16, \quad \sum_Z m_Z \deg Z \leq 4, \quad (1)$$

where D runs over the curve components. Let $e = \sum_Z \deg Z \leq 4$. Recall that an irreducible nondegenerate variety has degree at least its codimension plus one [EH87]; so an irreducible surface of degree 2 is a quadric surface in a $\mathbb{P}^3$, with at most one singular point, and by (F1) an irreducible surface of degree 3 in Y spans $\mathbb{P}^4$ and is a variety of minimal degree, hence a smooth cubic scroll or the cone over a twisted cubic [EH87, Thm. 1], with at most one singular point.

Subcase $e = 4$. Then all $m_Z = 1$ and (1) leaves no room for curves or points; moreover $B_2 = V(Q_2) \cdot [V(Q_1)]$, so $A_2 = 0$ and $V(Q_1, Q_2) = Y$. By Lemma 3.2(a) the complete intersection scheme $V(Q_1, Q_2)$ is reduced at the generic point of each component. It is Cohen–Macaulay, so its nilradical, whose associated points would be associated points of the scheme, vanishes; the scheme is reduced. The ideal (Q_1, Q_2) is unmixed of height 2 [Mat86, Thm. 17.6], hence saturated, so it is the ideal of Y . Every member of L vanishes on Y , so $L_{\mathbb{C}}$ is contained in the degree-2 part $\mathbb{C}Q_1 + \mathbb{C}Q_2$ of this ideal: L is a real pencil. Lemma 3.5 with $n = 4$ gives $|\Gamma(L)| \leq 8$.

Subcase $e \leq 3$. Non-real two-dimensional components come in pairs of equal degree, so they are planes, forming at most one pair, and there is at most one real component of degree 2 or 3; a pair of planes and such a component do not occur together. By Lemma 3.3, a point of $\Gamma(L)$ on a two-dimensional component lies on no real plane component (planes are smooth), and it is the singular point of a real component of degree 2 or 3, or it lies on $P \cap \bar{P}$ for a pair of planes. If $P \cap \bar{P}$ is a line, this line is real, lies in Y , and carries no point of $\Gamma(L)$. So at most one point of $\Gamma(L)$ lies on a two-dimensional component.

Let D be a curve component, $a = \deg D$. By (1) $2 \sum_D m_D \deg D \leq 16 - 4e \leq 12$; so $a \leq 6$ if D is real and $a \leq 3$ if not. We show $|\text{Sing } D| \leq 2 \leq 2m_D a$ and $|D \cap \overline{D}| \leq 3 \leq 4m_D a$. Let w be the dimension of the span of D . A plane curve component has degree at most 2, because by (F1) its plane would otherwise be a two-dimensional component containing it; so for $w \leq 2$, D is a line or a conic and is smooth. If $w = 3$, then (F2) gives $a \leq 4$ and Lemma 3.6(b) gives $|\text{Sing } D| \leq 1$. If $w = 4$, then $a \leq 6 < 8$ and $|\text{Sing } D| \leq a - 4 \leq 2$. For a non-real D : lines meet in at most one point; for conics, $P \neq \bar{P}$ by (F1) and $|D \cap \overline{D}| \leq 2$ as before; if $a = 3$, then D spans a hyperplane W , and $W \neq \overline{W}$, since otherwise (F2) would put the curve $D \cup \overline{D}$ of degree 6 into a curve of degree 4; then $W \cap \overline{W}$ is a plane not containing D , and $|D \cap \overline{D}| \leq |D \cap W \cap \overline{W}| \leq 3$.

By Lemma 3.3, $|\Gamma(L)| \leq 1 + |Y_0| + \sum_D 2m_D \deg D \leq 1 + 16 - 4e \leq 13$.

In all cases $|\Gamma(L)| \leq 16$, which proves Theorem 1.5; with Proposition 2.1 and Example 2.2 it gives $B'_{4,5} = \sharp(4, 4) = 16$. $\square$

8 Quadrics in higher dimension

Proposition 8.1. *Let L be a real linear system of quadrics on $\mathbb{P}^5$. If L has a fixed component, or $\dim Y \leq 1$, or L is a pencil, then $|\Gamma(L)| \leq 32$. More precisely $|\Gamma(L)| \leq 1$ in the first case and $|\Gamma(L)| \leq 10$ in the third.*

Proof. The first case is Lemma 3.4 as in Proposition 1.7, and the third is Lemma 3.5 with $n = 5$. Let $\dim Y \leq 1$. Choose real $Q_1, \dots, Q_4 \in L$ one after the other, each vanishing identically on no component of the intersection of the previous ones; these components have dimension at least 2 and are not contained in Y . Lemma 3.1 gives $2 \sum_D m_D \deg D + |Y_0| \leq 32$, and it suffices to bound, for each curve component D of degree a , $|\text{Sing } D|$ by $2m_D a$ (D real) and $|D \cap \overline{D}|$ by $4m_D a$ (D not real). If $m_D = 1$, Lemma 3.7 with $N = 5$, $k = 2$ gives $|\text{Sing } D| \leq 2a$ and $|D \cap \overline{D}| \leq 4a$.

Let $m_D \geq 2$, and let $W \cong \mathbb{P}^w$ be the span of D . If $w = 1$, then D is a line, which is smooth, and two conjugate lines meet in at most one point. Let $w \geq 2$. If W is real and $w \leq 4$, then $W \not\subset Y$, the curve D is a component of the base locus $Y \cap W$ of the real system $L|_W$, which has dimension at most one, and Lemma 3.1 on W , with members of $L|_W$ chosen one after the other as above, gives $2m_D^W a \leq 2^w$, where $m_D^W \geq 1$ is the coefficient of D for $L|_W$ (and $m_D^W = m_D^W$).

If D is real, then $a \leq 8$. For $w = 5$, Lemma 3.6(b) gives $|\text{Sing } D| \leq a - 5 \leq 3$. For $2 \leq w \leq 4$ we get $a \leq 2^{w-1}$, so D is a conic ($w = 2$), or $w = 3$, $a \leq 4$ and $|\text{Sing } D| \leq 1$, or $w = 4$ and $a \leq 8$. In the last case $|\text{Sing } D| \leq a - 4 \leq 3$ if $a \leq 7$, while for $a = 8$ we have $m_D^W = 1$ and Lemma 3.7 on W with $N = 4$ gives $|\text{Sing } D| \leq 12$. So $|\text{Sing } D| \leq 12 \leq 4a$ if $a = 8$ and $|\text{Sing } D| \leq 3$ otherwise; both are at most $2m_D a$.

If D is not real, then $m_D a \leq 8$, so $a \leq 4$ and $w \leq 4$. If $W \neq \overline{W}$, then $W \cap \overline{W}$ lies in a hyperplane of W not containing D , so $|D \cap \overline{D}| \leq a$. If $W = \overline{W}$, both D and $\overline{D}$ are components of $Y \cap W$, and Lemma 3.1 on W gives $4m_D^W a \leq 2^w$ with $a \geq w \geq 2$; this forces $w = a = 4$ and $m_D^W = 1$, and Lemma 3.7 on W gives $|D \cap \overline{D}| \leq 12$. So $|D \cap \overline{D}| \leq a$, or $a = 4$ and $|D \cap \overline{D}| \leq 12$; both are at most $4m_D a$. $\square$

So $B'_{4,6} = 32$ would follow from $T(2, 5)$ for the systems that have no fixed component, are not pencils and have $\dim Y \in \{2, 3\}$. For these, Y lies in no $V(Q_1, \dots, Q_4)$ of pure dimension one, so Lemma 3.7 is not available for the curve components either; in the proof of Theorem 1.5 this case needed (F1), (F2) and Lemma 3.6. One therefore needs bounds for the isolated real points on all components of the base locus of such a system, not only on those of dimension 2 and 3. We have not been able to prove such bounds.

Remark 8.2 (limits of the method). For quadrics on $\mathbb{P}^N$ Lemma 3.7 bounds a real component of multiplicity one by $\frac{1}{2}(N - 1) \deg D$ singular points, while its budget in Lemma 3.1 is $2 \deg D$. For

$N \geq 6$ the lemma is therefore too weak. There is also a structural obstacle. A smooth complete intersection curve of $N - 1$ quadrics in $\mathbb{P}^N$ has canonical bundle $\mathcal{O}(N - 3)$ and hence arithmetic genus $(N - 3)2^{N-2} + 1$, which exceeds 2^N for $N \geq 7$. Our arguments do not exclude a real nodal curve of this kind with more than 2^N isolated real nodes and no other real points; such a curve would contradict $T(2, N)$. We do not know whether such curves exist.

Verification. The proofs above are complete; the computations are checks of individual steps and do not replace any argument.

1. Exact scripts written when the result was found (Python standard library, rational arithmetic, fixed seeds; they reproduce their recorded outputs) check the following.
 - Lemma 5.2, iterated as in the proof of Proposition 5.1, on 95 random examples: lines and conics on surfaces of degree at most 3, $k \leq 5$, contact order at most 4; orders along D are measured on exact power series arcs.
 - The inequality $m_D(G_1, G_2) \leq m_D(F_1, F_2)$ of Lemma 5.3, on 54 random examples, all for the line $D = \{x_2 = x_3 = 0\}$: 18 random systems of forms vanishing to order $\mu \in \{2, 3\}$ along D , and 36 systems from structured families with degenerate tangent cones. Of these, 42 have $\mu \geq 2$; the other 12 have $\mu = 1$ and lie outside the hypothesis of Lemma 5.3. Intersection numbers are computed as orders of resultants.
 - The numerical inequalities, and the grid examples for $N \leq 4$, $k \leq 3$.
2. A separate script by the author, `lead_checks.py` (under one second), checks the following.
 - Lemma 5.2 on 66 further examples by a different method: explicit surfaces of degree 1, 2, 3 that are graphs over a coordinate plane, containing a line or a curve $\{x_1 = x_2^2\}$, with orders computed by exact substitution. It also checks the bound “at least $t - 1$ ” for combinations of the minors with index 0 alone.
 - The monomial case of Steps 2–3 of Lemma 5.3: $e(I) \geq e(K) + 3$ for 400 random monomial ideals $I \subset \mathfrak{m}^2$ in two variables, with K generated by the partial derivatives of the generators. Multiplicities are computed from Newton polygons, and the area formula is checked against the colength function.
 - All rows of the case tables in the proofs of Proposition 1.7, Theorem 1.5 and Proposition 8.1, and the agreement of Lemma 3.6(b) with Castelnuovo’s numbers.
 - The numerical inequalities of Sections 5 and 6 for $k, r < 400$, and the grid examples for $N, k \leq 4$.
3. Two independent verification runs, both AI-assisted, checked the proofs.
 - The first run re-read the sources [FLOS18, Sha15, Dre21, BHO⁺12] and checked every step of an earlier version of these proofs; it wrote no code. It found no mathematical gap and asked for changes of presentation and attribution, which are included. Since that version the curve case of Theorem 1.5 uses Lemmas 3.6 and 3.7 instead of Castelnuovo’s bound, Lemma 5.3 is written out in full, and the proofs of Propositions 1.7 and 8.1 were rewritten with Lemma 3.7. The script `lead_checks.py` of item 2 checks the case tables of these proofs and the monomial case of Lemma 5.3; the second run checked the proofs themselves.

- The second run checked the version that preceded the present one line by line, including Lemmas 3.6, 3.7 and 5.3, Propositions 1.7, 5.1 and 8.1 and Theorem 1.5, and found no gap. It re-read [FLOS18, Sha15, Dre21, BHO⁺12], and it checked the statements cited from [CLR80] and [LJT08] in their texts (for [CLR80] a scan of the journal version). Before reading the scripts of items 1 and 2 it wrote its own code from the text of the paper: exact tests of Lemma 5.2 in 105 cases and of Lemma 5.3 in 232 systems with 1223 pairs (F_1, F_2) , along lines, conics, twisted cubics and nodal cubics, with 696 checks of Step 2 by Rees' multiplicity criterion; the construction of the case $\mu = 1$ of Proposition 5.1 in 30 cases, on a nodal cubic and on a non-real line that meets its conjugate; and heuristic numerical counts of isolated real zeros for $(N, k) = (3, 2), (3, 3), (4, 2)$, none of them above k^N . It also reran the scripts of items 1 and 2, which reproduced their recorded outputs. The present version differs from the checked one only by the clarifications and small corrections that the run asked for or suggested: the index convention of [FLOS18], the hypothesis of Lemma 3.2(a) in the proof of Lemma 3.7, the remark after Proposition 8.1, the description of the second check of item 1 and of the two runs, the credit for the case $l = 2$, and some wording.

Scope and priority. Theorem 1.4 settles the row $m = 4$ of Conjecture 1.1 and the row $l = 3$ of Conjecture 1.2, and Theorem 1.5 settles the cases $(n, m) = (4, 5)$ and $(k, l) = (2, 4)$. Both conjectures remain open for all $m \geq 5$ and $n \geq 4$ (resp. $l \geq 4$ and $k \geq 2$) except $(n, m) = (4, 5)$ (resp. $(k, l) = (2, 4)$), for example for $B'_{6,5}$ and $B'_{4,6}$; Proposition 8.1 is a partial result for $B'_{4,6}$. We read [CLR80] in full, in a scan of the journal version. Besides the results quoted in the introduction it contains no upper bound for $B'_{n,m}$ with $m \geq 4$, and its remark on $B'_{4,4} = 8$ comes without details. We read the source files of [FLOS18, Sha15, Dre21, BHO⁺12]. We searched arXiv, Crossref, OpenAlex, zbMATH and Semantic Scholar, including the works citing [CLR80, FLOS18, Sha15, Dre21, BHO⁺12], and the web. We found no proof of $B'_{2k,4} \leq k^3$ for $k \geq 3$, of $B'_{4,5} \leq 16$, or of Conjecture 1.2 for some $l \geq 3$. Barone and Basu [BB16] note that the Bézout inequality fails over the reals, and they prove a weaker real analogue for connected components. We do not know Dressler's source for $B'_{4,4} = 8$. The weighted Bézout inequality and the other tools of Section 3 are standard. This negative search is not a proof of priority. The two conjectures appear as records AMR-014-0019 and AMR-021-0007 of the UnsolvedMath dataset [ula26].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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