

A Geometric Count of Fourth-Power Decompositions of Binary Octics

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Abstract

A general complex binary octic has exactly 76 decompositions as a sum of three fourth powers of quadratic forms, modulo permutations and independent fourth-root-of-unity rescalings. We give a geometric proof of this count, previously obtained numerically by Kowalczyk and Vill. Restriction to a conic realizes the problem as a projection of the degree-112 third secant variety of the quartic Veronese surface. Its scheme-theoretic base locus is a ribbon on a rational quartic, with nilpotent line bundle of degree -4 . The resulting intersection correction is $112 - 64 + 28 = 76$. Keeping the nonreduced base is essential. The ordered affine Waring map consequently has degree 29184; quotienting only by permutations gives degree 4864. This settles the counting problem for the perfect pair $(k, d) = (4, 2)$, not the general perfect-pair problem.

1 The count and its conventions

Write $S_m = \mathbb{C}[x, y]_m$ for the space of binary forms of degree m . Consider the polynomial map between nine-dimensional affine spaces

$$W : S_2^3 \longrightarrow S_8, \quad (q_1, q_2, q_3) \longmapsto q_1^4 + q_2^4 + q_3^4.$$

The group $\mu_4^3 \rtimes \mathfrak{S}_3$ acts on each fiber by rescaling the three quadratic forms by fourth roots of unity and permuting them. A *powered-summand decomposition* means a fiber element modulo this group. Throughout, general means outside a proper Zariski-closed subset over $\mathbb{C}$.

Theorem 1.1. *A general binary octic has exactly 76 powered-summand decompositions of length three. The map W is dominant and generically finite of degree $76 \cdot 3! \cdot 4^3 = 29184$. Its quotient by permutations alone has generic degree $76 \cdot 4^3 = 4864$.*

Problem E of Fröberg, Lundqvist, Oneto and Shapiro [1, Section 1.4] asks for the degree of the binary Waring map for all perfect pairs. Theorem 1.1 treats only the pair $(k, d) = (4, 2)$; the quotient convention is made explicit to distinguish the counts 76, 4864 and 29184.

Kowalczyk and Vill [2, Remark 3.14] found the count 76 by numerical homotopy continuation, while distinguishing verified solutions from a certification that all solutions had been found. They also use the restriction of ternary quartics to a conic and ask for a geometric explanation in [2, Question 5.3]. Our contribution is the intersection-theoretic derivation of this previously observed value. We do not classify real decompositions or give a closed-form construction of every summand. No absolute priority claim is made.

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2 Restriction of the ternary secant variety

Let $T_m = \mathbb{C}[X, Y, Z]_m$, put $Q = XZ - Y^2$, and denote by

$$\mathcal{S} = \sigma_3(v_4(\mathbb{P}^2)) \subset \mathbb{P}(T_4) = \mathbb{P}^{14}$$

the closure of the ternary quartics that are sums of three fourth powers of linear forms. This irreducible variety has dimension eight. Indeed, the projective secant parameterization has dimension at most eight, and the affine tangent spaces at X^4, Y^4, Z^4 span nine distinct degree-four monomials, giving the reverse inequality.

We use the following established properties of $\mathcal{S}$. Its degree is 112 [3, Proposition 2.4.2(1)], a value credited there to Ellingsrud and Strømme. Its projective scheme is defined by the 4×4 minors of the middle catalecticant [3, Theorem 3.2.1(1)]. For $d = 4$ and $n = 2$, the additional $(1, 3)$ flattening in that theorem has only three rows and contributes no 4×4 minors.

The linear restriction map

$$T_4 \longrightarrow S_8, \quad F \longmapsto F(x^2, xy, y^2)$$

is surjective and has kernel QT_2 . Thus it induces a rational projection $\rho : \mathcal{S} \dashrightarrow \mathbb{P}^8$, whose scheme-theoretic base is

$$B = \mathcal{S} \cap \mathbb{P}(QT_2).$$

Write $H = c_1(\mathcal{O}_{\mathcal{S}}(1))$ for the hyperplane class.

Lemma 2.1. *The maps W and ρ are dominant and generically finite.*

Proof. At $(x^2, y^2, (x+y)^2)$, the image of the differential of W is $x^6 S_2 + y^6 S_2 + (x+y)^6 S_2$. After removing the first and last three monomials of S_8 , the remaining coefficient matrix is

$$\begin{pmatrix} 20 & 15 & 6 \\ 15 & 20 & 15 \\ 6 & 15 & 20 \end{pmatrix}, \quad \det = 980 \neq 0.$$

Hence W is dominant. Every point of its image arises by restriction from $\mathcal{S}$, so ρ is dominant as well. The respective source and target dimensions agree. $\square$

3 The nonreduced projection base

Write a point of the projection center as $F = Qh$, where

$$h = aX^2 + bXY + cXZ + dY^2 + eYZ + fZ^2.$$

In the quadratic differential basis $(X^2, XY, XZ, Y^2, YZ, Z^2)$, normalize the catalecticant by using fourth derivatives. Its matrix is

$$M(Qh) = \begin{pmatrix} 0 & 0 & 6a & -4a & 2b & 4c \\ 0 & -4a & 2b & -6b & 2(d-c) & 2e \\ 6a & 2b & 4c & 2(d-c) & 2e & 6f \\ -4a & -6b & 2(d-c) & -24d & -6e & -4f \\ 2b & 2(d-c) & 2e & -6e & -4f & 0 \\ 4c & 2e & 6f & -4f & 0 & 0 \end{pmatrix}. \quad (1)$$

This is a fixed nonzero scalar multiple of the usual divided-coefficient matrix, so its rank equations define the same scheme.

Proposition 3.1. *The scheme B is a ribbon on the smooth rational quartic*

$$C = \{[Q(s^2X + 2stY + t^2Z)^2] : [s : t] \in \mathbb{P}^1\}.$$

Its nilpotent ideal $J = I_C/I_B$, regarded as a line bundle on $C \simeq \mathbb{P}^1$, is $\mathcal{O}_{\mathbb{P}^1}(-4)$.

Proof. On the chart $f = 1$, the principal block of (1) with indices $(2, 4, 5)$, counted from zero, has constant determinant 144. The rank condition is therefore exactly the vanishing of its symmetric 3×3 Schur complement. Put $u = e$ and $\epsilon = d - u^2/4$. Its six equations generate the ideal

$$\left(c - \frac{u^2}{8} + 24\epsilon, \quad b - \frac{u^3}{16} + \frac{23}{2}u\epsilon, \quad a - \frac{u^4}{256} + \frac{23}{16}u^2\epsilon, \quad \epsilon^2 \right). \quad (2)$$

For an explicit polynomial verification in both directions, see Appendix A. Thus this chart is $\text{Spec } \mathbb{C}[u, \epsilon]/(\epsilon^2)$. Exchanging X and Z gives the analogous chart $a = 1$.

These two charts cover the projective scheme. To check the support, set $a = f = 0$. The principal minors with indices $(0, 1, 3, 4)$ and $(1, 3, 4, 5)$ become $144b^4$ and $144e^4$, respectively. Hence $b = e = 0$ at every remaining support point. The minors with indices $(0, 1, 4, 5)$, $(0, 2, 3, 5)$ and $(1, 2, 3, 4)$ then become

$$64c^2(c - d)^2, \quad 64c^2(c^2 + 22cd + d^2), \\ 16(c - d)^2(c^2 + 22cd + d^2).$$

If $c \neq 0$, the first forces $d = c$, contradicting the second. If $c = 0$, the last forces $d = 0$. There is no projective point left. An open cover of the support is an open cover of the scheme, so the two explicit ribbon charts account for all of B .

Setting $\epsilon = 0$ in (2) gives exactly the stated rational quartic. On the overlap, normalize by a and use the coordinates on the second chart. Modulo ϵ^2 one obtains

$$u' = \frac{b}{a} = \frac{16}{u} + \frac{2944\epsilon}{u^3}, \quad \epsilon' = \frac{d}{a} - \frac{1}{4} \left(\frac{b}{a} \right)^2 = \frac{256\epsilon}{u^4}. \quad (3)$$

The reduced coordinates satisfy $u' = 16/u$, and the nilpotent generator has transition a nonzero constant times u^{-4} . This identifies J with $\mathcal{O}_{\mathbb{P}^1}(-4)$. $\square$

In particular, the fundamental cycle is $[B] = 2[C]$, and

$$\deg(H \cap [B]) = 2 \deg C = 8. \quad (4)$$

4 Normal bundle degrees

Lemma 4.1. *The variety $\mathcal{S}$ is smooth along C , and $\deg N_{C/\mathcal{S}} = 10$.*

Proof. At QZ^2 , the catalecticant has rank three. For the general ternary quartic, the six Schur equations used above have independent differentials there. One explicit check orders the fifteen monomials as $X^i Y^j Z^{4-i-j}$, first increasing i and then increasing j . For the upper-triangular Schur entries in row order, the Jacobian columns numbered $(4, 8, 11, 12, 13, 14)$ have determinant -331776 . Here the Schur complement is used without multiplying its equations by the pivot determinant. This proves smoothness at QZ^2 . The natural SL_2 action preserves Q and is transitive on C , so the same conclusion holds along the entire curve.

Put $L = \mathcal{O}_{\mathcal{S}}(1)|_C$, of degree four. The catalecticant is an L -valued symmetric form on $\mathcal{O}_C^6$. Let K be its rank-three kernel bundle and put $E = \mathcal{O}_C^6/K$. The induced nondegenerate form identifies $E \simeq E^* \otimes L$, whence

$$2 \deg E = 3 \deg L = 12, \quad \deg K = -6.$$

The differential of the rank condition restricts a varying symmetric form to K . Its kernel is the tangent space to $\mathcal{S}$, and the independence just checked makes it surjective. Therefore

$$N_{\mathcal{S}/\mathbb{P}^{14}}|_C \simeq \mathrm{Sym}^2 K^* \otimes L.$$

For a rank-three bundle, $c_1(\mathrm{Sym}^2 K^*) = -4c_1(K)$, so this rank-six normal bundle has degree $24 + 6 \cdot 4 = 48$. The Euler sequence gives $\deg T_{\mathbb{P}^{14}}|_C = 15 \cdot 4 = 60$. The tangent and normal sequences now give

$$\deg T_{\mathcal{S}}|_C = 12, \quad \deg N_{C/\mathcal{S}} = 12 - \deg T_{\mathbb{P}^1} = 10. \quad (5)$$

□

Lemma 4.2. *The embedding $B \hookrightarrow \mathcal{S}$ is regular of codimension seven, and*

$$\deg(N_{B/\mathcal{S}}|_C) = 14, \quad \deg(c_1(N_{B/\mathcal{S}}) \cap [B]) = 28.$$

Proof. In a completed smooth local ring of $\mathcal{S}$, the quotient defined by B is $\mathbb{C}[[u, \epsilon]]/(\epsilon^2)$. Lifting its two tangent parameters and completing to regular parameters puts its ideal in the form $(z_1, \dots, z_6, \epsilon^2)$. This is a regular sequence.

Let V be the kernel of the conormal surjection $N_{C/\mathcal{S}}^* \rightarrow J$. Restricting the conormal bundle of B to C gives a filtration with pieces V and $J^{\otimes 2}$: the six simple transverse equations give the former, and the squared nilpotent parameter gives the latter. Thus

$$\det(N_{B/\mathcal{S}}^*|_C) = \det V \otimes J^{\otimes 2} = \det(N_{C/\mathcal{S}}^*) \otimes J.$$

Equations (5) and $\deg J = -4$ give degree 14 for the normal determinant. This argument does not require the ribbon to split globally. Finally, $[B] = 2[C]$ doubles the Chern-class degree. □

5 The projection degree and actual decompositions

Blow up the scheme-theoretic base, writing $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$ and denoting its exceptional divisor by D . The nine restriction coordinates generate the base ideal. They therefore induce a projective morphism

$$\tilde{\rho} : \tilde{\mathcal{S}} \longrightarrow \mathbb{P}^8, \quad \tilde{\rho}^* \mathcal{O}_{\mathbb{P}^8}(1) = \pi^* \mathcal{O}_{\mathcal{S}}(1) \otimes \mathcal{O}(-D).$$

The blow-up is integral and birational over $\mathcal{S}$, so the generic degree of this morphism equals that of ρ .

For a regular embedding, the Segre-class and blow-up identities are

$$s(B, \mathcal{S}) = c(N_{B/\mathcal{S}})^{-1} \cap [B], \quad i_* s(B, \mathcal{S}) = \sum_{j \geq 1} (-1)^{j-1} \pi_*(D^j);$$

see, for example, [4, p. 2]. In codimension seven they yield

$$\begin{aligned} \pi_* D^j &= 0 \quad (1 \leq j < 7), & \pi_* D^7 &= [B], \\ \pi_* D^8 &= c_1(N_{B/\mathcal{S}}) \cap [B]. \end{aligned}$$

Regular embedding, rather than reducedness, is the relevant hypothesis. All normal calculations are local near B , where $\mathcal{S}$ is smooth. Lemmas 2.1 and 4.2 and (4) give

$$\deg \rho = \int_{\mathcal{S}} (\pi^* H - D)^8 = 112 - 8 \cdot 8 + 28 = 76. \quad (6)$$

It remains to justify that this degree counts distinct decompositions of the stated kind, rather than border-rank points or multiplicities. A dense open subset $U \subset \mathcal{S}$ consists of sums of three fourth powers of linearly independent linear forms. It is the projective linear group orbit of $X^4 + Y^4 + Z^4$, of dimension eight. On this orbit the decomposition is unique up to the stated symmetries. Indeed, the quadratic apolar kernel of $X^4 + Y^4 + Z^4$ is generated by XY, XZ, YZ in dual coordinates; its common zero set consists of the three coordinate points. In any other three-term decomposition, the rank-three catalecticant factors as $A \operatorname{diag}(\lambda_i) A^t$, where the columns of A are the squared linear forms. Rank three forces independent columns and nonzero coefficients, so its kernel is the kernel of A^t . The apolar quadrics therefore vanish at each summand point, forcing the same three coordinate points. Their coefficients are then fixed.

The complement $\mathcal{S} \setminus U$ has dimension at most seven, and so does the exceptional divisor in the blow-up. A general target point avoids their images. Since the ground field has characteristic zero, the dominant generically finite map is separable. Shrinking the target to a nonempty open subset makes its proper fibers finite and reduced, each consisting of 76 points. Each point is consequently in U and has the unique genuine decomposition just described.

Restriction identifies ternary linear forms with binary quadratics. For a fixed nonzero binary octic, a projective preimage has a unique affine representative restricting to that octic. This identifies the generic projection fiber with its powered-summand decompositions. The quadratics in a general decomposition are nonzero and no two are proportional, so $\mu_4^3 \rtimes \mathfrak{S}_3$ acts freely on the ordered triples. Multiplication by $3!4^3$ gives the affine degree, and quotienting only by permutations gives the other count. This proves Theorem 1.1.

A Exact equations for the ribbon chart

For completeness, the six Schur numerators on $f = 1$, divided by the common nonzero factor -16 , are the following polynomials in a, b, c, d, u :

$$\begin{aligned} p_1 &= 72ac - 9b^2 + 12bcu - 16c^3 - 4c^2u^2, \\ p_2 &= 18au + 21bc - 9bd + 3bu^2 - 14c^2u + 6cdu - 2cu^3, \\ p_3 &= 21bu + 4c^2 + 12cd - 14cu^2, \\ p_4 &= 36a + 12bu - 9c^2 + 18cd - 10cu^2 - 9d^2 + 6du^2 - u^4, \\ p_5 &= 42b - 19cu + 27du - 7u^3, \\ p_6 &= 8c + 192d - 49u^2. \end{aligned}$$

In $\mathbb{Q}[a, b, c, d, u]$, with $\epsilon = d - u^2/4$, their ideal equals (2). One can eliminate c, b, a successively using p_6, p_5, p_4 and reduce the remaining equations. Conversely, substitution of the four relations in (2) makes all six polynomials zero. These are identities of ideals, not only of their radicals. The supplementary exact-arithmetic program checks both inclusions, the five support minors, the two transition formulas, the smoothness minor and the dominance determinant. It also verifies the catalecticant normalization by direct differentiation of all fifteen quartic monomials. These checks supplement the geometric degree argument; a computation at a single specialized octic would not by itself prove the generic count.

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