Adjoining a Haar-Generic Matrix to a Parabolic-Free Subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$

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August 29, 2026

Abstract

Let Γ be a countable subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ containing no noncentral element of trace 2 or -2. We prove that, outside a Haar-null set of $g \in \mathrm{SL}_2(\mathbb{Q}_p)$, the generated group $\langle \Gamma, g\rangle$ has the same property. Thus adjoining one random element preserves the absence of parabolics almost surely, answering Question 4 in Miklós Abért's 2010 list. The key elementary lemma classifies the problematic one-variable generalized words: if all constants lie in a parabolic-free subgroup of SL_2 and the trace of the word is identically 2 or -2, then the word map is identically I or $-I$. The proof evaluates the word on $I + tN$ over the nilpotent cone. Its two highest coefficients force a central endpoint product and cancellation of the outer exponents, reducing the word by conjugation and induction. This manuscript is unrefereed and makes no absolute priority claim.

1 The probabilistic statement

For $h \in \mathrm{SL}_2(\mathbb{Q}_p)$, the characteristic polynomial is

$$T^2 - \mathrm{tr}(h)T + 1.$$

We call h *parabolic* when $\mathrm{tr}(h) = 2$ or -2 but $h \notin \{I, -I\}$. Equivalently, h or $-h$ is a nontrivial unipotent.

Since $\mathrm{SL}_2(\mathbb{Q}_p)$ is noncompact, it has no normalized Haar probability measure. Accordingly, "a random element" means a random variable whose law is absolutely continuous with respect to Haar measure. This includes, for example, normalized Haar measure on any compact open subset.

Theorem 1.1. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Q}_p)$ be countable and contain no parabolic elements. Then there is a Haar-null set $E \subset \mathrm{SL}_2(\mathbb{Q}_p)$ such that*

$$\langle \Gamma, g\rangle \quad \textit{contains no parabolic element for every } g \notin E.$$

Consequently, the conclusion holds almost surely for every probability law absolutely continuous with respect to Haar measure.

This is the natural measure-class formulation of Question 4 in [1]. Its proof is reduced to the algebraic lemma below.

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2 A generalized-word lemma

Let K be an algebraically closed field of characteristic zero. A subgroup $H \leq \mathrm{SL}_2(K)$ will be called parabolic-free if

$$\{h \in H : \mathrm{tr}(h) \in \{2, -2\}\} \subseteq \{I, -I\}. \tag{1}$$

A one-variable generalized word with constants in H has the form

$$w(x) = a_0 x^{e_1} a_1 x^{e_2} \cdots a_{k-1} x^{e_k} a_k, \qquad a_i \in H, \quad e_i \in \mathbb{Z} \setminus \{0\}. \tag{2}$$

By absorbing central signs and merging adjacent powers, we may assume that every interior coefficient $a_1, \ldots, a_{k-1}$ is noncentral. We call this form reduced.

Lemma 2.1. *Let $H \leq \mathrm{SL}_2(K)$ satisfy* (1)*, and let* (2) *be reduced. If*

$$\mathrm{tr}(w(x)) \equiv 2 \quad \text{or} \quad \mathrm{tr}(w(x)) \equiv -2 \qquad (x \in \mathrm{SL}_2(K)), \tag{3}$$

then $w(x)$ is identically I or identically $-I$.

Proof. We induct on the number k of occurrences of powers of x. The case $k = 0$ follows immediately from (1).

Let $N \in \mathfrak{sl}_2(K)$ be nilpotent. Since $N^2 = 0$,

$$(I + tN)^e = I + etN \qquad (e \in \mathbb{Z}) \tag{4}$$

for every $t \in K$. For every nonzero nilpotent N and every $A \in M_2(K)$, the rank-one identity

$$NAN = \mathrm{tr}(AN)N \tag{5}$$

holds.

Substitute $x = I + tN$ in (2). The coefficient of t^k in its trace is, by cyclicity and (5),

$$[t^k]\,\mathrm{tr}\,w(I + tN) = \left(\prod_{i=1}^{k} e_i\right) \mathrm{tr}(a_k a_0 N) \prod_{i=1}^{k-1} \mathrm{tr}(a_i N). \tag{6}$$

It vanishes on the nilpotent cone because the trace is constant.

The nilpotent cone in $\mathfrak{sl}_2(K)$ is irreducible. Moreover, if $A \in \mathrm{SL}_2(K)$ and $\mathrm{tr}(AN)$ vanishes for every nilpotent N, then A is central: nilpotents span $\mathfrak{sl}_2(K)$ and the trace pairing is nondegenerate there. Since the interior coefficients in a reduced word are noncentral, (6) forces

$$a_k a_0 = \varepsilon I, \qquad \varepsilon \in \{1, -1\}. \tag{7}$$

If $k = 1$, then

$$w(x) = \varepsilon a_0 x^{e_1} a_0^{-1}.$$

Its trace is $\varepsilon\,\mathrm{tr}(x^{e_1})$, which is nonconstant on the diagonal torus because $e_1 \neq 0$. Hence (3) cannot occur for $k = 1$.

Assume $k \geq 2$. After (7), the coefficient of t^{k-1} is

$$[t^{k-1}]\,\mathrm{tr}\,w(I + tN) = \varepsilon(e_1 + e_k) \left(\prod_{i=2}^{k-1} e_i\right) \prod_{i=1}^{k-1} \mathrm{tr}(a_i N). \tag{8}$$

Indeed, among the k terms obtained by omitting one nilpotent factor, all interior omissions retain the identity gap between the last and first factors and hence contain $\operatorname{tr} N = 0$. Only omission of the first or last factor survives, giving (8).

No factor $\operatorname{tr}(a_i N)$ vanishes identically on the irreducible nilpotent cone, and every middle exponent is nonzero. Constancy of the trace therefore implies

$$e_k = -e_1. \tag{9}$$

Put

$$v(x) = a_1 x^{e_2} a_2 \cdots a_{k-2} x^{e_{k-1}} a_{k-1},$$

with the evident interpretation $v = a_1$ when $k = 2$. Equations (7) and (9) give the identity of maps

$$w(x) = \varepsilon a_0 x^{e_1} v(x) x^{-e_1} a_0^{-1}. \tag{10}$$

Thus $\operatorname{tr} v(x) = \varepsilon \operatorname{tr} w(x)$ is again identically 2 or -2. After reducing v if necessary, it has strictly fewer x-factors and all its constants remain in H. The induction hypothesis makes $v(x) \equiv \delta I$ for some $\delta \in \{1, -1\}$. Equation (10) then gives $w(x) \equiv \varepsilon\delta I$, completing the induction. $\square$

Remark 2.2. The parabolic-free hypothesis in lemma 2.1 is essential. If $u \neq I$ is unipotent, then $w(x) = xux^{-1}$ has trace identically 2 but is not a constant map. More generally, the coefficient argument reduces a constant-trace generalized word to nested conjugations of a constant of the same trace. Condition (1) is exactly what makes the last constant central.

The dichotomy between constant central functions and functions taking every value is studied more generally for word maps with constants in [4]. The elementary coefficient induction above classifies the constant trace-± 2 case needed here under (1).

3 Proof of the probabilistic theorem

Every element of $\langle \Gamma, g \rangle$ is $w(g)$ for a generalized word (2) with coefficients in Γ. After the reductions used above, there are only countably many such words because Γ is countable.

Fix one word involving x, and consider the regular function

$$F_w : \mathrm{SL}_2 \longrightarrow \mathbb{A}^1, \qquad F_w(x) = \operatorname{tr}(w(x)). \tag{11}$$

Negative powers cause no difficulty: inversion is regular on SL_2.

If F_w is identically 2 or -2 on $\mathrm{SL}_2(\mathbb{Q}_p)$, Zariski density of the $\mathbb{Q}_p$-points extends the identity to $\mathrm{SL}_2(\overline{\mathbb{Q}}_p)$. Applying lemma 2.1 to the same coefficient subgroup in the algebraic closure shows that w is identically I or $-I$, so this word never produces a parabolic.

Otherwise,

$$Z_w = \{x \in \mathrm{SL}_2(\mathbb{Q}_p) : F_w(x) = 2 \text{ or } F_w(x) = -2\} \tag{12}$$

is the set of $\mathbb{Q}_p$-points of a proper Zariski-closed subset of SL_2. A proper algebraic subset of the three-dimensional p-adic analytic manifold $\mathrm{SL}_2(\mathbb{Q}_p)$ has Haar measure zero. This follows, for example, by covering with analytic coordinate charts and applying the standard fact that the zero set of a nonzero polynomial over a nonarchimedean local field has measure zero.

Constant words belong to Γ and are nonparabolic by hypothesis. Taking the union of (12) over the remaining countable family of words gives a Haar-null set E. For $g \notin E$, no element of $\langle \Gamma, g \rangle$ is parabolic. Absolute continuity of the chosen probability law now proves theorem 1.1.

4 Context

Abért's list records the analogous rooted-tree assertion and phrases the p-adic statement as "Show that" [1]. The rooted-tree randomness setting is developed by Abért and Virág [2]. Aoun's results on random subgroups of linear groups [3] are related but do not by themselves exclude parabolics: an abstractly free linear group can contain unipotent elements.

Targeted searches through August 29, 2026 did not locate this exact proof or a later status update for Question 4. Since the question was presented as an exercise-like request and the argument is elementary, the present note claims no priority beyond the proof written here.

AI-assisted tools were used for source discovery, algebraic exploration, exact-arithmetic falsification checks, adversarial proof review, and manuscript preparation. Every mathematical claim used in the theorem is accompanied by an explicit argument or a cited theorem. The named author is responsible for the final manuscript.

References

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