

A Cayley Graph with Chromatic Number Three and No Invariant Random Three-Coloring

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Abstract

We construct a connected seven-regular Cayley graph of $(C_5 * (C_2 \times C_2)) \times C_2$ with chromatic number three but no translation-invariant random proper three-coloring. A central involution partitions each of two finite subgroup graphs into pairs. In any proper three-coloring, a blown-up five-cycle forces at least four equal-colored pairs out of five, whereas a four-pair crown graph permits at most one. Translation invariance would therefore require the same probability to be both at least $4/5$ and at most $1/4$. We give a proper three-coloring by an explicit reduced-word rule, and an invariant random four-coloring from a finite quotient. Thus the invariant chromatic number of this graph is exactly four. This gives a negative answer to the optimal invariant random coloring question of Lyons and Schramm, also recorded as Question 31 in Abért's problem list.

1 The question and the graph

For a finite symmetric generating set S of a countable group Γ , we use the right Cayley graph: g is adjacent to gs for $s \in S$. Left translations are graph automorphisms. The space $\text{Col}(X, k)$ of proper colorings $V(X) \rightarrow \{0, \dots, k-1\}$ is a closed subspace of the product space $\{0, \dots, k-1\}^\Gamma$. An invariant random proper k -coloring is a Borel probability measure on this space invariant under

$$(h \cdot f)(g) = f(h^{-1}g).$$

Write $\chi_{\text{inv}}(X, \Gamma)$ for the least such k . No mixing, ergodicity or factor-of-IID condition is imposed.

Lyons and Schramm asked whether every Cayley graph admits an invariant random coloring with its ordinary chromatic number of colors; see Lyons–Nazarov [4, Section 5], Abért [1, Question 31], and Kechris–Marks [3, Problem 6.66]. For amenable groups the answer is positive by the invariant-measure property of continuous actions on compact spaces. The degree bound supplied by measurable Brooks theory [2] is a different statement from optimality at the ordinary chromatic number.

Let

$$V = \langle b, c \mid b^2 = c^2 = 1, bc = cb \rangle \cong C_2 \times C_2, \quad H = \langle a \mid a^5 = 1 \rangle * V, \quad \Gamma = H \times \langle t \mid t^2 = 1 \rangle.$$

In particular t commutes with a, b, c . Set

$$S = \{a, a^{-1}, at, a^{-1}t, bt, ct, bct\}, \quad X = \text{Cay}(\Gamma, S). \tag{1}$$

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Theorem 1. *The graph X is connected, simple, infinite and seven-regular, with*

$$\chi(X) = 3, \quad \chi_{\text{inv}}(X, \Gamma) = 4.$$

In particular $\text{Col}(X, 3)$ is nonempty but carries no Γ -invariant Borel probability measure.

The generators in (1) are distinct and nonidentity by free-product normal form. They are inverse-closed: a, a^{-1} are paired, as are $at, a^{-1}t$, and the last three are involutions. They generate Γ , since $t = a^{-1}(at)$, $b = (bt)t$ and $c = (ct)t$. Finally the reduced words $(ab)^n$ are distinct. These facts prove the graph-theoretic assertions except the two chromatic equalities.

2 Two finite restrictions

Let $A = \langle a, t \rangle \cong C_5 \times C_2$ and $B = \langle b, c, t \rangle \cong C_2^3$. The induced graph on a left coset of A has five two-vertex parts

$$\{ga^i, ga^i t\}, \quad i \in \mathbb{Z}/5\mathbb{Z}.$$

Consecutive parts have all four possible edges and there are no other edges. Thus this graph is the blowup $C_5[\overline{K_2}]$. The induced graph on a left coset of B has vertices gvt^ϵ , where $v \in V$ and $\epsilon \in \{0, 1\}$. There is an edge from gv to gwt precisely when $v \neq w$. It is the crown graph $K_{4,4}$ minus a perfect matching. Both descriptions follow by comparing right quotients with S ; generators from the other free factor create no additional edges inside either coset.

Call a pair $\{g, gt\}$ *equal* if its vertices share a color.

Lemma 2. *Every proper three-coloring of $C_5[\overline{K_2}]$ has at least four equal pairs among its five two-vertex parts.*

Proof. If a pair has distinct colors x, y , each neighboring pair must be monochromatic in the third color z . The two remaining pairs are adjacent and must avoid z . If either used both x and y , the other would have no available color. Thus they are monochromatic in different colors x and y . There is therefore at most one unequal pair. If every pair is equal the conclusion already holds. $\square$

Lemma 3. *Every proper three-coloring of the four-pair crown graph has at most one equal pair, where pairs are the nonedges of the removed perfect matching.*

Proof. Suppose two pairs are monochromatic. Their colors must be distinct because opposite-side vertices from different pairs are adjacent. Every vertex in the other two pairs is adjacent to vertices of both these colors, so both remaining pairs must be monochromatic in the third color. They have an edge between them, a contradiction. $\square$

Proposition 4. *There is no Γ -invariant random proper three-coloring of X .*

Proof. Assume such a law μ exists. Put $I_g(f) = \mathbf{1}_{\{f(g)=f(gt)\}}$ and $q = \mathbb{E}_\mu I_1$. Translation invariance gives $\mathbb{E}_\mu I_g = q$ for every g . Applying Lemmas 2 and 3 to the two subgroups through the identity gives, pointwise,

$$\sum_{i=0}^4 I_{a^i} \geq 4, \quad \sum_{v \in V} I_v \leq 1.$$

Taking expectations yields $5q \geq 4$ and $4q \leq 1$, which are incompatible. In integer form the contradiction is

$$0 \leq 4(5q - 4) + 5(1 - 4q) = -11.$$

$\square$

3 A proper three-coloring

We now verify that the coloring space used above is nonempty. The construction is rooted and is not claimed to be equivariant. Each $h \in H = C_5 * V$ has a unique reduced word of alternating nonidentity syllables from its two factors. Define a color pair $F(h) = (F_0(h), F_1(h))$ and color ht^ϵ by $F_\epsilon(h)$.

For $r = (0, 1, 0, 1, 2)$ start with

$$F(a^i) = (r_i, r_i), \quad 0 \leq i \leq 4. \quad (2)$$

This includes $F(1) = (0, 0)$. Extend along reduced words as follows. If w is empty or ends in a C_5 -syllable, its pair is equal, say $F(w) = (x, x)$. For each $v \in V \setminus \{1\}$ put

$$F(wv) = (y, z), \quad \{y, z\} = \{0, 1, 2\} \setminus \{x\}, \quad y < z. \quad (3)$$

If w ends in a V -syllable, its pair is unequal, say $F(w) = (x, y)$, and let z be the third color. Put

$$F(wa) = F(wa^4) = (z, z), \quad F(wa^2) = (x, x), \quad F(wa^3) = (y, y). \quad (4)$$

The stated equal/unequal input types hold by induction on syllable length. Unique normal form makes these rules unambiguous.

Proposition 5. *The function $f(ht^\epsilon) = F_\epsilon(h)$ is a proper three-coloring of X .*

Proof. It is important to check edges that cancel a syllable as well as edges that append one. Each left coset of C_5 in H has a unique representative w which is empty or ends in a V -syllable. For the empty representative the five pairs are given by (2), a proper cyclic sequence. Otherwise they are

$$(x, y), (z, z), (x, x), (y, y), (z, z),$$

where x, y, z are distinct. Consecutive pairs have disjoint color sets, including the last and first. This checks every edge corresponding to $a, a^{-1}, at, a^{-1}t$.

Likewise each left coset of V has a unique representative w which is empty or ends in a C_5 -syllable. Its identity pair is (x, x) and its other three pairs are all (y, z) , with x, y, z distinct. A crown edge either meets the identity pair or joins opposite coordinates of two other pairs. Its endpoints always have different colors. This checks all bt, ct, bct edges. Centrality of t makes the same checks valid on either sheet. $\square$

The vertices $1, a, a^2, a^3, a^4$ form an odd cycle, so two colors are impossible. Proposition 5 proves $\chi(X) = 3$.

4 Four colors suffice invariantly

Let $\pi : \Gamma \rightarrow Q = C_5 \times V$ be the homomorphism which kills t and makes the two free factors commute. The simple quotient graph is $C_5 \square K_4$, and no generator maps to the identity. Choose any bijection $j : V \rightarrow \{0, 1, 2, 3\}$ and set

$$D(i, v) = r_i + j(v) \pmod{4}, \quad r = (0, 1, 0, 1, 2).$$

The values of r differ at consecutive positions of the five-cycle, including the cyclic boundary, and j differs on distinct elements of V . Hence D properly colors the quotient. Its pullback $D \circ \pi$ properly colors X .

Choose $q \in Q$ uniformly and use the coloring $g \mapsto D(q^{-1}\pi(g))$. Translating by $h \in \Gamma$ replaces q with $\pi(h)q$, which remains uniform. This produces a Γ -invariant random proper four-coloring. Together with Proposition 4 this completes the proof of Theorem 1.

Remark 6. The obstruction concerns arbitrary invariant laws, not merely factors of IID. It also rules out an automorphism-invariant random proper three-coloring. Uniformly translating one coloring over the finite color-permutation group alone would not make its distribution invariant under translations of Γ .

The accompanying standard-library Python checker exhausts the 3^{10} and 3^8 assignments on the two finite pieces, checks all boundary color types, and tests the reduced-word formula on 818,104 directed Cayley edges, including edges leaving the enumerated word set. It also verifies a twenty-element orbit of proper four-colorings. These checks are supplementary; the proof above applies to every reduced word. The manuscript was developed with AI assistance and self-audited. It is an unrefereed preprint, not a proof-assistant certification, and makes no claim of independent review or absolute priority.

References

- [1] M. Abért, *Some questions*, 2 November 2010, Question 31, p. 5. <https://www.renyi.hu/~abert/questions.pdf>.
- [2] C. T. Conley, A. S. Marks and R. D. Tucker-Drob, Brooks’s theorem for measurable colorings, *Forum of Mathematics, Sigma* **4** (2016), e16. [doi:10.1017/fms.2016.14](https://doi.org/10.1017/fms.2016.14).
- [3] A. S. Kechris and A. S. Marks, *Descriptive Graph Combinatorics*, preliminary version, 9 October 2020, Problem 6.66, p. 81. <https://math.berkeley.edu/~marks/papers/combinatorics20book.pdf>.
- [4] R. Lyons and F. Nazarov, Perfect matchings as IID factors on non-amenable groups, *European Journal of Combinatorics* **32** (2011), 1115–1125, Section 5. Author preprint: <https://rdlyons.pages.iu.edu/pdf/factor.pdf>.