

Explicit Rotation and Arithmetic Obstructions for Null Chains on a Lorentz Ellipsoid

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Abstract

For the ellipsoid $x^2/a + y^2/b + z^2/c = 1$ in Minkowski space, we compute the rotation lift of the equator return map formed by alternating null geodesics. A global conformal cylinder gives the lift $B/(2A)$, and a contour calculation gives $A + B = \pi$, where A is an explicit real integral. Closure with prescribed period and winding is therefore equivalent to a single integral equation, with exactly one positive vertical parameter c for fixed a, b and positive target rotation. For positive algebraic a, b, c with $a \neq b$, an application of the analytic subgroup theorem to a twice-punctured elliptic curve proves that the rotation is transcendental. The rotational case $a = b$ has an elementary rationality criterion. Fixed rational-rotation loci on $a > b > 0$ are consequently not semialgebraic over the real algebraic numbers and are Zariski dense over $\overline{\mathbb{Q}}$. We distinguish equator iterations from literal tropic-to-tropic legs, for which closure also requires an even number of legs.

1 The closure problem and the main results

Give $\mathbb{R}^3$ the metric $dx^2 + dy^2 - dz^2$ and consider

$$\mathcal{E}_{a,b,c} : \quad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1, \quad a, b, c > 0. \quad (1)$$

The parameters are the denominators in this equation, not the semiaxes themselves. The induced metric is Lorentzian in an equatorial belt and degenerates on its two boundary curves, the *tropics*. The two null line fields determine unparametrized geodesics in the belt. Following a right null geodesic from the equator to the northern tropic and returning along a left null geodesic defines the equator map T of Genin, Khesin and Tabachnikov [1, Section 5]. We orient the motion so that its angular lift increases.

The invariant angular measure and porism are established in [1, Lemma-Definition 4.2 and Theorem 5.1]. Their Problem 5.2 asks for the relation on a, b, c specifying period and winding. Tabachnikov [2, Section 7, Problem 4] poses the corresponding question for chains of tropic-to-tropic null geodesics. We evaluate the rotation and give a precise obstruction to finite algebraic conditions in the axes.

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Put $f(t)^2 = a \sin^2 t + b \cos^2 t$ and define

$$A(a, b, c) = 2 \int_0^{\pi/2} \frac{f(t)}{\sqrt{c + f(t)^2}} dt, \quad (2)$$

$$B(a, b, c) = 2c \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{(a + c \sin^2 t)(b + c \sin^2 t)}} dt. \quad (3)$$

Theorem 1. *For all $a, b, c > 0$, one has*

$$A + B = \pi, \quad \rho_T = \frac{B}{2A} = \frac{1}{2} \left(\frac{\pi}{A} - 1 \right). \quad (4)$$

Here $\rho_T > 0$ is the unreduced rotation lift recording actual turns. The map T closes after k iterations with r turns if and only if

$$A(a, b, c) = \frac{\pi k}{k + 2r}, \quad k, r \in \mathbb{Z}_{>0}. \quad (5)$$

A literal chain of n tropic-to-tropic legs closes with r turns if and only if n is even and $A = \pi n / (n + 2r)$. If $\rho_T = p/q$ in lowest terms, its least physical number of legs is $\text{lcm}(q, 2)$, whereas the least period of T is q .

For fixed $a, b > 0$ and $\rho > 0$, exactly one $c > 0$ satisfies $\rho_T = \rho$. It depends real analytically on a, b, ρ and obeys

$$4 \min(a, b) \rho (1 + \rho) \leq c \leq 4 \max(a, b) \rho (1 + \rho), \quad (6)$$

with strict inequalities when $a \neq b$.

Theorem 2. *If a, b, c are positive real algebraic numbers and $a \neq b$, then both $A(a, b, c)/\pi$ and ρ_T are transcendental. In particular no null chain closes. For a positive algebraic triple with $a = b = \alpha$, a closed chain exists if and only if $\sqrt{1 + c/\alpha}$ is rational. More precisely,*

$$\rho_T = \frac{\sqrt{1 + c/\alpha} - 1}{2}, \quad c = 4\alpha\rho_T(1 + \rho_T). \quad (7)$$

Corollary 3. *Fix $\rho \in \mathbb{Q}_{>0}$ and let*

$$S_\rho = \{(a, b, c) \in \mathbb{R}^3 : a > b > 0, c > 0, \rho_T(a, b, c) = \rho\}.$$

For fixed algebraic $a > b > 0$, the unique closing parameter c is transcendental. The locus S_ρ is not semialgebraic over $\mathbb{R} \cap \overline{\mathbb{Q}}$, and no nonzero polynomial in $\overline{\mathbb{Q}}[a, b, c]$ vanishes on it. In particular, no finite Boolean combination of polynomial equalities and inequalities with algebraic coefficients characterizes this rational-rotation locus in the axes alone.

The arithmetic input is Wüstholz's analytic subgroup theorem, used in the period-vector formulation of Huber and Wüstholz [3, Theorem 6.2 of the author preprint]. Generalized Jacobians are classical, due to Rosenlicht and presented by Serre [4, Chapter V]; we use [3, Section 4.5]. An arithmetic approach to this null-chain problem was already announced in Wüstholz's Ha Long lectures [5, final five slides], and [6] develops related elliptic and abelian period theory. Thus we do not claim the first arithmetic approach or a new transcendence theorem. The proofs below isolate the explicit rotation formula and its consequences for the real parameter locus.

2 A global conformal cylinder

Take $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$ and $-\pi/2 < \chi < \pi/2$, and put $v = c \cos^2 \chi$. Consider

$$x = \sqrt{\frac{a(a+v)}{a+c}} \cos \phi, \quad y = \sqrt{\frac{b(b+v)}{b+c}} \sin \phi, \quad (8)$$

$$z = c \sin \chi \sqrt{\frac{c + f(\phi)^2}{(a+c)(b+c)}}. \quad (9)$$

Lemma 4. *Equations (8)–(9) parametrize the open Lorentzian belt once and extend to a homeomorphism of the closed annulus onto the closed belt. In the open belt the induced metric is*

$$g = (v + f^2) \left(\frac{f^2}{c + f^2} d\phi^2 - \frac{c^2 \cos^4 \chi}{(a+v)(b+v)} d\chi^2 \right). \quad (10)$$

Proof. Substitution gives (1), and the Lorentz normal $(x/a, y/b, -z/c)$ has squared length

$$N = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{vf(\phi)^2}{abc}. \quad (11)$$

Thus $v = 0$ gives the tropics and $v = c$ the equator.

For a point of the open belt, $N > 0$, and the function

$$H(v) = \frac{x^2(a+c)}{a(a+v)} + \frac{y^2(b+c)}{b(b+v)}$$

strictly decreases on $[0, c]$. Indeed x, y cannot both vanish there. Using (1) gives $H(c) = 1 - z^2/c \leq 1$ and $H(0) = 1 + cN > 1$. There is a unique solution of $H(v) = 1$ in $(0, c]$. The values of $\cos \phi, \sin \phi$ are then unique, and the sign of z selects the sign of χ . The same argument with $N = 0$ gives the unique boundary value $v = 0$. This proves bijectivity on the closed belt; compactness gives the claimed homeomorphism. Direct differentiation yields zero mixed metric coefficient and (10). Its two diagonal factors are nonzero in the open belt, including $\chi = 0$, so the open-belt parametrization is regular. No smooth nondegeneracy is asserted at the tropics. $\square$

Define increasing coordinates

$$X(\phi) = 2 \int_0^\phi \frac{f(t)}{\sqrt{c + f(t)^2}} dt, \quad (12)$$

$$Y(\chi) = 2c \int_0^\chi \frac{\cos^2 t}{\sqrt{(a + c \cos^2 t)(b + c \cos^2 t)}} dt. \quad (13)$$

Their derivatives are positive in the open belt. The circumference of the X coordinate is $4A$ and the range of Y is $(-B, B)$. Equation (10) becomes

$$g = \frac{v + f^2}{4} (dX^2 - dY^2). \quad (14)$$

This is a global conformal cylinder, with a continuous identification of its boundary circles with the tropics.

On a Lorentz surface, each null line field consists of unparametrized geodesics. In fact, if V is a nonzero null tangent field, metric compatibility gives $g(\nabla_V V, V) = 0$. In dimension two, $V^\perp$

is precisely the line spanned by V , so $\nabla_V V$ is proportional to V . The null geodesics in (14) are therefore the lines $X - Y = \text{constant}$ and $X + Y = \text{constant}$.

Starting at $Y = 0$, the two segments to $Y = B$ and back, both with increasing X , advance X by $2B$. The boundary homeomorphism in Lemma 4 identifies their unique common endpoint with the tropic joining used to define T . Thus

$$T : X \mapsto X + 2B \quad \text{on } \mathbb{R}/(4A\mathbb{Z}). \quad (15)$$

The angular coordinate has degree one around the z axis. Consequently the actual positive rotation lift is $B/(2A)$.

A full tropic-to-tropic leg also advances X by $2B$, but changes the sign of Y . A physical closed chain therefore needs an even number n of legs, as well as $2nB = 4rA$. If $B/(2A) = p/q$ in lowest terms, the least such n is $\text{lcm}(q, 2)$ and its winding is np/q . Repeated orbits are allowed in the closure statements. There is no zero-winding chain in the positive orientation; reversing orientation reverses winding without changing the parameter conditions.

3 A complementary period identity

Lemma 5. *The integrals in (2)–(3) satisfy $A + B = \pi$.*

Proof. First assume $a > b$ and set

$$P(z) = (z + a)(z + b)(c - z), \quad Q(z) = z(z + a)(z + b)(z - c) = -zP(z).$$

The substitutions $u = -a \sin^2 t - b \cos^2 t$ and $v = c \sin^2 t$ give

$$A = \int_{-a}^{-b} \sqrt{\frac{u}{P(u)}} du, \quad B = \int_0^c \sqrt{\frac{v}{P(v)}} dv, \quad (16)$$

where the displayed square roots are positive on their integration intervals. In the upper half-plane use the product

$$R(z) = \sqrt{z + a} \sqrt{z + b} \sqrt{z} \sqrt{z - c}, \quad F(z) = \frac{z}{R(z)},$$

with each square root having argument in $(0, \pi/2)$. Then $R^2 = Q$, $R(z)/z^2 \rightarrow 1$ on large upper semicircles, and $F(z) = 1/z + O(1/z^2)$.

On $(-a, -b)$ the upper boundary value is $R_+(u) = -i\sqrt{-Q(u)}$, since three factors have argument $\pi/2$. On $(0, c)$ it is $R_+(v) = i\sqrt{-Q(v)}$. Thus

$$F_+(u) = -i\sqrt{u/P(u)}, \quad -a < u < -b, \quad (17)$$

$$F_+(v) = -i\sqrt{v/P(v)}, \quad 0 < v < c. \quad (18)$$

On the remaining real intervals F_+ is real.

Integrate from $-L$ to L along the real boundary, with small upper indentations around the branch points, and then along the upper semicircle from L to $-L$. The indentation integrals tend to zero: the singularity is at most inverse square root at $-a, -b, c$, and is of square-root order at 0. There is no interior pole. The large arc integral tends to $i\pi$, while (17)–(18) contribute $-i(A + B)$ to the imaginary part of the real-boundary integral. Cauchy's theorem proves $A + B = \pi$. Taking imaginary parts on finite contours avoids separately integrating divergent real tails. Symmetry exchanges a, b , and continuity handles $a = b$. $\square$

Completion of Theorem 1. Combine (15) with Lemma 5 to obtain (4) and the two closure conventions. For fixed a, b , differentiation under the integral gives

$$\frac{\partial A}{\partial c} = - \int_0^{\pi/2} \frac{f(t)}{(c + f(t)^2)^{3/2}} dt < 0. \quad (19)$$

The integral is real analytic on the positive parameter region. Dominated convergence gives $A \rightarrow \pi$ as $c \downarrow 0$ and $A \rightarrow 0$ as $c \rightarrow \infty$. Hence ρ_T strictly increases from 0 to ∞ . Existence, uniqueness and real analytic dependence of c follow from the intermediate value and implicit function theorems.

Let $m = \min(a, b)$ and $M = \max(a, b)$. Monotonicity of $x/(c + x)$ gives

$$\pi \sqrt{\frac{m}{m + c}} \leq A \leq \pi \sqrt{\frac{M}{M + c}}.$$

At $A = \pi/(1 + 2\rho)$ these inequalities rearrange to (6), strictly when $a \neq b$. Scaling all three parameters by the same positive number leaves A, B, ρ_T unchanged. If $a = b = \alpha$, direct integration yields $A = \pi\sqrt{\alpha/(\alpha + c)}$ and (7). $\square$

4 Periods on a twice punctured elliptic curve

Assume now that $a > b > 0$ and $c > 0$ are algebraic. Let E be the smooth projective curve with affine equation

$$y^2 = Q(z) = z(z + a)(z + b)(z - c). \quad (20)$$

Its four distinct branch points give genus one. Its two points at infinity P_+, P_- , where $y/z^2 \rightarrow +1, -1$, are algebraic and distinct. On $U = E \setminus \{P_+, P_-\}$ put

$$\omega = \frac{z dz}{y}, \quad \eta = \frac{dz}{y}. \quad (21)$$

The form η is holomorphic and nonzero. The only poles of ω are simple poles at P_+, P_- with residues $-1, +1$: the local coordinate $w = 1/z$ gives $\omega = -dw/w + O(dw)$ at P_+ , and the opposite residue at P_- . In particular ω is of the third kind, not a holomorphic Abel–Jacobi differential.

Let γ_A go from $-a$ to $-b$ on the sheet $y = -i\sqrt{-Q(z)}$ and return on the opposite sheet. Then

$$\int_{\gamma_A} \omega = -2iA, \quad \int_{\gamma_A} \eta = 2i \int_{-a}^{-b} \frac{dz}{\sqrt{-Q(z)}} \neq 0. \quad (22)$$

Let γ_R go from $-b$ to 0 on $y = -\sqrt{Q(z)}$ and return on $y = +\sqrt{Q(z)}$. Since $Q > 0$ on this interval,

$$\int_{\gamma_R} \omega = 2 \int_{-b}^0 \frac{-z}{\sqrt{Q(z)}} dz > 0. \quad (23)$$

These are closed cycles on U : the branch points are regular points of the smooth curve, not removed punctures. The integrals are finite. Finally, a positively oriented small loop τ around P_- has

$$\int_{\tau} \omega = 2\pi i, \quad \int_{\tau} \eta = 0. \quad (24)$$

Choose an algebraic base point of U . Its generalized Jacobian $G = J(U)$ is a semi-abelian variety over $\overline{\mathbb{Q}}$ of dimension two, with an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G \xrightarrow{p} E \longrightarrow 0. \quad (25)$$

Here E is identified with its Jacobian using an algebraic origin. The Abel map $j : U \rightarrow G$ identifies invariant differentials on G with differentials on E having at most simple poles at the two punctures [3, Section 4.5]. Write Ω for the invariant algebraic form with $j^*\Omega = \omega$.

After choosing the sign of the torus coordinate, the restriction of Ω to $\mathbb{G}_m$ is dt/t . Indeed, j_* identifies the integral first homology groups, and the puncture loop is a primitive generator of the kernel over $H_1(E, \mathbb{Z})$. Its period is $2\pi i$ by (24). This fixes the coefficient of dt/t , rather than leaving an unspecified algebraic or complex multiple.

For each closed cycle γ on U , let $v_\gamma \in \text{Lie}(G)_\mathbb{C}$ be its period vector. It satisfies

$$\exp_G(v_\gamma) = 1, \quad \Omega(v_\gamma) = \int_\gamma \omega. \quad (26)$$

The projection to $\text{Lie}(E)_\mathbb{C}$ is detected by $\int_\gamma \eta$. In particular v_{γ_A} has nonzero projection, whereas v_τ has zero projection.

Lemma 6. *There is no elliptic subgroup $H \subset G$ on which Ω vanishes.*

Proof. Such an H would project isogenously onto E . Otherwise it would lie in the affine torus, impossible for a positive-dimensional complete group. The addition homomorphism $\mathbb{G}_m \times H \rightarrow G$ is then an isogeny, and pulls Ω back to dt/t . The image of its period lattice has finite index in the period lattice of G . Thus every period of Ω is a rational multiple of $2\pi i$. This contradicts the strictly positive real period (23). $\square$

5 Transcendence and the closure locus

We use the following form of the analytic subgroup theorem [3, Theorem 6.2 of the author preprint]. If G_0 is a connected commutative algebraic group over $\overline{\mathbb{Q}}$, V is a $\overline{\mathbb{Q}}$ -linear subspace of $\text{Lie}(G_0)$, and $0 \neq u \in V_\mathbb{C}$ satisfies $\exp_{G_0}(u) \in G_0(\overline{\mathbb{Q}})$, then there is a connected positive-dimensional algebraic subgroup $H_0 \subset G_0$ such that

$$u \in \text{Lie}(H_0)_\mathbb{C}, \quad \text{Lie}(H_0) \subset V. \quad (27)$$

This includes a nonzero period with exponential equal to the identity. In that case division by a sufficiently large integer gives a nonzero algebraic torsion point, without changing the annihilating algebraic linear forms.

Proof of Theorem 2. First suppose $a > b$. If $A/\pi = p_0/q_0 \in \mathbb{Q}$, with $q_0 > 0$, the period vector

$$v = q_0 v_{\gamma_A} + p_0 v_\tau$$

lies in $\ker \Omega$ by (22)–(24). Its projection to $\text{Lie}(E)_\mathbb{C}$ is nonzero, so $v \neq 0$. Apply (27) with the one-dimensional algebraic space $V = \ker \Omega$. The resulting subgroup has dimension one and projects nontrivially to E . A connected one-dimensional subgroup of a semi-abelian variety is a torus or an elliptic curve, and a torus cannot map nontrivially to E . The subgroup is therefore elliptic and annihilated by Ω , contrary to Lemma 6. Thus A/π is irrational.

Suppose next that $\alpha = A/\pi$ is algebraic and irrational. In $G' = G \times \mathbb{G}_m$ the period vector

$$v' = (v_{\gamma_A}, 2\pi i)$$

is annihilated by the algebraic form $\Omega' = \Omega + \alpha dt/t$. Its projection to $\text{Lie}(E)_\mathbb{C}$ is nonzero. Apply (27) with $V = \ker \Omega'$, a two-dimensional space. The resulting subgroup H' surjects onto E and has dimension one or two.

If $\dim H' = 1$, it is an elliptic curve. Its projection to the last $\mathbb{G}_m$ is trivial, since a morphism from a complete variety to an affine torus is constant. Its image in G is an elliptic subgroup annihilated by Ω , again contradicting Lemma 6.

If $\dim H' = 2$, the identity component of its kernel over E is a one-dimensional subtorus of $\mathbb{G}_m^2$. Such a subtorus has, up to isogeny, a parametrization $t \mapsto (t^m, t^n)$ with integers m, n not both zero. Restriction of Ω' to it gives

$$(m + \alpha n) dt/t = 0.$$

For $n = 0$ this forces $m = 0$; otherwise it forces $\alpha = -m/n$. Both conclusions are impossible. Hence A/π is transcendental. Its nonconstant rational fractional-linear transform $\rho_T = (1/(A/\pi) - 1)/2$ is transcendental as well.

Exchange a, b to handle $b > a$. When $a = b$, use (7) instead of the elliptic curve argument, since (20) is then singular. The closure criterion in Theorem 1 finishes the classification. $\square$

Proof of Corollary 3. Fix algebraic $a > b > 0$ and rational $\rho > 0$. The unique positive c from Theorem 1 cannot be algebraic, by Theorem 2. The set S_ρ is nonempty but contains no triple of real algebraic numbers. If it were semialgebraic over $\mathbb{R} \cap \overline{\mathbb{Q}}$, the real-closed-field transfer principle would give such a triple, a contradiction.

Suppose a nonzero $F \in \overline{\mathbb{Q}}[a, b, c]$ vanished on S_ρ . Regard F as a polynomial in c . At least one coefficient is a nonzero polynomial in a, b . Choose positive algebraic $a > b$ at which that coefficient does not vanish; algebraic pairs are dense in the open chamber. The unique closing c is then a root of a nonzero polynomial over $\overline{\mathbb{Q}}$, contradicting its transcendence. This argument also allows complex algebraic coefficients of F . $\square$

The coefficient restriction in Corollary 3 is essential to its statement. The corollary does not exclude conditions using transcendental period data, infinitely many conditions, or arbitrary transcendental coefficients. In particular it does not exclude the analytic criterion (5). The nonzero residues in (21) explain why replacing the invariant circle coordinate by an ordinary holomorphic Abel–Jacobi coordinate would not justify an algebraic torsion or division-polynomial criterion.

6 Numerical examples and reproducibility

The bounds (6) and strict monotonicity permit bisection for c . Table 1 gives numerical illustrations; their decimals are not exact closure certificates. The distinction between return-map period and physical chain length is visible in the odd-period rows.

a	b	k	r	Approximate c	Least legs	Turns
4	1	2	1	6.856275463644	2	1
4	1	3	1	3.989247257453	6	2
5	2	5	2	7.478530409046	10	4
3	1	1	1	15.295729795235	2	2

Table 1: Approximate vertical parameters for the target rotation r/k . The physical chain counts include the even-leg condition.

The supplementary source contains a symbolic checker for ten rational identities in the coordinate and metric calculations, and residue checks for the third-kind differential. It also retains nine

55-digit quadrature regressions and four bisection examples. These finite computations check algebra and branch conventions; the proofs of closure, transcendence and nonsemialgebraicity above do not depend on numerical tests. Floating-point bounds are not interval certificates.

This is an unrefereed manuscript. AI assistance was used in deriving and checking the arguments, literature research, symbolic computation and manuscript preparation. The author is responsible for the statements. No independent review, proof-assistant verification or absolute priority certification is claimed.

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