

No Uniform Hanna Neumann Bound for Convex-Core Volume

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Abstract

There is no universal constant K such that $V(A \cap B) \leq KV(A)V(B)$ for torsion-free convex-cocompact Kleinian groups, where V denotes three-dimensional convex-core volume. We give an explicit deformation of rank-two classical Schottky groups with positive core volumes tending to zero and constant systole $\operatorname{arccosh}(31)$. Two independent parity kernels are distinct proper index-two subgroups of rank three; their intersection has rank five, and their volume ratio is exactly the reciprocal of the ambient core volume. Thus fixed ranks, fixed indices and a fixed positive systole do not restore a uniform bound. We also record the exact finite-cover double-coset identity and prove a neighborhood-volume bound by counting lifts in covering spaces. The counterexample answers the universal-constant interpretation of Swarup's question in Bestvina's list; it does not settle arbitrary ambient-dependent bounds for infinite-index subgroups.

1 The precise statement

For a nonelementary discrete subgroup $H < \operatorname{Isom}^+(\mathbb{H}^3)$, let

$$\mathcal{C}(H) = \operatorname{Hull}_{\mathbb{H}^3}(\Lambda H), \quad V(H) = \operatorname{vol}_3(\mathcal{C}(H)/H).$$

Volume is measured in ambient dimension three. In particular, a planar Fuchsian core has volume zero even when its intrinsic area is positive.

Question 3.2 in Bestvina's list [1], attributed there to Swarup, asks for convex-core-volume analogues of the Hanna Neumann surface-area inequality. The wording allows a worse constant without specifying its permitted dependence. The following theorem rules out a universal constant, including one depending only on dimension. The record AMR-010-0302 in *UnsolvedMath* [4] motivates this distinction.

Theorem 1. *For every $K > 0$, there exist a torsion-free, rank-two classical Schottky group $G < \operatorname{PSL}(2, \mathbb{C})$ and distinct proper normal subgroups $A, B < G$ such that*

$$[G : A] = [G : B] = 2, \quad [G : A \cap B] = 4, \quad V(A \cap B) > KV(A)V(B).$$

All three volumes are positive. The free ranks of $A, B, A \cap B$ are 3, 3, 5, respectively. Moreover, these examples can all satisfy $\operatorname{sys}(\mathbb{H}^3/G) = \operatorname{arccosh}(31)$.

The underlying ingredients are classical Schottky construction, Bonahon's regularity theorem for core volume [2], and finite-sheeted covering theory. We give explicit generators and track the subgroup data. No claim of a new volume-continuity theorem is involved.

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2 An explicit fixed-systole deformation

Put $r = 1/4$. For $0 \leq t < \pi$, set

$$a_t(z) = -1 - \frac{e^{it}}{16(z+3)}, \quad b(z) = 3 - \frac{1}{16(z-1)}, \quad G_t = \langle a_t, b \rangle. \quad (1)$$

Lemma 2. *The groups G_t are classical Schottky groups of rank two. If $v(t) = V(G_t)$, then*

$$v(0) = 0, \quad 0 < v(t) < \infty \quad (0 < t < \pi), \quad \lim_{t \downarrow 0} v(t) = 0.$$

Proof. Consider the four closed round disks of radius r , centered at $-3, -1, 1, 3$. They are pairwise disjoint. Directly from (1),

$$|a_t(z) + 1| = \frac{r^2}{|z+3|}, \quad |b(z) - 3| = \frac{r^2}{|z-1|}.$$

Consequently a_t takes the exterior of the first disk to the interior of the second, and b pairs the third and fourth in the same way. The classical Schottky construction gives discreteness, freeness, torsion freedom and convex cocompactness; see [3, Section 7]. The complement P of the four open hemiballs above the disks is a fundamental polyhedron in $\mathbb{H}^3$, independent of t .

At $t = 0$, both generators are represented by real matrices of positive determinant. Their group preserves a hyperbolic plane, so its convex core has zero three-dimensional volume. For $t > 0$, a matrix representing a_t is

$$M_t = \begin{pmatrix} -1 & -3 - e^{it}/16 \\ 1 & 3 \end{pmatrix}, \quad \frac{\text{tr}(M_t)^2}{\det M_t} = 64e^{-it}.$$

This normalized squared trace is nonreal for $0 < t < \pi$. A Möbius transformation preserving a generalized circle is conjugate to a projective real matrix, for which the same invariant is real. Thus G_t preserves no generalized circle.

If the infinite limit set of G_t were contained in a generalized circle, that circle would be unique and invariant. Hence the limit set contains four points spanning a nondegenerate ideal tetrahedron. Its convex hull has nonempty interior, and a small interior ball injects locally into the quotient. Therefore $v(t) > 0$; convex cocompactness gives finiteness.

The fixed disjoint faces of P , with analytically varying side pairings, give a smooth cusp-free convex-cocompact deformation at $t = 0$. Bonahon's Main Theorem [2] supplies a finite right derivative for convex-core volume along such a deformation, also when the initial core is planar. In particular $v(t) \rightarrow v(0) = 0$, indeed $v(t) = O(t)$. Only this continuity consequence is needed. $\square$

Lemma 3. *For every $0 \leq t < \pi$,*

$$\text{sys}(\mathbb{H}^3/G_t) = \delta, \quad \delta = \text{arcosh}(31).$$

Proof. Two disjoint hemispherical face planes of radius r , whose Euclidean centers are distance d apart, have hyperbolic separation

$$\text{arcosh}\left(\frac{d^2 - 2r^2}{2r^2}\right).$$

Here $d \geq 2$ and $r = 1/4$, so distinct faces of P are at least δ apart.

Cut a closed geodesic at the side-paired faces and lift the resulting segments to P . At least one segment is necessary, since the interior of P is simply connected. Each segment

joins distinct face planes: a geodesic cannot enter and leave the same half-space through its boundary plane. Thus its length is at least δ . This is also the usual cyclically reduced face-itinerary argument for a Schottky axis. An axis cannot lie in a face plane, because its ideal endpoints are limit points, whereas the Schottky boundary circles lie in the domain of discontinuity.

Finally, the real loxodromic element b has normalized squared trace 64, and therefore translation length

$$2 \operatorname{arccosh}(4) = \operatorname{arccosh}(31) = \delta.$$

Its axis projects to a closed geodesic, giving equality. $\square$

3 Two proper subgroups

Lemma 4. *If H is a finite-index subgroup of a convex-cocompact Kleinian group G , then*

$$\Lambda H = \Lambda G, \quad V(H) = [G : H] V(G).$$

Proof. A finite-index subgroup has the same limit set: a finite coset decomposition makes each ambient orbit a finite union of translates of an H -orbit, and the finite-index normal core gives invariance under the ambient group. Equivalently, a positive power of every element of G lies in the normal core, and loxodromic fixed points are dense in the limit set. Thus the convex hulls agree. The quotient $\mathcal{C}(G)/H \rightarrow \mathcal{C}(G)/G$ is a finite covering of degree $[G : H]$; integration over fundamental domains proves the volume identity. $\square$

Proof of Theorem 1. Define an epimorphism

$$\phi_t : G_t \longrightarrow (\mathbb{Z}/2)^2, \quad \phi_t(a_t) = (1, 0), \quad \phi_t(b) = (0, 1).$$

Let A_t and B_t be the kernels of its first and second coordinates, and put $I_t = \ker \phi_t = A_t \cap B_t$. They are proper normal subgroups of indices 2, 2, 4. Since $a_t \in B_t \setminus A_t$ and $b \in A_t \setminus B_t$, the first two are distinct and incomparable. Schreier's formula gives ranks 3, 3, 5. Finite index also implies convex cocompactness and quasiconvexity.

By Lemma 4,

$$V(A_t) = 2v(t), \quad V(B_t) = 2v(t), \quad V(I_t) = 4v(t),$$

so

$$\frac{V(I_t)}{V(A_t)V(B_t)} = \frac{1}{v(t)} \longrightarrow \infty \quad (t \downarrow 0). \quad (2)$$

Choose $t > 0$ with $v(t) < 1/K$, using Lemma 2. Lemma 3 gives the asserted constant systole. $\square$

Remark 5. The parameter choice is existential, through Bonahon's theorem. No numerical volume estimate or explicit computational threshold for t is needed or claimed. Simply taking $t = 0$ would give zero volumes and would not be a counterexample.

4 What finite covers require of an ambient constant

Proposition 6. *Suppose $0 < V(G) < \infty$, and $A, B < G$ have finite indices a, b . Then*

$$\frac{V(A \cap B)}{V(A)V(B)} = \frac{[G : A \cap B]}{ab V(G)} \leq \frac{1}{V(G)}.$$

More precisely,

$$\sum_{[g] \in A \backslash G/B} V(A \cap gBg^{-1}) = \frac{V(A)V(B)}{V(G)}. \quad (3)$$

Proof. The map $G/(A \cap B) \rightarrow G/A \times G/B$ is injective, so $[G : A \cap B] \leq ab$. Apply Lemma 4 for the first assertion.

The diagonal action on the finite set $G/A \times G/B$ has orbits indexed by $A \backslash G/B$, with stabilizers conjugate to $A \cap gBg^{-1}$. The sum of their orbit sizes is ab . Multiplying that index identity by $V(G)$ proves (3). $\square$

In the parity examples equality holds even for the single intersection. Thus an ambient-dependent constant, if valid for all subgroup pairs, must be at least $1/V(G_t)$. This exact covering calculation says nothing about an upper bound for arbitrary infinite-index subgroups of a fixed G .

5 A positive bound for neighborhood volumes

The collapse in (2) suggests changing the functional. The next elementary estimate makes that change explicit. For a nontrivial subgroup H , put

$$W_r(H) = \text{vol}_n(N_r(\mathcal{C}(H))/H) \quad (r > 0),$$

using the axis when H is cyclic, and set $W_r(1) = 0$. Write $\beta_n(s)$ for the volume of a radius- s ball in $\mathbb{H}^n$.

Proposition 7. *Let $G < \text{Isom}(\mathbb{H}^n)$ be discrete and torsion-free, with $\text{inj}(\mathbb{H}^n/G) \geq s > 0$, and let A, B be convex-cocompact subgroups. For $r > 0$,*

$$\sum_{[g] \in A \backslash G/B} W_r(A \cap gBg^{-1}) \leq \frac{\min\{W_{r+s}(A)W_r(B), W_r(A)W_{r+s}(B)\}}{\beta_n(s)}.$$

Proof. Put $M = \mathbb{H}^n/G$. Let $m_H(x)$ be the number of lifts of $x \in M$ in $N_r(\mathcal{C}(H))/H$, viewed inside the cover $\mathbb{H}^n/H$. Open s -balls about distinct such lifts are embedded and pairwise disjoint. Indeed, a path of length less than $2s$ between distinct lifts would project to a nontrivial loop based at x , contradicting the injectivity bound. All these balls lie in $N_{r+s}(\mathcal{C}(H))/H$. Hence

$$m_H(x)\beta_n(s) \leq W_{r+s}(H), \quad \int_M m_H(x) dx = W_r(H). \quad (4)$$

The fiber product over M of the two r -neighborhoods decomposes into double-coset pieces

$$(N_r(\mathcal{C}(A)) \cap gN_r(\mathcal{C}(B)))/(A \cap gBg^{-1}).$$

Since $\mathcal{C}(A \cap gBg^{-1})$ is contained in both relevant convex hulls, its r -neighborhood embeds into this piece. Consequently the left side of the desired inequality is at most $\int_M m_A(x)m_B(x) dx$. Bound either factor using (4) and integrate the other. All terms are nonnegative, so Tonelli's theorem justifies the countable decomposition. $\square$

This estimate uses enlarged neighborhoods and an ambient injectivity bound. It is not an inequality for the unthickened volume V .

6 Scope and verification

Theorem 1 is a complete counterexample to a universal multiplicative volume bound in dimension three. It does not answer the broader fixed-ambient question: whether each fixed G admits some finite $C(G)$ valid for all its quasiconvex subgroups. Accordingly, the broad source record is not counted here as wholly resolved. Nor does embedding these groups in $\mathbb{H}^n$, $n > 3$, give positive n -dimensional volumes.

The accompanying exact-arithmetic check verifies the matrix identity, face-separation constants, the parity-cover indices and ranks, and the double-coset index formula for all 36 subgroup pairs of S_3 . These checks support the written argument; they do not numerically compute core volumes or constitute formal verification. The geometric proof depends on the cited classical results.

The manuscript was prepared with AI assistance in derivation, literature search, algebra checks and exposition. It is an unrefereed, self-audited preprint. A bounded search found no earlier explicit application of this family to Bestvina’s Question 3.2, but no absolute priority claim is made.

References

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