

Counterexamples to a Conjecture of Fröberg, Lundqvist, Oneto and Shapiro on Maximal k -Ranks of Binary Forms

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Abstract

For a complex binary form f of degree kd , the k -rank of f is the least number of k -th powers of binary forms of degree d whose sum is f . Fröberg, Lundqvist, Oneto and Shapiro conjectured (Arnold Math. J. 4 (2018), Conjecture 1.9) that for all k and d the maximal k -rank of binary forms of degree kd equals k , and that the forms $\ell_1 \ell_2^{kd-1}$, with non-proportional linear forms ℓ_1, ℓ_2 , attain it. We show that xy^{kd-1} is a sum of fewer than k k -th powers of forms of degree d for every $d \geq 2$ when $k \in \{5, 6, 7, 8, 9\}$ or k is odd with $7 \leq k \leq 201$; so the conjecture is false for all these (k, d) . For $k = 7$ the counterexample is a closed-form identity whose proof needs no computer: xy^{13} is a linear combination of six 7th powers of explicit binary quadratic forms, built from $\sqrt{165}$, a cube root, two square roots and a cube root of unity, and hence a sum of six 7th powers. The same construction gives $k - 1$ terms for odd $k \geq 7$ whenever three conditions on two explicit integer polynomials hold; they were checked by exact computer calculation for $7 \leq k \leq 201$. This construction is a variant of Reznick's "synching" of powers of quadratic forms. For $k = 5, 6, 8, 9$ the identities, with 4, 5, 6, 7 terms, are computer-assisted: an interval Newton test of Krawczyk type, carried out in exact rational and interval arithmetic by several independent programs, proves that an exact solution exists next to an approximate one. By results of Lundqvist, Oneto, Reznick and Shapiro, of Carlini and Oneto, and of Bik, Draisma, Oneto and Ventura, the conjecture holds at $(k, d) = (3, 2), (3, 3), (3, 4), (4, 2)$, so $(5, 2)$ is the first failure in the order of k for $d = 2$; the case $k = 4, d \geq 3$ remains open, and so does the first clause alone (maximal k -rank equal to k) outside the known cases. The same identities answer Problem D of the source, on k -ranks of binary monomials, in the negative, and they give upper bounds below N in Heilbronn's problem on sums of N -th powers of entire functions and in its analogue for polynomials, the Waring number of $\mathbb{C}[t]$. We found no such bound in the literature accessible to us, but two primary sources could not be read. This is an unrefereed note.

1 Introduction

1.1 The conjecture

Let $S_e = \mathbb{C}[x, y]_e$ be the space of complex binary forms of degree e . Following [FLOS18, Definition 1.5], for positive integers k, d and $f \in S_{kd}$, the k -rank $\text{rk}_k(f)$ is the least s such that

$$f = g_1^k + \cdots + g_s^k \quad \text{with } g_1, \dots, g_s \in S_d.$$

Over $\mathbb{C}$ a coefficient in front of a k -th power can be absorbed into it, so linear combinations of k -th powers give the same notion. The *maximal k -rank* $\text{rk}_k^{\max}(kd, 2)$ of binary forms of degree kd is the least m such that every $f \in S_{kd}$ is a sum of at most m k -th powers of forms in S_d [FLOS18, Definition 1.8]; it is the maximum of rk_k on S_{kd} .

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Conjecture 1.1 ([FLOS18, Conjecture 1.9]). *For all positive integers k and d :*

- (A) $\text{rk}_k^{\max}(kd, 2) = k$; and
- (B) *the binary forms $\ell_1 \ell_2^{kd-1}$, where ℓ_1 and ℓ_2 are non-proportional linear forms, have the maximal k -rank: $\text{rk}_k(\ell_1 \ell_2^{kd-1}) = \text{rk}_k^{\max}(kd, 2)$.*

As printed, Definition 1.8 of [FLOS18] speaks of “every form of degree kd in $n + 1$ variables”, whereas the generic k -rank $\text{rk}_k^{\circ}(kd, n)$ is defined there for forms in n variables, Conjecture 1.9 speaks of binary forms, and the case $k = 2$ is proved there by factoring a binary form [FLOS18, (5)]. The literal reading, with ternary forms for the argument 2, cannot be meant: it would make the conjecture false already for $(k, d) = (2, 1)$, since $x^2 + y^2 + z^2$ is not a sum of two squares of linear forms. We therefore read $\text{rk}_k^{\max}(kd, 2)$ as the maximal k -rank of binary forms, as in [LORS19], where it is written $\text{rk}_k^{\max}(2, kd)$. Clause (A) is B. Shapiro’s conjecture of 2014 [LORS19, Conjecture 1.4]. Clause (B) goes back to the sentence “One can also suspect that $\text{rk}_k^{\max}(\ell_1 \ell_2^{kd-1}) = k$ ” in [LORS19, Section 1], where $\text{rk}_k^{\max}$ applied to a form stands for rk_k . This reading of (B), $\text{rk}_k(\ell_1 \ell_2^{kd-1}) = k$, and the reading in Conjecture 1.1 agree whenever (A) holds. Our results refute the conjunction of (A) and (B) and the reading $\text{rk}_k(\ell_1 \ell_2^{kd-1}) = k$ of (B); they do not decide which clause of Conjecture 1.1 fails. The conjecture is record AMR-014-0004 of the UnsolvedMath collection [ula26], where it is called source-order Conjecture 4 and listed as partially solved.

1.2 What was known

For $k = 1$ the conjecture is trivial. For $k = 2$ it follows from [FLOS18, (5)]: a binary form of degree $2d$ factors as $g_1 g_2$ with $g_1, g_2 \in S_d$, and $g_1 g_2 = \left(\frac{1}{2}(g_1 + g_2)\right)^2 + \left(\frac{i}{2}(g_1 - g_2)\right)^2$, while xy^{2d-1} is not a square. For $d = 1$ (classical Waring rank) the maximal rank of binary forms of degree k is k , attained exactly at the forms $\ell_1 \ell_2^{k-1}$ [Rez13, Theorems 4.9 and 5.4], as quoted in [LORS19, Section 1].

For binary monomials, Carlini and Oneto proved $\text{rk}_k(x^a y^b) \leq \max([a]_k, [b]_k) + 1$, where $[a]_k$ is the remainder of a modulo k [CO15, Proposition 3.4]; in particular $\text{rk}_k(xy^{kd-1}) \leq k$. They also proved that for $k \geq 3$ a monomial which is not a k -th power has k -rank at least 3 [CO15, Theorem 3.2], so that $\text{rk}_3(xy^{3d-1}) = 3$ for all d [CO15, Corollary 3.6]. The value $\text{rk}_4(xy^7) = 4$ was computed with CoCoA [CO15, Remark 3.7] and with Macaulay2 [LORS19, Example 4.7]. Every binary sextic is a sum of three cubes of quadratic forms [LORS19, Theorem 3.1], and every binary form of degree kd is a sum of k k -th powers of forms of degree d for $(k, d) = (3, 3), (3, 4), (4, 2)$, by a computer-assisted proof of Bik, Draisma, Oneto and Ventura [BDOV20, Theorem 1.6]. Together with the lower bounds just quoted, these give clause (A) at $(3, 2), (3, 3), (3, 4)$ and $(4, 2)$, and then clause (B) as well. So before this note Conjecture 1.1 was known for $k \leq 2$, for $d = 1$, and at $(k, d) = (3, 2), (3, 3), (3, 4), (4, 2)$. For $k \geq 5$ no case was known; for $k \geq 5$ Carlini and Oneto write that “we can only obtain partial results” [CO15, Section 3].

1.3 Results

All identities below are identities in $\mathbb{C}[t]$ or $\mathbb{C}[x, y]$; by Lemma 2.1 an identity $t = \sum_{i \leq s} g_i(t)^k$ with $\deg g_i \leq 2$ is the same as an identity $xy^{2k-1} = \sum_{i \leq s} G_i^k$ with $G_i(x, y) = y^2 g_i(x/y) \in S_2$.

Theorem 1.2 ($k = 7$, closed form). *Let $s \in \mathbb{C}$ with $s^2 = 165$, put $I_{\mp} = (-15 \mp s)/10$ and choose $\beta, q, p \in \mathbb{C}$ with*

$$\beta^2 = I_-, \quad q^3 = -\frac{15053 - 420s}{14053}, \quad p^2 = I_+ q.$$

Then $\beta, p, q \neq 0$ and $q^6 \neq 1$. Put $h(t) = 1 + \beta t + t^2$, $g(t) = 1 + pt + qt^2$, $A = -\beta/(pq^6)$ and $\sigma = 7\beta(1 - q^{-6})$, and let ω be a primitive cube root of unity. Then

$$t = \frac{1}{3\sigma} \sum_{j=0}^2 \omega^{-j} \left(h(\omega^j t)^7 + A g(\omega^j t)^7 \right). \quad (1)$$

Since every complex number is a 7th power, t is a sum of six 7th powers of polynomials of degree exactly 2, and

$$xy^{13} = \sum_{j=0}^2 \frac{\omega^{-j}}{3\sigma} (y^2 + \beta \omega^j xy + \omega^{2j} x^2)^7 + \sum_{j=0}^2 \frac{A \omega^{-j}}{3\sigma} (y^2 + p \omega^j xy + q \omega^{2j} x^2)^7 \quad (2)$$

is a sum of six 7th powers of binary quadratic forms.

Theorem 1.3 (odd k). *For every odd k with $7 \leq k \leq 201$, t is a sum of $k - 1$ k -th powers of polynomials of degree 2; equivalently, xy^{2k-1} is a sum of $k - 1$ k -th powers of binary quadratic forms.*

The proof (Section 4) is a construction that works for every odd $k \geq 7$ for which three conditions (C1)–(C3) on two explicit integer polynomials hold; these conditions were checked by exact computer calculation for $7 \leq k \leq 201$.

Theorem 1.4 (computer-assisted). *There are binary quadratic forms $G_i \in S_2$ such that*

$$xy^9 = \sum_{i=1}^4 G_i^5, \quad xy^{11} = \sum_{i=1}^5 G_i^6, \quad xy^{13} = \sum_{i=1}^6 G_i^7, \quad xy^{15} = \sum_{i=1}^6 G_i^8, \quad xy^{17} = \sum_{i=1}^7 G_i^9.$$

For $k = 7$ this is a second identity, different from Theorem 1.2.

Corollary 1.5. *Let $d \geq 2$, and let $k \in \{5, 6, 7, 8, 9\}$ or k odd with $7 \leq k \leq 201$. Put $s(k) = 4, 5, 6, 6, 7$ for $k = 5, 6, 7, 8, 9$ and $s(k) = k - 1$ otherwise. Then $\text{rk}_k(\ell_1 \ell_2^{kd-1}) \leq s(k) < k$ for all non-proportional linear forms ℓ_1, ℓ_2 . Hence Conjecture 1.1 is false at (k, d) , and clause (B) is false at (k, d) in the reading $\text{rk}_k(\ell_1 \ell_2^{kd-1}) = k$. For $d = 2$ the conjecture holds for $k \leq 4$ and fails for $k = 5$, so $(5, 2)$ is the first failure in the order of k for $d = 2$.*

We do not decide which clause fails: if (A) holds at (k, d) , then (B) fails there. Clause (A) alone is not decided at any (k, d) with $k \geq 5$, $d \geq 2$, and the case $k = 4$, $d \geq 3$ remains open (Section 7). In Section 6 we show that Problem D of [FLOS18], on k -ranks of binary monomials, has a negative answer (Proposition 6.1), and we derive bounds below N for Heilbronn’s problem on entire functions and for its analogues, among them the Waring number of $\mathbb{C}[t]$ (Proposition 6.2).

1.4 What is new and what is not

The averaging over roots of unity in Theorems 1.2 and 1.3 is classical ([HL19, Problem 2.26], [Vol15, Theorem 3]). For powers of quadratic trinomials, with the middle coefficient chosen as a root of a coefficient polynomial so that unwanted coefficients vanish, it is Reznick’s *synching* [Rez19, Theorem 2.1 and Corollary 2.3], with which he constructs linearly dependent sets of d -th powers of binary quadratic forms; [Rez19] does not treat xy^{2k-1} or Conjecture 1.1. What we add is a choice of two quadratic trinomials for which the averaged sum is exactly t . The existence test of Section 5 is that of Krawczyk and Moore [Kra69, Moo77]. To our knowledge, and relative to the literature found, the counterexamples to Conjecture 1.1 and the explicit identities are new. No answer to the conjecture was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and we claim no priority (see “Scope and priority” in Section 8).

2 Preliminaries

Lemma 2.1. *Let k, d, s be positive integers.*

- (a) *If ℓ_1, ℓ_2 are non-proportional linear forms, then $\text{rk}_k(\ell_1 \ell_2^{kd-1}) = \text{rk}_k(xy^{kd-1})$.*
- (b) *For $G_1, \dots, G_s \in S_d$ put $g_i(t) = G_i(t, 1)$. Then $xy^{kd-1} = \sum_i G_i^k$ if and only if $t = \sum_i g_i(t)^k$. Every $g \in \mathbb{C}[t]$ with $\deg g \leq d$ is $G(t, 1)$ for $G(x, y) = y^d g(x/y) \in S_d$.*
- (c) *If $xy^{2k-1} = \sum_{i \leq s} G_i^k$ with $G_i \in S_2$, then $xy^{kd-1} = \sum_{i \leq s} (y^{d-2} G_i)^k$ for every $d \geq 2$.*
- (d) *If $\text{rk}_k(xy^{kd-1}) < k$, then Conjecture 1.1 fails at (k, d) .*

Proof. (a) Let T be the linear substitution with $\ell_1 \circ T = x$ and $\ell_2 \circ T = y$. Composition with T is a ring automorphism of $\mathbb{C}[x, y]$ that preserves degrees, so $f = \sum g_i^k$ with $g_i \in S_d$ if and only if $f \circ T = \sum (g_i \circ T)^k$ with $g_i \circ T \in S_d$. (b) Setting $y = 1$ gives one direction. Conversely, both sides of $xy^{kd-1} = \sum G_i^k$ are forms of degree kd , and a form F of degree kd satisfies $F(x, y) = y^{kd} F(x/y, 1)$ for $y \neq 0$, so it is determined by $F(t, 1)$. (c) $(y^{d-2} G_i)^k = y^{k(d-2)} G_i^k$ and $(2k-1) + k(d-2) = kd-1$. (d) If (A) holds at (k, d) , then $\text{rk}_k(xy^{kd-1}) < k = \text{rk}_k^{\max}(kd, 2)$ and (B) fails; otherwise (A) fails. $\square$

Powers of a quadratic. For $k \geq 1$ and $b, c \in \mathbb{C}$ let $Q_n(b, c)$ be the coefficient of t^n in $(1 + bt + ct^2)^k$. Expanding the product of k factors,

$$Q_n(b, c) = \sum_{\substack{j+2l=n \\ j+l \leq k}} \frac{k!}{j! l! (k-j-l)!} b^j c^l. \quad (3)$$

The bijection $(j, l) \mapsto (j, k-j-l)$ between the index sets for n and for $2k-n$ preserves the multinomial coefficients, and $c^{k-n} b^j c^l = b^j c^{k-j-l}$ when $j+2l=n$; hence the *reflection rule*

$$Q_{2k-n}(b, c) = c^{k-n} Q_n(b, c) \quad (0 \leq n \leq k). \quad (4)$$

All terms of Q_n have $j \equiv n \pmod{2}$, so there are polynomials $\varphi_n \in \mathbb{Z}[I]$ (depending on k) with

$$Q_n(b, c) = b^{n \bmod 2} c^{\lfloor n/2 \rfloor} \varphi_n(b^2/c) \quad (c \neq 0), \quad (5)$$

namely $\varphi_n(I) = \sum_j \frac{k!}{j! l! (k-j-l)!} I^{(j-e)/2}$ with $e = n \bmod 2$, $l = (n-j)/2$, summed over $j \equiv n \pmod{2}$ with $j+l \leq k$. For $1 \leq n \leq k$ all coefficients of φ_n are positive integers, $\varphi_n(0) > 0$ (the term $j = e$), and φ_n has degree $\lfloor n/2 \rfloor$ and leading coefficient $\binom{k}{n}$ (the term $j = n$).

Lemma 2.2. *Let $m \geq 1$, let λ be a primitive m -th root of unity and $P = \sum_n P_n t^n \in \mathbb{C}[t]$. Then $\frac{1}{m} \sum_{j=0}^{m-1} \lambda^{-j} P(\lambda^j t) = \sum_{n \equiv 1 \pmod{m}} P_n t^n$.*

Proof. The coefficient of t^n on the left is $P_n \cdot \frac{1}{m} \sum_j \lambda^{j(n-1)}$, which is P_n if $m \mid n-1$ and 0 otherwise. $\square$

3 A closed form for $k = 7$

Proof of Theorem 1.2. Step 1: the constants. The quadratic $5I^2 + 15I + 3$ has discriminant $225 - 60 = 165$, so its roots are $I_{\mp} = (-15 \mp s)/10$. Let $f(I) = I^3 + 42I^2 + 210I + 140$. On the roots, $I^2 = -3I - \frac{3}{5}$ and $I^3 = \frac{42}{5}I + \frac{9}{5}$, hence $f(I) = \frac{1}{5}(462I + 583)$ and

$$f(I_{\mp}) = -22 \mp \frac{231}{25} s. \quad (6)$$

Multiplying numerator and denominator by $-550 + 231s$ and dividing by the common factor 605,

$$\frac{f(I_+)}{f(I_-)} = \frac{-550 + 231s}{-550 - 231s} = -\frac{9107065 - 254100s}{8502065} = -\frac{15053 - 420s}{14053},$$

so $q^3 = f(I_+)/f(I_-)$. Since s is irrational, $I_{\mp} \neq 0$ and $f(I_{\mp}) \neq 0$; hence $\beta \neq 0$, $q \neq 0$ and $p \neq 0$. If $q^6 = 1$, then $f(I_+)^2 = f(I_-)^2$; but $f(I_+)^2 - f(I_-)^2 = (f(I_+) - f(I_-))(f(I_+) + f(I_-)) = \frac{462}{25}s \cdot (-44) \neq 0$. So $q^6 \neq 1$, and A and $\sigma \neq 0$ are defined.

Step 2: five coefficients. For $k = 7$ and $I = b^2/c$, formula (3) gives

$$Q_1 = 7b, \quad Q_4 = 35b^4 + 105b^2c + 21c^2 = 7c^2(5I^2 + 15I + 3),$$

$$Q_7 = b^7 + 42b^5c + 210b^3c^2 + 140bc^3 = bc^3f(I),$$

and (4) gives $Q_{10} = c^3Q_4$ and $Q_{13} = c^6Q_1 = 7bc^6$. For h we have $(b, c) = (\beta, 1)$ and $I = \beta^2 = I_-$; for g we have $(b, c) = (p, q)$ and $I = p^2/q = I_+$. Both values of I are roots of $5I^2 + 15I + 3$, so the coefficients of $t^1, t^4, t^7, t^{10}, t^{13}$ are

$$h^7 : 7\beta, 0, \beta f(I_-), 0, 7\beta; \quad g^7 : 7p, 0, pq^3f(I_+), 0, 7pq^6.$$

Let $P = h^7 + Ag^7 = \sum_n P_n t^n$. Since $Apq^6 = -\beta$,

$$P_1 = 7\beta(1 - q^{-6}) = \sigma, \quad P_4 = P_{10} = 0, \quad P_7 = \beta(f(I_-) - q^{-3}f(I_+)) = 0, \quad P_{13} = 7\beta + 7Apq^6 = 0.$$

Step 3: averaging. P has degree at most 14, and the exponents $n \leq 14$ with $n \equiv 1 \pmod{3}$ are 1, 4, 7, 10, 13. By Lemma 2.2 the right-hand side of (1) is $\sigma^{-1} \sum_{n \equiv 1 \pmod{3}} P_n t^n = t$.

Step 4: powers. Each of the six coefficients $\omega^{-j}/(3\sigma)$ and $A\omega^{-j}/(3\sigma)$ is a non-zero complex number, hence equal to γ^7 for some $\gamma \in \mathbb{C}$, and $\gamma^7 h(\omega^j t)^7 = (\gamma h(\omega^j t))^7$. The polynomials $\gamma h(\omega^j t)$ and $\gamma g(\omega^j t)$ have leading coefficients $\gamma \omega^{2j}$ and $\gamma q \omega^{2j}$, which are non-zero. The homogeneous form (2) follows by Lemma 2.1(b), since $y^2 h(\omega^j x/y) = y^2 + \beta \omega^j xy + \omega^{2j} x^2$ and similarly for g . $\square$

Numerical values. For $s = +\sqrt{165}$, the real cube root q , $\omega = e^{2\pi i/3}$ and the roots β, p with positive imaginary, respectively real, part, one has $\beta \approx 1.6686890836i$, $q \approx -0.8824824684$, $p \approx 0.4360670215$, $A \approx -8.1018773317i$ and $\sigma \approx -13.0499070329i$. Taking principal 7th roots of the six coefficients, (2) becomes $xy^{13} = \sum_{i=1}^6 (a_i y^2 + b_i xy + c_i x^2)^7$ with the values of Table 1 (rounded to six decimals). In double precision the coefficients of $\sum_i (a_i + b_i t + c_i t^2)^7$ differ from those of t by at most $4.5 \cdot 10^{-14}$, for each of the 48 choices of the roots s, ω, β, q, p (`check_note.py`, part [6]). This is a sanity check only.

i	a_i	b_i	c_i
1 ($h, j = 0$)	0.577351 + 0.131777 <i>i</i>	-0.219894 + 0.963419 <i>i</i>	0.577351 + 0.131777 <i>i</i>
2 ($g, j = 0$)	0.798482	0.348192	-0.704647
3 ($h, j = 1$)	0.590543 - 0.044255 <i>i</i>	-0.890333 - 0.428762 <i>i</i>	-0.333597 - 0.489297 <i>i</i>
4 ($g, j = 1$)	0.763008 - 0.235357 <i>i</i>	-0.077480 + 0.339462 <i>i</i>	0.516543 + 0.479281 <i>i</i>
5 ($h, j = 2$)	0.551262 - 0.216354 <i>i</i>	0.616130 - 0.772602 <i>i</i>	-0.088263 + 0.585584 <i>i</i>
6 ($g, j = 2$)	0.763008 + 0.235357 <i>i</i>	-0.077480 - 0.339462 <i>i</i>	0.516543 - 0.479281 <i>i</i>

Table 1: The six forms $a_i y^2 + b_i xy + c_i x^2$ of (2), for the roots chosen in the text.

Exact verification. The proof above needs no computer. As a supplement, the identity was also checked by exact computation. Multiplying (1) by $3\sigma pq^6 = 21\beta p(q^6 - 1)$ gives

$$\sum_{j=0}^2 \omega^{-j} \left(pq^6 h(\omega^j t)^7 - \beta g(\omega^j t)^7 \right) = 21\beta p(q^6 - 1)t. \quad (7)$$

The coefficients of both sides lie in the ring

$$\Lambda = \mathbb{Q}[s, \omega, \beta, q, p] / (s^2 - 165, \omega^2 + \omega + 1, \beta^2 - I_-, q^3 - r, p^2 - I_+ q), \quad r = -\frac{15053-420s}{14053}.$$

The five relations are monic in s, ω, β, q, p respectively, and each involves only variables that come earlier in this order. So Λ is a free $\mathbb{Q}$ -module with basis $s^a \omega^b \beta^c q^d p^e$ ($a, b, c, e \leq 1, d \leq 2$), every element has a unique normal form, and each choice of the roots in Theorem 1.2 defines a ring homomorphism $\Lambda \rightarrow \mathbb{C}$. All 15 coefficients of the difference of the two sides of (7) have normal form zero, and $q^6 - 1 = r^2 - 1 = \frac{58212000-12644520s}{197486809} \neq 0$. Hence (7) holds for every choice of the roots. This was computed by five programs: the program with which the identity was found (with SymPy), the first two independent verification runs (each with its own arithmetic in Λ written with Python fractions), `check_note.py` of this note (part [2]), and the verification run on the final text, which built Λ as a tower of quotient rings $\mathbb{Q} \subset \mathbb{Q}(s) \subset \dots$ and also evaluated the six-term identity in ball arithmetic for all 48 choices of the roots. Corrupted versions of the identity are rejected by these programs (Section 8).

4 Odd k

Throughout this section $k = 2m+1$ with $m \geq 3$, and φ_n are the polynomials (5) for this k . Theorem 1.2 is the case $m = 3$ of the following construction, with $\varphi_4 = 7(5I^2 + 15I + 3)$ and $\varphi_7 = f$. In the terminology of [Rez19, Section 2] it synchs the k -th powers of two quadratic trinomials at once.

Proposition 4.1. *Let I_0, I_1 be roots of φ_{m+1} with $\varphi_k(I_0)\varphi_k(I_1) \neq 0$ and $\varphi_k(I_0)^2 \neq \varphi_k(I_1)^2$. Choose $\beta, q, p \in \mathbb{C}$ with*

$$\beta^2 = I_0, \quad q^m = \varphi_k(I_1)/\varphi_k(I_0), \quad p^2 = I_1 q,$$

put $h = 1 + \beta t + t^2$, $g = 1 + pt + qt^2$, $A = -\beta/(pq^{2m})$, $\sigma = k\beta(1 - q^{-2m})$, and let λ be a primitive m -th root of unity. Then $\beta, p, q, \sigma \neq 0$ and

$$t = \frac{1}{m\sigma} \sum_{j=0}^{m-1} \lambda^{-j} \left(h(\lambda^j t)^k + A g(\lambda^j t)^k \right), \quad (8)$$

so t is a sum of $2m = k - 1$ k -th powers of polynomials of degree exactly 2.

Proof. Since $\varphi_{m+1}(0) > 0$, the roots I_0, I_1 are non-zero, so $\beta \neq 0$; q^m is finite and non-zero, so $q \neq 0$ and $p \neq 0$; and $q^{2m} = \varphi_k(I_1)^2/\varphi_k(I_0)^2 \neq 1$, so $\sigma \neq 0$. The exponents $n \leq 2k = 4m + 2$ with $n \equiv 1 \pmod{m}$ are 1, $m + 1$, $2m + 1 = k$, $3m + 1 = 2k - (m + 1)$ and $4m + 1 = 2k - 1$. By (5) and (4), Q_{m+1} and $Q_{3m+1} = c^m Q_{m+1}$ vanish when b^2/c is a root of φ_{m+1} , $Q_k = bc^m \varphi_k(b^2/c)$, $Q_1 = kb$ and $Q_{2k-1} = c^{2m} Q_1 = kbc^{2m}$. For h , $(b, c) = (\beta, 1)$ and $b^2/c = I_0$; for g , $(b, c) = (p, q)$ and $b^2/c = I_1$. So the coefficients of $P = h^k + Ag^k$ are $P_{m+1} = P_{3m+1} = 0$,

$$P_k = \beta \varphi_k(I_0) + Apq^m \varphi_k(I_1) = \beta(\varphi_k(I_0) - q^{-m} \varphi_k(I_1)) = 0,$$

$P_{2k-1} = k\beta + Akpq^{2m} = 0$ and $P_1 = k\beta + Akp = \sigma$. By Lemma 2.2 the right-hand side of (8) is t . As in Step 4 of the proof of Theorem 1.2, each of the $2m$ terms is a k -th power of a polynomial whose leading coefficient is non-zero. $\square$

The hypothesis of Proposition 4.1 follows from three conditions that can be checked by a finite computation with integers:

- (C1) φ_{m+1} is squarefree (it has degree $\lfloor (m+1)/2 \rfloor \geq 2$);
- (C2) φ_{m+1} and φ_k have no common root;
- (C3) the remainder of φ_k^2 upon division by φ_{m+1} is not a constant.

Lemma 4.2. (i) *If (C1)–(C3) hold, then the hypothesis of Proposition 4.1 holds.*

- (ii) *Let $\Pi > k$ be a prime and let bars denote reduction modulo Π . If $\gcd(\bar{\varphi}_{m+1}, \bar{\varphi}'_{m+1}) = 1$ and $\gcd(\bar{\varphi}_{m+1}, \bar{\varphi}_k) = 1$ in $\mathbb{F}_\Pi[I]$, and the remainder of $\bar{\varphi}_k^2$ upon division by $\bar{\varphi}_{m+1}$ in $\mathbb{F}_\Pi[I]$ has positive degree, then (C1)–(C3) hold.*

Proof. (i) Let $D = \deg \varphi_{m+1} \geq 2$. By (C1), φ_{m+1} has D distinct roots, and by (C2) φ_k vanishes at none of them. If φ_k^2 took one value v at all of them, the remainder R of φ_k^2 modulo φ_{m+1} , of degree less than D , would take the value v at D points, so $R = v$, against (C3).

(ii) The leading coefficient of φ_{m+1} is $\binom{k}{m+1}$, whose prime factors are at most $k < \Pi$; so $\deg \bar{\varphi}_{m+1} = \deg \varphi_{m+1}$. Suppose φ_{m+1} had a multiple root, or a root in common with φ_k . Let $\mu \in \mathbb{Z}[I]$ be the primitive integer multiple of the minimal polynomial over $\mathbb{Q}$ of that root. By Gauss's lemma, μ^2 divides φ_{m+1} in $\mathbb{Z}[I]$, respectively μ divides φ_{m+1} and φ_k in $\mathbb{Z}[I]$. The leading coefficient of μ divides that of φ_{m+1} , so $\bar{\mu}$ has positive degree, and $\bar{\mu}$ divides $\gcd(\bar{\varphi}_{m+1}, \bar{\varphi}'_{m+1})$, respectively $\gcd(\bar{\varphi}_{m+1}, \bar{\varphi}_k)$: a contradiction. Finally, since the leading coefficient of φ_{m+1} is a unit in the localisation $\mathbb{Z}_{(\Pi)}$, division gives $\varphi_k^2 = H\varphi_{m+1} + R$ with $H, R \in \mathbb{Z}_{(\Pi)}[I]$ and $\deg R < \deg \varphi_{m+1}$. Reducing modulo Π gives the division in $\mathbb{F}_\Pi[I]$, so $\bar{R}$ is the remainder computed there; if $\bar{R}$ has positive degree, so has R . $\square$

Proof of Theorem 1.3. For each of the 98 odd k with $7 \leq k \leq 201$, the hypotheses of Lemma 4.2(ii) hold for the prime $\Pi = 2^{61} - 1$; this was computed by `check_note.py` (part [4], 0.2 seconds). By Lemma 4.2, Proposition 4.1 and Lemma 2.1(b), the theorem follows. $\square$

The computation of (C1)–(C3), the only step that needs a computer, was also done exactly over $\mathbb{Q}$ with SymPy (gcd with the derivative, resultant, remainder) by the program with which the construction was found, for all 98 values; by the first verification run exactly and modulo $2^{61} - 1$, with φ_n obtained by expanding $(1 + bt + t^2)^k$; by the second modulo two primes larger than k and exactly for $k \leq 61$; by `check_note.py` exactly for $k \leq 61$; and by the verification run on the final text exactly over $\mathbb{Q}$ (with FLINT) and modulo two further primes for all 98 values, together with a direct check of the hypothesis of Proposition 4.1 by rigorous isolation of the roots of φ_{m+1} . All agree.

Example 4.3 ($k = 9$). Here $m = 4$, $\varphi_5 = 126(I^2 + 4I + 2)$ with roots $I_0 = -2 - \sqrt{2}$ and $I_1 = -2 + \sqrt{2}$, and $\varphi_9 = I^4 + 72I^3 + 756I^2 + 1680I + 630$. Modulo $I^2 + 4I + 2$ one has $\varphi_9 \equiv -384I - 334$, so $\varphi_9(I_0) = 434 + 384\sqrt{2}$ and $\varphi_9(I_1) = 434 - 384\sqrt{2}$. Hence

$$q^4 = \frac{434 - 384\sqrt{2}}{434 + 384\sqrt{2}} = -\frac{120817 - 83328\sqrt{2}}{26639},$$

$\lambda = i$, and (8) writes t , and so xy^{17} , as a sum of eight 9th powers of quadratic polynomials, respectively forms. The analogue of (7), $\sum_{j=0}^3 i^{-j} (pq^8 h(i^j t)^9 - \beta g(i^j t)^9) = 36\beta p(q^8 - 1)t$, was checked exactly in the corresponding ring by `check_note.py` (part [3]) and by the original program.

Remark 4.4. For $k = 5$ ($m = 2$) the construction gives nothing: $\varphi_3 = 10(I + 2)$ has only one root, so $I_0 = I_1$ and $\sigma = 0$. For even k the reflection (4) does not respect the classes modulo m of the exponents, and we have no closed form. The second verification run also checked the hypotheses of

Lemma 4.2(ii) for all odd k with $203 \leq k \leq 1001$; we record this, but state Theorem 1.3 only for $k \leq 201$, where the conditions were checked by several programs. Whether (C1)–(C3) hold for every odd $k \geq 7$ is open; if they do, $\text{rk}_k(xy^{kd-1}) \leq k-1$ for every odd $k \geq 7$ and every $d \geq 2$.

5 Certified identities for $k = 5, 6, 8, 9$

The equations. Fix k and s , write $g_i(t) = c_{i0} + c_{i1}t + c_{i2}t^2$ for $i = 1, \dots, s$, and collect the $3s$ coefficients in a vector $c \in \mathbb{C}^{3s}$. Then $t = \sum_i g_i(t)^k$ if and only if $F(c) = 0$, where

$$F_n(c) = [t^n] \sum_{i=1}^s g_i(t)^k - \delta_{n,1} \quad (n = 0, 1, \dots, 2k), \quad \frac{\partial F_n}{\partial c_{ij}} = k [t^{n-j}] g_i(t)^{k-1}. \quad (9)$$

These are $2k+1$ equations in $3s$ unknowns. The solution set is invariant under $g_i(t) \mapsto \mu g_i(\lambda t)$ with $\lambda \in \mathbb{C}^*$ and $\mu^k \lambda = 1$, which can be used to give c_{10} any non-zero value. A *certificate* fixes $3s - (2k+1)$ coordinates to short exact values (Table 2), which leaves a square system of $2k+1$ equations in the $2k+1$ free coordinates, and gives a centre c^0 whose real and imaginary parts are exact decimal numbers with 60 significant digits, and the radius $r = 2^{-100} \approx 7.9 \cdot 10^{-31}$. Let X be the box of all c whose fixed coordinates have the prescribed values and whose free coordinates satisfy $|\text{Re}(c_j - c_j^0)| \leq r$ and $|\text{Im}(c_j - c_j^0)| \leq r$. Any zero of F in X gives an identity $t = \sum_{i \leq s} g_i(t)^k$.

certificate	(k, s)	free unknowns	fixed coordinates
cert_k5_d2_s4_A	(5, 4)	11	$c_{10} = 0.61$
cert_k6_d2_s5_A	(6, 5)	13	$c_{10} = 0.47, c_{21} = -0.14 - 0.11i$
cert_k6_d2_s5_B	(6, 5)	13	$c_{10} = 0.67, c_{21} = -0.27 - 0.40i$
cert_k7_d2_s6_A	(7, 6)	15	$c_{10} = 0.82, c_{21} = 0.26 - 0.14i, c_{31} = -0.81 + 0.83i$
cert_k8_d2_s6_A	(8, 6)	17	$c_{10} = 0.72$
cert_k9_d2_s7_A	(9, 7)	19	$c_{10} = 0.68, c_{21} = -0.07 - 0.31i$

Table 2: The six certificates; each has a centre with 60 significant digits and the radius 2^{-100} .

Lemma 5.1 (existence test). *Let $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map, $c^0 \in \mathbb{C}^n$, $r > 0$, and $X = \{x : |\text{Re}(x_j - c_j^0)| \leq r, |\text{Im}(x_j - c_j^0)| \leq r \ (1 \leq j \leq n)\}$. For all a, b let $K_{ab} \subset \mathbb{C}$ be a closed convex set with $\partial_b F_a(x) \in K_{ab}$ for all $x \in X$, and let $\mathcal{J}$ be the set of matrices M with $M_{ab} \in K_{ab}$. Let Y be a complex $n \times n$ matrix such that*

(a) $\|I - YM\|_\infty < 1$ for every $M \in \mathcal{J}$, and

(b) $-YF(c^0) + (I - YM)h$ lies in the interior of $X - c^0$ for every $M \in \mathcal{J}$ and every $h \in X - c^0$.

Then F has exactly one zero x^* in X , and $DF(x^*)$ is invertible.

Proof. For $x \in X$ put $h = x - c^0$. The segment from c^0 to x lies in X , and by the chain rule for holomorphic polynomials $F(x) - F(c^0) = M_x h$, where $(M_x)_{ab} = \int_0^1 \partial_b F_a(c^0 + \tau h) d\tau$. This integral is a limit of averages of values in the closed convex set K_{ab} , so $M_x \in \mathcal{J}$. The map $\phi(x) = x - YF(x)$ satisfies $\phi(x) - c^0 = -YF(c^0) + (I - YM_x)h$, so $\phi(X) \subset X$ by (b). Since X is compact and convex, Brouwer's theorem gives $x^* \in X$ with $YF(x^*) = 0$. By (a) with $M = DF(c^0) \in \mathcal{J}$, the matrix $YDF(c^0)$ is invertible, so Y is invertible and $F(x^*) = 0$. If $x_1, x_2 \in X$ are zeros, then in the same way $0 = F(x_1) - F(x_2) = M(x_1 - x_2)$ with $M \in \mathcal{J}$, and YM is invertible by (a); so $x_1 = x_2$. Finally $DF(x^*) \in \mathcal{J}$ is invertible. $\square$

This is the test of Krawczyk [Kra69] and Moore [Moo77] in a form suited to complex boxes. Its hypotheses were verified for each certificate by three programs of this note, which differ in the sets K_{ab} and in the arithmetic.

- **check_krawczyk.py** (standard library only): K_{ab} are rectangles obtained by evaluating $k[t^{a-j}]g_i^{k-1}$ over X in interval arithmetic with endpoints in $2^{-512}\mathbb{Z}$ and outward rounding; $F(c^0)$ is computed exactly; Y is a floating-point inverse of $DF(c^0)$ rounded to the grid, hence an exact matrix; (a) is checked through row sums of upper bounds of $|\operatorname{Re}| + |\operatorname{Im}|$, and (b) by strict inequalities for the real and imaginary parts.
- **check_krawczyk_mpiv.py**: the same test for the real $2n \times 2n$ form of the system, in the interval arithmetic of mpmath with 400 bits, with $F(c^0)$ enclosed by interval evaluation.
- **check_note.py**, part [5] (standard library only): K_{ab} is the closed disc around $DF(c^0)_{ab}$ of radius e_{ab} , where $e_{ab} = k([t^{a-j}](U_i + \eta_i)^{k-1} - [t^{a-j}]U_i^{k-1})$ for the free coordinate $b = c_{ij}$; here U_i is the polynomial with coefficients $|\operatorname{Re} c_{il}^0| + |\operatorname{Im} c_{il}^0|$, and η_i has coefficient $2r$ at the free and 0 at the fixed coordinates of g_i . This bounds $|\partial_b F_a(x) - \partial_b F_a(c^0)|$ on X , because a polynomial Φ with non-negative coefficients satisfies $|\Phi(z+w) - \Phi(z)| \leq \Phi(u+v) - \Phi(u)$ whenever $|z_l| \leq u_l$ and $|w_l| \leq v_l$. With Y a floating-point inverse converted to exact rationals and $B_{ab} = |(I - YDF(c^0))_{ab}| + \sum_c |Y_{ac}|e_{cb}$, the program checks in exact rational arithmetic (moduli bounded by $|\operatorname{Re}| + |\operatorname{Im}|$) that $\rho = \max_a \sum_b B_{ab} < 1$, which gives (a), and that $|YF(c^0)|_a + \frac{3}{2}r \sum_b B_{ab} < r$ for all a , which gives (b), since $|h_b| \leq \sqrt{2}r < \frac{3}{2}r$ on $X - c^0$ and a complex number of modulus less than r has real and imaginary parts in $(-r, r)$.

The results are in Table 3. The margins are large: all bounds in the columns (b) are below $6 \cdot 10^{-12}r$. The three verification runs of Section 8 checked the six certificates with programs of their own: exact rational majorants, a Newton–Kantorovich test on polydisks, and the test of Lemma 5.1 in ball arithmetic.

certificate	$\max F(c^0) $	check_krawczyk.py		check_krawczyk_mpiv.py		check_note.py	
		(a)	(b)	(a)	(b)	ρ	(b)
$k = 5$	$3.5 \cdot 10^{-60}$	$7.5 \cdot 10^{-15}$	$5.9 \cdot 10^{-45}$	$7.5 \cdot 10^{-28}$	$6.0 \cdot 10^{-58}$	$1.0 \cdot 10^{-14}$	$1.2 \cdot 10^{-44}$
$k = 6 \text{ A}$	$1.3 \cdot 10^{-59}$	$3.8 \cdot 10^{-13}$	$3.0 \cdot 10^{-43}$	$7.6 \cdot 10^{-26}$	$6.0 \cdot 10^{-56}$	$2.7 \cdot 10^{-13}$	$3.2 \cdot 10^{-43}$
$k = 6 \text{ B}$	$5.5 \cdot 10^{-59}$	$1.5 \cdot 10^{-13}$	$1.2 \cdot 10^{-43}$	$4.3 \cdot 10^{-26}$	$3.4 \cdot 10^{-56}$	$1.4 \cdot 10^{-13}$	$1.7 \cdot 10^{-43}$
$k = 7$	$1.6 \cdot 10^{-58}$	$3.1 \cdot 10^{-12}$	$2.5 \cdot 10^{-42}$	$6.9 \cdot 10^{-25}$	$5.4 \cdot 10^{-55}$	$3.5 \cdot 10^{-12}$	$4.2 \cdot 10^{-42}$
$k = 8$	$2.8 \cdot 10^{-59}$	$2.9 \cdot 10^{-12}$	$2.3 \cdot 10^{-42}$	$8.5 \cdot 10^{-25}$	$6.7 \cdot 10^{-55}$	$2.9 \cdot 10^{-12}$	$3.5 \cdot 10^{-42}$
$k = 9$	$2.7 \cdot 10^{-59}$	$1.2 \cdot 10^{-13}$	$9.3 \cdot 10^{-44}$	$1.8 \cdot 10^{-25}$	$1.4 \cdot 10^{-55}$	$1.3 \cdot 10^{-13}$	$1.5 \cdot 10^{-43}$

Table 3: Verification of Lemma 5.1. $\max |F(c^0)|$: largest real or imaginary part of the exact residual at the centre. (a): bounds for $\|I - YM\|_\infty$ on $\mathcal{J}$ (must be < 1). (b): bounds for the real and imaginary parts, respectively the moduli (**check_note.py**), of $-YF(c^0) + (I - YM)h$ (must be $< r = 7.9 \cdot 10^{-31}$). All tests pass.

Proof of Theorem 1.4. For each certificate the hypotheses of Lemma 5.1 hold (Table 3), so the square system has a zero in X . Together with the fixed coordinates it gives polynomials g_i of degree at most 2 with $t = \sum_{i \leq s} g_i(t)^k$, where $(k, s) = (5, 4), (6, 5), (7, 6), (8, 6), (9, 7)$. By Lemma 2.1(b) the forms $G_i = y^2 g_i(x/y)$ give the identities of the theorem. (On X every $|c_{i2}|$ is at least 0.10, so all g_i have degree exactly 2; and the coefficients are algebraic numbers, since the zero is a non-singular zero of a system over $\mathbb{Q}(i)$.) $\square$

The case $k = 5$. Rounded to 16 significant digits, the certified identity $t = g_1^5 + g_2^5 + g_3^5 + g_4^5$ has

$$\begin{aligned} g_1 &= 0.61 + 0.7485551526772892t - 0.4969158600467957t^2, \\ g_2 &= -0.6015985488651935 + 0.7555844907567810t + 0.4869814442853431t^2, \\ g_3 &= (-0.05004956086299882 + 0.3299314893237855i) + (-0.2551307739278631 - 0.1850216470009282i)t \\ &\quad + (-0.6139653223997975 + 0.2012452055819655i)t^2, \\ g_4 &= (0.2344198089905820 + 0.2375017880833938i) + (0.09765213643366455 - 0.2996477630717563i)t \\ &\quad + (0.6149973437139760 - 0.1980689705733045i)t^2. \end{aligned}$$

The certificate itself gives 60 digits, and the exact coefficients lie within $7.9 \cdot 10^{-31}$ of them in real and imaginary part.

Remark 5.2. Here g_1, g_2 are real and $g_4 = \zeta \bar{g}_3$ with $\zeta = e^{4\pi i/5}$ (bar: conjugate coefficients), so $t = g_1^5 + g_2^5 + 2\operatorname{Re}(g_3^5)$. Indeed, $\tau(g_1, g_2, g_3, g_4) = (\bar{g}_1, \bar{g}_2, \zeta \bar{g}_4, \zeta \bar{g}_3)$ maps solutions to solutions, fixes c_{10} , and moves the centre by less than 10^{-60} (`check_real_structure_k5.py`); as the certified zero lies within $5.9 \cdot 10^{-45}$ of the centre, uniqueness in Lemma 5.1 shows that τ fixes it. This is not used elsewhere.

How the certificates were found (not part of any proof). Damped Gauss–Newton iterations from random complex starts gave approximate solutions of (9); points with a Jacobian of full rank were rescaled, given short exact values in the fixed coordinates, refined by Newton’s method at 90 digits and rounded to 60 digits. Most runs end at ill-conditioned near-solutions with small leading coefficients, because $xy^{2k-1} = \lim_{\varepsilon \rightarrow 0} ((y^2 + \varepsilon xy)^k - y^{2k})/(k\varepsilon)$ is a limit of sums of two k -th powers. So a small residual proves nothing by itself, and Lemma 5.1 is needed.

6 Consequences

Proof of Corollary 1.5. By Theorems 1.2 to 1.4, xy^{2k-1} is a sum of $s(k)$ k -th powers of quadratic forms. By Lemma 2.1(c) and (a), $\operatorname{rk}_k(\ell_1 \ell_2^{kd-1}) = \operatorname{rk}_k(xy^{kd-1}) \leq s(k) < k$ for every $d \geq 2$, and by Lemma 2.1(d) Conjecture 1.1 fails at (k, d) ; the reading $\operatorname{rk}_k(\ell_1 \ell_2^{kd-1}) = k$ of (B) fails as well. For $d = 2$ and $k \leq 4$ the conjecture holds by the results quoted in Section 1.2: for $k \leq 2$ by [FLOS18, (5)], at $(3, 2)$ by [LORS19, Theorem 3.1] and [CO15, Corollary 3.6], and at $(4, 2)$ by [BDOV20, Theorem 1.6] together with $\operatorname{rk}_4(xy^7) = 4$ [CO15, LORS19]. At $(5, 2)$ it fails. $\square$

The statement at $(4, 2)$ rests on the computer calculations of [BDOV20, CO15, LORS19]. “First failure” refers to the order of k for $d = 2$: for $k = 4$ and $d \geq 3$ the conjecture is open.

6.1 Monomials

Problem D of [FLOS18] (record AMR-014-0006 of [ula26]) asks, for $k \geq 3$ and a monomial $x^a y^b$ of degree kd , whether the bound $\operatorname{rk}_k(x^a y^b) \leq \max([a]_k, [b]_k) + 1$ of [CO15, Proposition 3.4] is an equality.

Proposition 6.1. *The answer to Problem D of [FLOS18] is negative. For k, d and $s(k)$ as in Corollary 1.5, $\max([1]_k, [kd-1]_k) + 1 = k$, while $\operatorname{rk}_k(xy^{kd-1}) \leq s(k) < k$. The smallest example here is $\operatorname{rk}_5(xy^9) \leq 4$; an example with a proof that needs no computer is $\operatorname{rk}_7(xy^{13}) \leq 6$.*

Proof. $[1]_k = 1$ and $[kd-1]_k = k-1$, so the bound is k ; the rest is Corollary 1.5. $\square$

Equality does hold in some cases, by the results of Carlini and Oneto: for $k = 3$, where every binary monomial that is not a cube has 3-rank 3 [CO15, Corollary 3.6]; for $k = 4$ in the class of exponents $\equiv 2$ modulo 4, e.g. $x^6 y^2$ has 4-rank 3 [CO15, Remark 3.7]; and for xy^7 and $x^3 y^5$, which have 4-rank 4 by a CoCoA computation reported in [CO15, Remark 3.7].

6.2 Heilbronn's problem and the Waring number of $\mathbb{C}[t]$

Let C be one of the classes P, E, R, M of polynomials, entire, rational and meromorphic functions in the plane, and for $N \geq 2$ let $g_C(N)$ be the least k such that every $f \in C$ is a sum of k N -th powers of elements of C [HL19, Update 2.26]. Heilbronn's problem [HL19, Problem 2.26] asks for $k(N) = g_E(N)$, and $g_P(N)$ is the Waring number of $\mathbb{C}[t]$, denoted $v(0, N)$ in [Vol15]. The identity $z = N^{-2} \sum_{\nu} (1 + \omega_{\nu} z)^N / \omega_{\nu}$, summed over the N -th roots of unity ω_{ν} , shows $k(N) \leq N$, and $k(N) = N$ for $N \leq 3$ [HL19, Problem 2.26]. The Update to this problem derives lower bounds from a theorem of Cartan, among them $k(k-1) > N$ in P for $N > 2$, and states that “for large k , no upper bounds substantially better than $k \leq N$ are known”, for any of the four classes [HL19, Update 2.26]. Voloch quotes $\sqrt{n} < v(0, n) \leq n$ from Newman and Slater [NS79] and asks whether $v(0, n) = n/2 + O(1)$ [Vol15].

Proposition 6.2. *For $C \in \{P, E, R, M\}$: $g_C(5) \leq 4$, $g_C(6) \leq 5$, $g_C(7) \leq 6$, $g_C(8) \leq 6$, $g_C(9) \leq 7$, and $g_C(N) \leq N - 1$ for every odd N with $7 \leq N \leq 201$.*

Proof. By Theorems 1.2 to 1.4, $t = \sum_{i \leq s} g_i(t)^N$ with polynomials g_i and s as stated. For $f \in C$ we get $f = \sum_i g_i(f)^N$ with $g_i \circ f \in C$. $\square$

With the lower bound quoted above, $g_P(5) \in \{3, 4\}$ and $4 \leq g_P(N) \leq s(N)$ for $N = 6, \dots, 9$; the exact values are not known. We found no upper bound below N for any $N \geq 4$ in the literature accessible to us. This is weak evidence of novelty only: the sentence of [HL19] quoted above concerns large k and “substantially better” bounds, so it does not show that bounds such as $N - 1$ for particular N were unknown, and the primary sources [NS79] and [Hay85] could not be accessed and were not read. Our results do not decide Voloch's question. Finally, $\text{rk}_k(xy^{kd-1}) \geq 3$ for $k \geq 3$ [CO15, Theorem 3.2], and since $\text{rk}_k(xy^{kd-1}) \geq g_P(k)$ by Lemma 2.1(b), the bound quoted above gives $\text{rk}_k(xy^{kd-1}) \geq 4$ for $6 \leq k \leq 9$; these lower bounds are cited, not proved here, and the exact values for $k \geq 5$ are unknown.

7 Open questions

1. Clause (A). The equality $\text{rk}_k^{\max}(kd, 2) = k$ is known for $k \leq 2$, for $d = 1$ and at $(3, 2)$, $(3, 3)$, $(3, 4)$, $(4, 2)$, and, as far as we know, open otherwise. At $(5, 2)$, the generic 5-rank of binary forms of degree 10 is $\lceil 11/3 \rceil = 4$ [LORS19, Theorem 2.3], and the maximal rank is at most twice the generic rank ([BT15], as recalled in [FLOS18, Remark 1.10]); so $4 \leq \text{rk}_5^{\max}(10, 2) \leq 8$. Whether some binary form of degree 10 has 5-rank 5 is unknown; we only show that xy^9 is not such a form.

2. The case $k = 4$. Is $\text{rk}_4(xy^{4d-1}) = 4$ for all $d \geq 3$? A parameter count (below) predicts that three terms do not suffice, but this is not a proof.

3. All $k \geq 5$. Is $\text{rk}_k(xy^{kd-1}) < k$ for every $k \geq 5$ and $d \geq 2$? For odd $k > 201$ this would follow from (C1)–(C3) (Remark 4.4); for even $k \geq 10$ we have no construction. With $d = 2$ there are $3s$ unknowns, $2k + 1$ equations and a one-parameter group, so solutions are expected when $3s - (2k + 1) \geq 1$, that is, $s \geq (2k + 2)/3$; for $d \geq 3$ the corresponding count $s(d + 1) \geq kd + 2$ is worse. The certified values $s(k) = 4, 5, 6, 6, 7$ for $k = 5, \dots, 9$ are equal to $\lceil (2k + 2)/3 \rceil$. The second verification run found no regular solution with $(k, s) = (4, 3)$ or $(5, 3)$ in 20,000 random starts each, or with $(7, 5)$ in 6,000 starts. This is evidence, not proof, that these $s(k)$ are the least numbers of k -th powers of quadratic forms with sum xy^{2k-1} .

4. Closed forms. We have no closed form for $k = 5, 6, 8$; an integer-relation search for the invariants $c_{i1}^2 / (c_{i0}c_{i2})$ of the $k = 5$ solution (degree at most 16, 250 digits) found none.

8 Computations and reproducibility

Only two steps are computer-assisted: (C1)–(C3) for $7 \leq k \leq 201$ (Theorem 1.3), and the hypotheses of Lemma 5.1 for the six certificates (Theorem 1.4). The reproducibility package has three parts.

Programs of this note. `check_note.py` (standard library, about 5 seconds) re-checks formulas (3)–(5) for $k \leq 21$, the exact parts of Theorem 1.2 and Example 4.3 (the numbers (6), q^3 , $q^6 - 1$ and all coefficients of (7) in Λ), the hypotheses of Lemma 4.2(ii) for $\Pi = 2^{61} - 1$ and all odd $k \leq 201$ ((C1)–(C3) also exactly over $\mathbb{Q}$ for $k \leq 61$), the six certificates, three corrupted versions of each certificate (a free coordinate moved by $10r$, a fixed value changed by 10^{-20} , the centre truncated to 20 decimals; all 18 fail), and the numerical checks of Section 3. `rejection_tests_k7_exact.py` (SymPy, 1 second) tests four corrupted versions of (7): q^3 multiplied by 1.001; the roots $I_{\mp}$ exchanged; exponent 6 instead of 7 in the g -terms; ω replaced by a primitive 4th root of unity. All four are rejected. The program that first produced this recorded output was not kept; this one reproduces it byte for byte. `run_quick.sh` re-runs the quick part of the package and compares with the recorded outputs.

Programs with which the results were first obtained (original/): the six certificates and the two checkers of Section 5; `rejection_tests.py`, which corrupts each certificate in six ways (a free coordinate moved by $10r$ in its real or imaginary part, a fixed value changed by 10^{-20} , the radius replaced by 2^{-230} , two centre values exchanged, the centre truncated to 20 decimals) and runs both checkers, which reject all 36 corrupted certificates; `check_k7_exact.py` (the exact check of (7)); `exact_symmetric.py` ((C1)–(C3) over $\mathbb{Q}$ for odd $k \leq 201$, and the identities for $k = 7, 9$; 44 seconds); `check_real_structure_k5.py` (Remark 5.2); `show_identity.py`, which prints the identity of a certificate and its exact residual (the recorded files `identity_*.txt` were made with the precision argument 22, as in `show_identity.py cert.json 22`; the default is 25 digits); and `make_cert.py` and the search programs, which are not part of any proof. Both checkers also pass at lower precision (grid 2^{-256} , respectively 250 bits). SHA-256 hashes of the certificates and checkers are recorded.

Verification runs. The programs and recorded outputs of the three verification runs described in the next paragraph are in the folders `verification_run_A/`, `verification_run_B/` and `independent_run_2/` (the run on the final text). All programs of the package with recorded outputs were run again from a copy of the package when this note was written, and again from the archive by the run on the final text; their outputs agree with the recorded ones, up to running times.

Verification. The results were first obtained, with proofs, with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two worked separately before this note was written, and both re-derived Lemma 2.1, the logic of the refutation, the proof of Theorem 1.2, Proposition 4.1 and Lemma 5.1.

The first run checked (7) exactly with its own arithmetic in Λ (five corrupted versions rejected) and at 100 digits for all 24 choices of the roots (relative error at most $7.4 \cdot 10^{-92}$); checked (C1)–(C3) as described in Section 4; verified the six certificates with its own checker (exact rational majorants, Y from a 50-digit inverse, $|\phi(X) - c^0| \leq 2.6 \cdot 10^{-54}$) and rejected all 35 corrupted certificates it built (seven corruptions of each of five certificates); and re-ran the original checkers and rejection tests at both precisions. It found [BDOV20], which the first version had missed, attributed $\text{rk}_3(xy^{3d-1}) = 3$ and the 4-rank of x^6y^2 to [CO15], and required corrections of wording (the range “odd k with $7 \leq k \leq 201$ ”, the quotations from [LORS19] and [HL19], Voloch’s question).

The second run checked (7) with its own exact arithmetic (five corrupted versions rejected) and with explicit 7th roots at 110 digits for all 48 choices of the roots (error at most $5.4 \cdot 10^{-109}$); checked (C1)–(C3) (Section 4); and verified the six certificates with a Newton–Kantorovich test on polydisks, based on the exact Taylor expansion of F at the centre (42 corrupted certificates rejected). Its own search (20,000 random starts for $(k, s) = (5, 4)$, 6,000 for $(6, 5)$ and for $(8, 6)$) found 74, 135 and 1 regular solutions; those for $k = 5$ lie in three orbits of the one-parameter group, one of which contains

our certificate. It certified one point of each of these orbits and the points for $k = 6, 8$ with its own programs, so Theorem 1.4 for $k = 5, 6, 8$ was also proved without our data. It re-ran all original programs, with identical outputs, and also found [BDOV20], the truncated quotation and the year of [HL19], the missing script behind one recorded output and the undocumented precision argument of `show_identity.py`.

All corrections required by the two runs were made; none of them changes an identity or a proof, and the list of known cases in Section 1.2 became longer. Neither run found an error in the identities, the certificates, Lemma 5.1, the checkers, the construction or the reduction.

The third run examined the final text of this note and the package. It read the cited passages of [FLOS18, LORS19, CO15, BDOV20, HL19, Vol15] again, re-derived every proof line by line, and checked the computational claims with programs of its own, written with FLINT and Arb: identity (7) in its own tower arithmetic for Λ (all 15 coefficients zero; a corrupted version rejected), and the six-term identity in ball arithmetic at 3000 bits for all 48 choices of the roots (largest coefficient error $1.6 \cdot 10^{-897}$), with the numbers of Table 1; (C1)–(C3) for all odd k with $7 \leq k \leq 201$ as described in Section 4, and (8) numerically for $k \leq 101$; the six certificates by the test of Lemma 5.1 in ball arithmetic (all pass, with the residuals of Table 3; 36 corrupted certificates rejected; $|c_{i2}| \geq 0.108$ on every box); and the smaller numerical claims of Sections 5 to 7, among them that the identity of Theorem 1.4 for $k = 7$ does not lie in the orbit of that of Theorem 1.2 under the group of Section 5. It re-ran the whole package from the archive, with the results stated above. It found no mathematical error. It found the paper [Rez19], whose synching is the technique of Theorem 1.2 and Proposition 4.1, and required five corrections, which were made: the credit to [Rez19] in Section 1.4, in Section 4 and below; the wording on Heilbronn’s problem in the abstract and in Section 1.3; two numbers of the first run in this paragraph; the words “as far as we know” in Section 7; and the description of this run.

Scope and priority. The note refutes Conjecture 1.9 of [FLOS18], as the conjunction of clauses (A) and (B), at the (k, d) of Corollary 1.5, and clause (B) in the reading of [LORS19]. It does not decide clause (A) at any new (k, d) , it does not determine $\text{rk}_k(xy^{kd-1})$ for $k \geq 5$, and it covers only $k = 5, \dots, 9$ and the odd k with $7 \leq k \leq 201$. Theorem 1.2 needs no computer, Theorem 1.3 needs the finite computation of (C1)–(C3), and Theorem 1.4 is computer-assisted. The conjecture, its notation and the known cases are due to Fröberg, Lundqvist, Oneto and Shapiro [FLOS18], to Lundqvist, Oneto, Reznick and Shapiro [LORS19], to Carlini and Oneto [CO15] (among them the bound $\text{rk}_k(xy^{kd-1}) \leq k$, the value $\text{rk}_3(xy^{3d-1}) = 3$ and the 4-rank 3 of x^6y^2) and to Bik, Draisma, Oneto and Ventura [BDOV20]. The averaging over roots of unity is classical [HL19, Vol15]; for powers of quadratic trinomials, with the middle coefficient chosen as a root of a coefficient polynomial, it is Reznick’s synching [Rez19]; and the existence test is that of Krawczyk and Moore [Kra69, Moo77]. To our knowledge and relative to the literature found, the counterexamples to Conjecture 1.9 and the explicit identities are new. No answer to the conjecture was found in the literature accessible to us; a search that finds nothing is not a proof of novelty, and we claim no priority.

What was read: [FLOS18] in the arXiv version (v1, TeX source), compared with the journal version, whose page numbers we give (Definition 1.8 on p. 141, Conjecture 1.9 and (5) on p. 142, Problem D on p. 143); [LORS19] in the arXiv version v1 (Section 1, Theorems 2.3 and 3.1, Section 4 with Example 4.7, Appendix B), whose numbering we use; [CO15] in the arXiv version (Sections 2 and 3); [BDOV20] in the arXiv version v2 (Section 1 with Theorem 1.6, and its proof in Section 4); [HL19] in the arXiv version (Problem 2.26 and its Update); [Vol15] completely; [Rez19] in the arXiv version v1 (Sections 1, 2 and 5), whose numbering we use. The numbering of the journal versions of [LORS19, CO15, BDOV20, Rez19] was not compared. Not read: [Rez13], whose Theorems 4.9 and 5.4 are quoted from [LORS19]; [BT15], quoted from [FLOS18, Remark 1.10]; [NS79] and [Hay85], which

could not be accessed (the publisher’s site refused automated access to [NS79], and we found no online copy of [Hay85]), so that what we say about them is taken from [HL19, Vol15]; and [Kra69, Moo77], cited for the origin of the test of Lemma 5.1, whose proof is given in full. Further papers on sums of powers of polynomials that came up in the searches were read by the verification runs, most of them only in their abstracts or reviews; none of them concerns representations of t by fewer than N N -th powers over $\mathbb{C}$.

The searches were made on 14 October 2026, when the results were found, by the three verification runs and when this note was written: queries to the arXiv (for k -ranks and maximal k -ranks of binary forms, sums of powers of forms and of polynomials, linearly dependent powers of quadratic forms, the Waring number of $\mathbb{C}[t]$, Heilbronn’s problem, the monic rank, and the authors of [FLOS18, LORS19]), to Crossref and to zbMATH; the lists of works citing [FLOS18], [LORS19], [CO15], [BDOV20] and [NS79] in OpenAlex and Semantic Scholar (all 39 titles for [NS79]); and nine web searches. No refutation of the conjecture, no decomposition of $\ell_1 \ell_2^{kd-1}$ with fewer than k terms, and no identity $t = \sum_{i < N} f_i(t)^N$ over $\mathbb{C}$ was found. [BDOV20] proves new cases of clause (A) but says nothing about clause (B) or about $k \geq 5$. Two preprints of 2026 by proposers of [FLOS18] that cite it treat other problems of the list and do not mention Conjecture 1.9.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [BDOV20] Arthur Bik, Jan Draisma, Alessandro Oneto, and Emanuele Ventura. The monic rank. *Mathematics of Computation*, 89(325):2481–2505, 2020. [arXiv:1901.11354](#), [doi:10.1090/mcom/3512](#).
- [BT15] Grigoriy Blekherman and Zach Teitler. On maximum, typical and generic ranks. *Mathematische Annalen*, 362(3-4):1021–1031, 2015. [doi:10.1007/s00208-014-1150-3](#).
- [CO15] Enrico Carlini and Alessandro Oneto. Monomials as sums of k th powers of forms. *Communications in Algebra*, 43(2):650–658, 2015. [arXiv:1305.4553](#), [doi:10.1080/00927872.2013.842247](#).
- [FLOS18] Ralf Fröberg, Samuel Lundqvist, Alessandro Oneto, and Boris Shapiro. Algebraic stories from one and from the other pockets. *Arnold Mathematical Journal*, 4(2):137–160, 2018. [arXiv:1801.01692](#), [doi:10.1007/s40598-018-0088-z](#).
- [Hay85] Walter K. Hayman. Waring’s Problem für analytische Funktionen. *Bayerische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, Sitzungsberichte*, 1984:1–13, 1985. Zbl 0567.30002.
- [HL19] Walter K. Hayman and Eleanor F. Lingham. *Research Problems in Function Theory: Fiftieth Anniversary Edition*. Problem Books in Mathematics. Springer, Cham, 2019. [arXiv:1809.07200](#), [doi:10.1007/978-3-030-25165-9](#).
- [Kra69] R. Krawczyk. Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken. *Computing*, 4(3):187–201, 1969. [doi:10.1007/BF02234767](#).

- [LORS19] Samuel Lundqvist, Alessandro Oneto, Bruce Reznick, and Boris Shapiro. On generic and maximal k -ranks of binary forms. *Journal of Pure and Applied Algebra*, 223(5):2062–2079, 2019. [arXiv:1711.05014](#), [doi:10.1016/j.jpaa.2018.08.015](#).
- [Moo77] R. E. Moore. A test for existence of solutions to nonlinear systems. *SIAM Journal on Numerical Analysis*, 14(4):611–615, 1977. [doi:10.1137/0714040](#).
- [NS79] D. J. Newman and Morton Slater. Waring’s problem for the ring of polynomials. *Journal of Number Theory*, 11(4):477–487, 1979. [doi:10.1016/0022-314X\(79\)90027-1](#).
- [Rez13] Bruce Reznick. On the length of binary forms. In *Quadratic and Higher Degree Forms*, volume 31 of *Developments in Mathematics*, pages 207–232. Springer, New York, 2013. [doi:10.1007/978-1-4614-7488-3_8](#).
- [Rez19] Bruce Reznick. Linearly dependent powers of binary quadratic forms. *Pacific Journal of Mathematics*, 303(2):729–755, 2019. [arXiv:1903.11569](#), [doi:10.2140/pjm.2019.303.729](#).
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-014-0004.
- [Vol15] José Felipe Voloch. Waring’s problem for polynomials in positive characteristic. Notes of a talk at the conference Silvermania, <https://math.colorado.edu/arithmetric2015/slides/Voloch.pdf>, 2015.