A Free-Uniform-Spanning-Forest Proof of Finite-Index Multiplicativity

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Abstract

An explicit question of Abért asks for a free-spanning-forest proof that the first L^2-Betti number is multiplicative under passage to a finite-index subgroup. We give such a proof using the free uniform spanning forest (FUSF). For $H \leq \Gamma$ of index k, lift a spanning tree of the finite Schreier quotient to a forest W in a labeled Cayley multigraph G, and contract its finite components to obtain a Cayley multigraph X of H. The closed finite-cycle space of G is the graph of a bounded H-equivariant operator over that of X. Equivariant dimensions therefore give the FUSF intensity identity

$$I_H(G) = (k-1) + I_H(X).$$

Combining this with the FUSF formula $I_K = 1 + \beta_1^{(2)}(K)$ yields $\beta_1^{(2)}(H) = k\beta_1^{(2)}(\Gamma)$ without importing finite-index multiplicativity of von Neumann dimension.

1 Introduction

Abért's Question 25 asks whether free spanning forests can prove basic properties of the first L^2-Betti number, "for instance, that it is multiplicative for taking a finite index subgroup" [1]. The preceding question and Lyons's expected-degree formula identify the intended process as the free uniform spanning forest (FUSF, also called FSF), not the free minimal spanning forest.

Theorem 1.1. *Let Γ be a finitely generated group and let $H \leq \Gamma$ have finite index $k = [\Gamma : H]$. Then*

$$\beta_1^{(2)}(H) = k\,\beta_1^{(2)}(\Gamma).$$

This identity has a proof through FUSF edge intensities and contraction of a lifted finite Schreier tree.

The standard multiplicativity theorem is of course classical. The point is the forest derivation requested in the source. Lyons supplies the two primary inputs: the determinantal cycle-space description of FUSF [3] and its L^2-Betti intensity formula [4]. A later use of FUSF imports finite-index multiplicativity separately from standard L^2 theory rather than deriving it from forests [2]; we make no priority claim for the argument below.

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2 Labeled Cayley multigraphs and FUSF trace

Choose a finite generating tuple $A = (a_1, \ldots, a_n)$ for a group K, with no identity entries. Let $Y = Y(K, A)$ be the labeled Cayley multigraph with vertex set K and one edge cell

$$e(g, i) : g \longrightarrow ga_i$$

for every $g \in K$ and label i. Orientations are used only for chains. Cells remain distinct for repeated labels, inverse labels, and involutions, so the left K-action is free on vertices and edge cells. Contraction may later create parallel edges and loops; a loop is itself a finite cycle and has FUSF inclusion probability zero.

Fix one orientation per edge cell and let $\partial : \ell^2(EY) \to \ell^2(VY)$ be the incidence operator. Put

$$\mathscr{Z}_c(Y) = \{z : z \text{ is finitely supported and } \partial z = 0\}, \qquad \mathscr{Z}(Y) = \overline{\mathscr{Z}_c(Y)}.$$

The determinantal description of FUSF is

$$\mathrm{FUSF}(Y) = \mathbf{P}_{\mathscr{Z}(Y)^\perp}; \tag{1}$$

in particular,

$$\mathbb{P}[e \in F] = \langle P_{\mathscr{Z}(Y)^\perp} \mathbf{1}_e, \mathbf{1}_e \rangle$$

[3, Section 3].

Suppose a group L acts freely and cocompactly on the edge cells. If D_E contains one representative of each L-edge orbit, define

$$I_L(Y) = \sum_{e \in D_E} \mathbb{P}[e \in F].$$

By (1), this is the equivariant trace

$$I_L(Y) = \dim_L \mathscr{Z}(Y)^\perp. \tag{2}$$

We shall use only normalization on finite sums of $\ell^2(L)$, additivity on orthogonal sums, and invariance under L-equivariant unitaries.

For an infinite finitely generated group K, Lyons's formula is

$$I_K(Y(K, A)) = 1 + \beta_1^{(2)}(K). \tag{3}$$

For completeness, let $\delta = \partial^*$ and $\mathscr{S}(Y) = \overline{\operatorname{im} \delta}$ be the star space. Finite cycles are orthogonal to stars, and the standard Cayley-graph model of reduced first L^2-cohomology gives

$$\beta_1^{(2)}(K) = \dim_K\bigl(\mathscr{Z}(Y)^\perp \cap \mathscr{S}(Y)^\perp\bigr).$$

If K is infinite, the only square-summable constant vertex function is zero. Polar decomposition of δ therefore gives $\dim_K \mathscr{S}(Y) = 1$, proving (3); compare [4, Corollary 4.12].

3 The lifted Schreier tree

Return to $H \leq \Gamma$ of index k, choose a finite generating tuple A for Γ, and set $G = Y(\Gamma, A)$. Let

$$Q = H \backslash G$$

be the finite labeled Schreier quotient, with k vertices. Choose an ordinary spanning tree $\overline{W}$ in its underlying finite multigraph and let $W \subseteq G$ be the full inverse image.

Because $G \to Q$ is a graph covering and a tree is simply connected, every component of W maps isomorphically to $\overline{W}$. Thus every component is a finite tree with k vertices and $k-1$ edges; H acts freely and transitively on the components; and W/H has $k-1$ edge cells.

Contract every component of W and write

$$X = G/W.$$

The H-action on $V(X)$ is free and transitive, so X is a labeled Cayley multigraph of H for a finite Schreier generating multiset. Naturally,

$$E(X) = E(G) \setminus E(W). \tag{4}$$

Since G/H has kn edge cells,

$$|E(X)/H| = kn - (k-1). \tag{5}$$

4 Bounded lifting of the closed cycle space

Lemma 4.1. *Under* (4)*, there is a bounded H-equivariant operator*

$$L : \mathscr{Z}(X) \longrightarrow \ell^2(EW)$$

such that

$$\mathscr{Z}(G) = \{(Lq, q) : q \in \mathscr{Z}(X)\} \subseteq \ell^2(EW) \oplus \ell^2(E(X)).$$

Proof. First let $q \in \mathscr{Z}_c(X)$ and regard it as a chain on $E(G) \setminus E(W)$. For a component C of W, put $d_C = (\partial q)|_{V(C)}$. The boundary of the contracted chain at the vertex represented by C is $\sum_{v \in V(C)} d_C(v)$, hence this sum is zero.

For a finite tree C, its incidence map is an isomorphism

$$\partial_C : \mathbb{C}^{E(C)} \longrightarrow \left\{ d \in \mathbb{C}^{V(C)} : \sum_{v \in V(C)} d(v) = 0 \right\}.$$

There is consequently a unique tree chain w_C with $\partial_C w_C = -d_C$. Set $Lq = \sum_C w_C$. It has finite support and $\partial(Lq + q) = 0$.

All components are H-translates of one fixed finite tree. The inverses ∂_C^{-1} therefore have one common norm bound M. Since the incidence operator on G is bounded,

$$\|Lq\|_2^2 \leq M^2 \sum_C \|d_C\|_2^2 = M^2 \|\partial q\|_2^2 \leq M^2 \|\partial\|^2 \|q\|_2^2.$$

Thus L extends boundedly to $\mathscr{Z}(X)$, and uniqueness makes it H-equivariant.

Conversely, decompose a finite cycle $z = w + q \in \mathscr{Z}_c(G)$ into its W and non-W parts. Summing $\partial z = 0$ over each component cancels the W-edge contributions and says precisely that q contracts to a cycle in X. Uniqueness on each tree forces $w = Lq$. Hence

$$\mathscr{Z}_c(G) = \{(Lq, q) : q \in \mathscr{Z}_c(X)\}.$$

The graph map $q \mapsto (Lq, q)$ is bounded and bounded below by $\|(Lq, q)\|_2 \geq \|q\|_2$, so its image of the closed space is closed. Taking closures proves the lemma. $\square$

5 The forest-contraction identity

If a bounded equivariant map between closed invariant Hilbert subspaces is bijective with bounded inverse, its polar decomposition is an equivariant unitary. Lemma 4.1 therefore gives

$$\dim_H \mathscr{Z}(G) = \dim_H \mathscr{Z}(X). \tag{6}$$

Moreover,

$$\ell^2(EG) = \ell^2(EW) \oplus \ell^2(E(X)), \qquad \dim_H \ell^2(EW) = k - 1.$$

All modules have finite H-dimension. Additivity and (6) yield

$$\begin{aligned}\dim_H \mathscr{Z}(G)^\perp &= (k-1) + \dim_H \ell^2(E(X)) - \dim_H \mathscr{Z}(X)\\ &= (k-1) + \dim_H \mathscr{Z}(X)^\perp.\end{aligned}$$

Using (2), we obtain the central forest identity

$$\boxed{I_H(G) = (k-1) + I_H(X).} \tag{7}$$

This is an equality of determinantal traces. It asserts neither a coupling nor a samplewise contraction rule between the two forests.

6 Proof of multiplicativity

Assume first that Γ is infinite. Every free Γ-edge orbit in G splits into k free H-edge orbits. Since the FUSF projection is Γ-invariant,

$$I_H(G) = kI_\Gamma(G). \tag{8}$$

Apply (3) to the Cayley multigraph G of Γ and the contracted Cayley multigraph X of H. Equations (7) and (8) give

$$k\big(1 + \beta_1^{(2)}(\Gamma)\big) = (k-1) + \big(1 + \beta_1^{(2)}(H)\big),$$

which is Theorem 1.1.

If Γ is finite, then H is finite and both positive-degree L^2-Betti numbers vanish. The forest identity is also consistent: if $|\Gamma| = N$, FUSF is the uniform spanning tree and

$$I_\Gamma(G) = 1 - \frac{1}{N}, \qquad I_H(X) = 1 - \frac{1}{|H|} = 1 - \frac{k}{N},$$

so $k(1 - 1/N) = (k-1) + (1 - k/N)$.

7 Resistance form and scope

For unit conductances, the FUSF inclusion probability $p_Y(e)$ equals the free effective resistance between the endpoints. Subtracting (7) from the exact edge-orbit counts gives

$$\sum_{e\in E(X)/H}(1-p_X(e)) = \sum_{e\in E(G)/H}(1-p_G(e)) = k\sum_{e\in E(G)/\Gamma}(1-p_G(e)).$$

Thus the summed nonforest-edge/free-resistance intensity also multiplies by the index.

The proof compares dimensions only between two H-modules. Its Γ-to-H step is the explicit orbit count (8); no finite-index theorem for von Neumann dimension or L^2-Betti numbers is used. Finite components of W are essential to boundedness in Lemma 4.1. The result concerns FUSF, not FMSF, and makes no cost claim.

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