

Property (τ) Need Not Survive Intersections of Normal Subgroups

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Abstract

We construct a family of finite-index normal subgroups of a fixed finitely generated free group which has property (τ) , whereas its prefix-intersection chain does not. The construction uses Kassabov's bounded-degree alternating-group expanders. Two quotient maps have identical expanding generators and send one additional generator to, respectively, the identity and a 3-cycle. Their joint image is the full product $\text{Alt}(N) \times \text{Alt}(N)$. In its action on N^2 ordered pairs, the diagonal provides a mean-zero test function with lazy Rayleigh quotient $3/[2r(N-1)]$, where r is the fixed number of generators. Pullback transfers this bound to the regular quotient and then to the prefix chain. Thus failure already occurs for a sequence of pairwise intersections, giving a negative answer to Question 9 in Abért's list.

1 The question and the result

For a finitely generated group Γ and a family (H_j) of finite-index normal subgroups, property (τ) means that the Cayley graphs of Γ/H_j , with generators induced by one fixed finite generating set of Γ , have a uniform positive spectral gap. Question 9 of Abért [1, p. 2] asks whether this implies property (τ) for the chain

$$K_m = \bigcap_{j=1}^m H_j.$$

The answer is negative.

Theorem 1. *There exist a positive integer r , the free group $\Gamma = F_r$, and finite-index normal subgroups $H_j \triangleleft \Gamma$ such that (H_j) has property (τ) but (K_m) does not. More precisely, for the lazy averaging operator defined below there is $\delta > 0$ with*

$$\text{gap}(\Gamma/H_j) \geq \delta \quad (j \geq 1), \quad \text{gap}(\Gamma/K_{2j}) \leq \frac{3}{2r(j+3)} \quad (j \geq 1).$$

The expansion input is Kassabov's theorem [3]; it is not a new expander construction. Abért and Elek [2, Theorem 8] show that intersecting a normal (τ) -family with one *fixed* finite-index subgroup preserves the relevant property. Their Corollary 9 gives a counterexample for nonnormal families, and the paragraph following it leaves the normal case open there. Our two intersected kernels both vary, with unbounded indices, so the fixed subgroup theorem does not preclude Theorem 1.

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2 Averaging and pullback

For an r -tuple $a = (a_1, \dots, a_r)$ acting on a finite set X , let $U(a_i)f(x) = f(a_i^{-1}x)$ on $\ell^2(X)$, with counting measure, and set

$$P_a = \frac{1}{2}I + \frac{1}{4r} \sum_{i=1}^r (U(a_i) + U(a_i)^*). \quad (1)$$

Then

$$\langle (I - P_a)f, f \rangle = \frac{1}{4r} \sum_{i=1}^r \|U(a_i)f - f\|^2. \quad (2)$$

For a transitive action, the gap is the infimum of the quotient of (2) by $\|f\|^2$ over nonzero mean-zero functions. We retain repeated and identity labels in a generating tuple. Their multiplicities are bounded by the fixed tuple length, so this convention does not change whether a family has property (τ) . Laziness avoids any issue at the negative end of the spectrum.

Lemma 2. *Let a finite group Q act transitively on X , and let a generate Q . Every Rayleigh quotient of a mean-zero function on X is a Rayleigh quotient for the regular action of Q with the same tuple. The same conclusion holds for pullback along a surjective homomorphism of finite groups with compatible tuples.*

Proof. Fix $x_0 \in X$. The orbit map $p : Q \rightarrow X$, $p(g) = gx_0$, is equivariant for left multiplication and has fibers of a common size b . Hence $f \circ p$ has mean zero if f does, and its squared norm and each squared displacement are b times the corresponding quantities for f . The Rayleigh quotient is unchanged. For a group homomorphism, use its equal-size coset fibers in the same argument. $\square$

3 The paired quotient construction

Kassabov's theorem [3, Theorem 2 and Section 4] supplies constants k and $\kappa > 0$ and generating tuples

$$S_N = (s_{1,N}, \dots, s_{k,N}) \text{ of } G_N = \text{Alt}(N), \quad N \geq 5,$$

such that every unitary representation π of G_N without invariant vectors satisfies

$$\max_{1 \leq i \leq k} \|\pi(s_{i,N})v - v\| \geq \kappa \|v\|. \quad (3)$$

Sets of smaller size are padded by identities. The constants are independent of N .

Take $r = k + 1$ and $\Gamma = F_r = \langle x_1, \dots, x_k, t \rangle$. For $c_N = (123) \in G_N$, define epimorphisms $\varphi_N^\pm : \Gamma \rightarrow G_N$ by

$$\varphi_N^-(x_i) = \varphi_N^+(x_i) = s_{i,N}, \quad \varphi_N^-(t) = 1, \quad \varphi_N^+(t) = c_N.$$

They are onto since the common generators already generate G_N . Write $M_N^\pm = \ker \varphi_N^\pm$. Applying (3) to the mean-zero regular representation and then using (2) gives

$$\text{gap}(\Gamma/M_N^\pm) \geq \frac{\kappa^2}{4r}. \quad (4)$$

The additional generator only contributes nonnegative energy.

The image Q_N of $(\varphi_N^-, \varphi_N^+)$ is generated by

$$(s_{1,N}, s_{1,N}), \dots, (s_{k,N}, s_{k,N}), (1, c_N).$$

It contains the full diagonal copy of G_N . Conjugating $(1, c_N)$ by that diagonal gives $(1, gc_N g^{-1})$ for every $g \in G_N$. All 3-cycles are conjugate in $\text{Alt}(N)$ for $N \geq 5$ and they generate $\text{Alt}(N)$. Thus Q_N contains $\{1\} \times G_N$, and consequently

$$Q_N = G_N \times G_N \cong \Gamma / (M_N^- \cap M_N^+). \quad (5)$$

No independence assumption on the two quotient maps has been used.

Consider the coordinatewise action of this product on $X_N = \{1, \dots, N\}^2$. It is transitive. Let

$$D_N = \{(a, a) : 1 \leq a \leq N\}, \quad f_N = \mathbf{1}_{D_N} - \frac{1}{N}.$$

Directly,

$$\sum_{X_N} f_N = 0, \quad \|f_N\|^2 = N - 1. \quad (6)$$

All the generators $(s_{i,N}, s_{i,N})$ fix f_N . The permutation $(1, c_N)$ takes three points of D_N outside D_N and brings three points into it. It follows that

$$\|U(1, c_N)f_N - f_N\|^2 = 6.$$

Equations (2) and (6), followed by Lemma 2, prove

$$\text{gap}(\Gamma / (M_N^- \cap M_N^+)) \leq \frac{3}{2r(N-1)}. \quad (7)$$

In particular, the paired intersection quotients are not an expander family even though both families of factors are.

Proof of Theorem 1. For $j \geq 1$, put $N = j + 4$ and enumerate $H_{2j-1} = M_N^-$, $H_{2j} = M_N^+$. Their uniform lower gap bound is (4). Since $K_{2j} \subseteq M_N^- \cap M_N^+$, there is a surjection from Γ / K_{2j} to the quotient in (7). Lemma 2 gives $\text{gap}(\Gamma / K_{2j}) \leq 3/[2r(j+3)]$, which tends to zero. This proves the claimed failure of property (τ) for the chain. $\square$

Corollary 3. *The ambient group can instead be chosen finitely generated and residually finite, with $\bigcap_j H_j = \{1\}$.*

Proof. Replace Γ by $\Gamma / \bigcap_j H_j$ and each H_j by its image. All the finite quotients and their marked averaging operators are unchanged. The resulting normal finite-index family separates points, proving residual finiteness. $\square$

This last ambient group is not asserted to be free or finitely presented. Trivial total intersection is not required by the original Question 9.

4 Checks and scope

The argument is a proof for arbitrary N , not a computational extrapolation. Two exact implementations accompany this note. A Python program generates $A_5 \times A_5$ and $A_6 \times A_6$ exhaustively, checks their orders 3,600 and 129,600, and verifies the lifted diagonal cut. It also checks the centered-matrix displacement identity for all 5,910 permutations in degrees 3 through 7. A separately written C++ program enumerates the even permutations of degree five, checks all product orbit fibers, and verifies the norm and 3-cycle displacement formulas in degrees 5 through 128. These are finite

regression checks; uniform expansion of the factors comes from Kassabov’s theorem, not from those programs.

Mimura [4, Remark 6.5] discusses a related spectral-gap question for diagonal products subject to additional LEF-approximation conditions. We do not assert that our paired sequence satisfies those conditions or resolves that narrower question. Nor do we assert that every generating set of $G_N \times G_N$ fails to expand: the correlated tuple in (5) is essential.

A targeted primary-literature search did not locate this particular normal-kernel construction; absolute priority is not claimed. The established alternating-group expansion theorem is explicitly credited. This unrefereed note was developed with AI assistance in the derivation, source checking, exact computations and drafting. No independent specialist review or proof-assistant verification is claimed.

References

- [1] M. Abért, *Some questions*, 2 November 2010, Question 9. <https://users.renyi.hu/~abert/questions.pdf>.
- [2] M. Abért and G. Elek, Dynamical properties of profinite actions, *Ergodic Theory Dynam. Systems* **32** (2012), 1805–1835. [doi:10.1017/S0143385711000654](https://doi.org/10.1017/S0143385711000654). [arXiv:1005.3188](https://arxiv.org/abs/1005.3188).
- [3] M. Kassabov, Symmetric groups and expander graphs, *Invent. Math.* **170** (2007), 327–354. [doi:10.1007/s00222-007-0065-y](https://doi.org/10.1007/s00222-007-0065-y). [arXiv:math/0505624](https://arxiv.org/abs/math/0505624).
- [4] M. Mimura, Amenability versus non-exactness of dense subgroups of a compact group, *J. Lond. Math. Soc.* **100** (2019), 592–622. [doi:10.1112/jlms.12229](https://doi.org/10.1112/jlms.12229). [arXiv:1805.01398](https://arxiv.org/abs/1805.01398).