

Cube Ranks of Monomials

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Version 2

Abstract

Let $M = \prod_i x_i^{a_i}$ have degree $3d > 0$, and let o be the number of odd exponents. We determine the least number of cubes of homogeneous degree- d forms whose sum is M . Over $\mathbb{C}$, this number is one if every exponent is divisible by three, three if M is not a cube and $o \leq d$, and four if $o > d$. The same formula holds over $\mathbb{R}$. The new lower bound uses an exponent-one coordinate, which must exist when $o > d$. Restricting a hypothetical three-cube expression to its coordinate hyperplane gives proportional forms, because the Fermat plane cubic has no nonconstant rational parametrization. Differentiation then forces a degree- $2d$ square divisor in a monomial whose square capacity is smaller than d . Classical constructions give the matching upper bounds. This version removes the support restriction of version 1; general monomial k -ranks for $k \geq 4$ are not determined here.

1 Statement and context

For a nonzero homogeneous polynomial F of degree kd over $K \in \{\mathbb{C}, \mathbb{R}\}$, define its k -rank by

$$\text{rk}_{k,K}(F) = \min\{r : F = F_1^k + \cdots + F_r^k, F_i \in K[x_1, \dots, x_n]_d\}.$$

These are actual polynomial decompositions, not border rank. For $k = 3$, a nonzero scalar multiple has the same rank, since every nonzero scalar in either field has a cube root.

Carlini and Oneto [1] proved that the complex cube rank of a noncube monomial is either three or four, and determined it in two and three variables. Their Remarks 3.9–3.10 singled out $x_0^3 x_1 x_2 x_3$ and $x_0^2 x_1 x_2 x_3 x_4$ as remaining possible rank-three tests in their analysis. Problem C of Fröberg, Lundqvist, Oneto and Shapiro [3] asks for the k -rank of every monomial, for all $k \geq 3$. We settle the complete case $k = 3$.

Theorem 1.1. *Let $M = \prod_{i=1}^n x_i^{a_i}$, where $a_i \geq 0$ and $\sum_i a_i = 3d > 0$, and put $o = \#\{i : a_i \text{ is odd}\}$. Then*

$$\text{rk}_{3,\mathbb{C}}(M) = \begin{cases} 1, & 3 \mid a_i \text{ for every } i, \\ 3, & M \text{ is not a cube and } o \leq d, \\ 4, & o > d. \end{cases} \quad (1)$$

The lower bound in the third case is the contribution of the paper. The other cases and upper constructions follow from the classical factorization and grouping methods of [1, 2]. Version 1 of this paper [5] established the third case under the additional condition $2N > 3d$, with N the number of occurring variables. The normal-derivative argument below removes that condition and supersedes the earlier support-counting proof.

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2 A square divisor from a normal derivative

Lemma 2.1. *Let H_1, H_2, H_3 be homogeneous polynomials of the same degree in $q \geq 1$ variables over $\mathbb{C}$, not all zero. If $H_1^3 + H_2^3 + H_3^3 = 0$, then $(H_1, H_2, H_3) = (v_1, v_2, v_3)Q$ for a constant nonzero vector $v \in \mathbb{C}^3$ and a nonzero polynomial Q .*

Proof. The ratios define a rational map from $\mathbb{P}^{q-1}$ to the smooth plane cubic

$$E : z_1^3 + z_2^3 + z_3^3 = 0.$$

This curve has genus one. Every rational map $\mathbb{P}^1 \dashrightarrow E$ extends to a morphism, since the source is a nonsingular curve and the target is projective. If nonconstant, the morphism is finite and separable, and Riemann–Hurwitz [4] gives

$$-2 = \deg(f)(2g(E) - 2) + \deg R = \deg R \geq 0,$$

a contradiction. A nonconstant rational map from a higher projective space would restrict to a nonconstant rational map on the line through two domain points with different images. Thus the original map is constant; for $q = 1$ this is immediate. Choose a nonzero coordinate v_j of its constant value and set $Q = H_j/v_j$. Equality on the dense domain implies equality of polynomials everywhere. In particular, Q has the common homogeneous degree. $\square$

Proposition 2.2. *Let $F = x_n G$ be a homogeneous polynomial of degree $3d > 0$ over $\mathbb{C}$, with $x_n \nmid G$ and $n \geq 2$. If F is a sum of three degree- d cubes, then $G|_{x_n=0}$ has a square divisor Q^2 , where Q is a nonzero homogeneous polynomial of degree d in the remaining variables.*

Proof. Write $F = F_1^3 + F_2^3 + F_3^3$ and let $H_i = F_i|_{x_n=0}$. The H_i cannot all vanish: otherwise x_n divides every F_i , so $x_n^3 \mid F$, contrary to $x_n \nmid G$. Their cubes sum to zero. Lemma 2.1 gives $H_i = v_i Q$ with $Q \neq 0$ and $\deg Q = d$. Differentiate the original identity and then restrict to $x_n = 0$:

$$G|_{x_n=0} = 3 \sum_{i=1}^3 H_i^2 \frac{\partial F_i}{\partial x_n} \Big|_{x_n=0} = 3Q^2 \sum_{i=1}^3 v_i^2 \frac{\partial F_i}{\partial x_n} \Big|_{x_n=0}. \quad (2)$$

The left side is nonzero. The displayed identity proves the divisibility in $\mathbb{C}[x_1, \dots, x_{n-1}]$. $\square$

3 Proof of the classification

Proof of Theorem 1.1. Unique factorization shows that a polynomial whose cube is a monomial is a scalar monomial. Thus rank one occurs exactly when every exponent is divisible by three.

We recall the rank-two exclusion of [1, Theorem 3.2]. If a noncube monomial were $A^3 - B^3$, its three factors $A - \zeta B$, for the cube roots ζ of one, would be nonzero scalar monomials. If A, B are dependent, the target is a cube. Otherwise these factors are pairwise nonproportional monomials in the two-dimensional span of A, B . Three distinct monomials are linearly independent, a contradiction.

For the upper bound, divide the $3d$ variable factors of M , with repetition, into three lists of length d . Their products A, B, C satisfy $M = ABC$ and

$$24ABC = (A + B + C)^3 - (A + B - C)^3 - (A - B + C)^3 + (A - B - C)^3. \quad (3)$$

Scalar coefficients can be absorbed into the cubes, giving rank at most four.

If $o \leq d$, then

$$\sum_i \left\lfloor \frac{a_i}{2} \right\rfloor = \frac{3d - o}{2} \geq d. \quad (4)$$

Choose integers $0 \leq b_i \leq \lfloor a_i/2 \rfloor$ whose sum is d , and put $V = x^b$ and $U = x^{a-2b}$. Both monomials have degree d , and the classical identity

$$6UV^2 = (U + V)^3 + (U - V)^3 - 2U^3 \quad (5)$$

gives rank at most three. This is the grouping construction of [1]; the exponent-capacity method occurs in [2, Appendix B]. Together with the first two paragraphs it settles the first and second cases of (1).

It remains to suppose $o > d$. Some exponent is one: if not, every odd positive exponent is at least three, and $3d = \sum_i a_i \geq 3o > 3d$, which is impossible. Relabel so that $a_n = 1$. There are at least two occurring variables, because $3d \geq 3$. Assume that M is a sum of at most three cubes, padding with zero forms if necessary. Proposition 2.2 implies that

$$Q^2 \mid \prod_{i < n} x_i^{a_i}, \quad \deg Q = d.$$

By unique factorization, Q is a scalar monomial. Therefore

$$d = \deg Q \leq \sum_{i < n} \left\lfloor \frac{a_i}{2} \right\rfloor = \frac{3d - o}{2} < d,$$

a contradiction. The missing floor term is zero because $a_n = 1$. Hence the four-cube upper bound is sharp. $\square$

Corollary 3.1. *For every real monomial of positive degree divisible by three, $\text{rk}_{3,\mathbb{R}}(M) = \text{rk}_{3,\mathbb{C}}(M)$; both ranks satisfy (1).*

Proof. Complex rank is a lower bound for real rank. Both (3) and (5) use real monomials and real coefficients, whose signed cube roots exist over $\mathbb{R}$. They attain the complex lower bounds. The rank-one case is immediate. $\square$

4 Examples and scope

The two sextics singled out in [1, Remarks 3.9–3.10] both have rank four. The degree-twelve monomial with exponent partition $(5, 3, 1, 1, 1, 1)$, excluded by the strict support inequality in version 1, also has rank four: here $d = 4$ and $o = 6$. The partition $(11, 1, 1, 1, 1, 1, 1)$ gives another rank-four example in degree eighteen. In contrast, $x_0^6 x_1 x_2 x_3$ has rank three, as already observed in [1, Example 2.5]. Multiplying by a monomial cube need not preserve cube rank.

Extra ambient variables cannot lower the answer: setting them to zero in a proposed decomposition gives a decomposition in the occurring variables. No independence or monomial assumption is imposed on the summands. The theorem does not determine general monomial k -ranks for $k \geq 4$, rational-coefficient ranks, or border ranks. In particular, the full all- k Problem C of [3] is not resolved by this paper.

The accompanying standard-library checker verifies (2) coefficientwise, the two upper-bound identities, and the exponent conditions for all 11,663 positive exponent partitions of degrees $3d$, $1 \leq d \leq 10$. Normal and optimized executions agree on 83,258 conditions per run; two deliberately incorrect variants are rejected. These tests supplement the written proof and do not establish the geometric lemma by finite computation.

AI systems assisted with proof development, literature comparison, computation and writing. This is an unrefereed preprint; self-audit and exact tests are not independent peer review or proof-assistant verification. No absolute-priority claim is made.

References

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