

A Genus-Two Algebraic Family with Vanishing Real Toledo Numbers and Nonzero Generic Witt Class

Alper Ferudun*

4 October 2026

Abstract

We give four explicit matrices defining a genus-two surface-group representation over $\mathbb{R}[s, t, 1/(st(1+t^2))]$. Every real specialization has zero Witt class and zero Toledo number. Over $\mathbb{R}(s, t)$, however, the Witt class is the anisotropic Pfister form $\langle 1, -(1+t^2) \rangle \otimes \langle 1, -s \rangle$, of additive order two. A nonzero second residue and a trace pole detect bad reduction at the nonreal divisor $1+t^2=0$. The calculation shows concretely that the real Toledo function does not determine the generic Witt-valued invariant. General realization results of Dymara and Januszkiewicz already imply existence of such examples; this note provides a compact matrix presentation and its direct verification. It is not a solution of the full open-ended AIM programme on Witt-valued Toledo invariants, and no priority claim is made.

1 The explicit family

Write $W(F)$ for the Witt group of nondegenerate symmetric bilinear forms over a field F of characteristic different from two. A diagonal form and its Witt class are denoted by the same symbol $\langle a_1, \dots, a_r \rangle$; $\langle 0 \rangle$ means zero when it occurs in a cocycle formula. We use $[A, B] = ABA^{-1}B^{-1}$.

Let Σ_2 be the closed oriented surface with presentation

$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle.$$

Set

$$z = 1 + t^2, \quad R = \mathbb{R}[s, t, 1/(stz)], \quad K = \mathbb{R}(s, t).$$

Thus the real points of $\text{Spec } R$ are exactly the pairs (s, t) with $st \neq 0$. Consider

$$A_1 = \begin{pmatrix} 0 & -t/z \\ z/t & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1/z & t^2/z^2 \\ -z & 1 \end{pmatrix}, \quad (1)$$

$$A_2 = \begin{pmatrix} 1/z & -t^2/(sz) \\ s & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & t/s \\ -s/t & 0 \end{pmatrix}. \quad (2)$$

Theorem 1.1. *The assignments $a_i \mapsto A_i$, $b_i \mapsto B_i$ define $\rho : \pi_1(\Sigma_2) \rightarrow \text{SL}_2(R)$. Its generic Witt class is*

$$w(\rho_K) = q := \langle 1, -z, -s, sz \rangle = \langle 1, -z \rangle \otimes \langle 1, -s \rangle \quad \text{in } W(K). \quad (3)$$

The form q is anisotropic and its Witt class has order exactly two. At every real point of $\text{Spec } R$, the specialized representation has zero Witt class and hence zero ordinary Toledo number. Nevertheless, no $\text{GL}_2(K)$ -conjugate of ρ_K takes values in $\text{SL}_2(\mathcal{O})$, where $\mathcal{O} = \mathbb{R}(s)[t]_{(t^2+1)}$.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Here w is the Witt class with the explicit cocycle convention fixed in Section 2. Reversing its sign, or the surface orientation, does not change (3), since $q = -q$. No division by an integer in $W(K)$ is involved.

Burger's Question 4.5 in the AIM surface-group list [1] asks for a study of the generic Witt-valued Toledo invariant of an algebraic family. Our family concerns its information beyond real specializations. It does not give a classification of these invariants or of algebraic families.

2 Computing the surface class

For an infinite field F of characteristic different from two, we use the Witt cocycle of Dymara–Januszkiewicz [2, (1.4)]:

$$w(G, H) = \langle -G_{21}(GH)_{21}H_{21} \rangle \quad (G, H \in \mathrm{SL}_2(F)). \quad (4)$$

In particular, this value is zero if G , H or GH is diagonal. The associated central extension has multiplication

$$(G, u)(H, v) = (GH, u + v + w(G, H)).$$

Write $\tilde{G} = (G, 0)$. Since $w(G, G^{-1}) = 0$, one has $\widetilde{G^{-1}} = (\tilde{G})^{-1}$. The central component of $[\tilde{A}_1, \tilde{B}_1][\tilde{A}_2, \tilde{B}_2]$ evaluates the characteristic class on the oriented surface relator. This also specifies the orientation convention used here.

For $C \neq 0$, put

$$X_C = \begin{pmatrix} 0 & -t/(Cz) \\ Cz/t & 0 \end{pmatrix}, \quad Y_C = \begin{pmatrix} 1/z & t^2/(Cz^2) \\ -Cz & 1 \end{pmatrix}, \quad Z = \mathrm{diag}(z, z^{-1}). \quad (5)$$

Lemma 2.1. *One has $\det X_C = \det Y_C = 1$ and $[X_C, Y_C] = Z$. The central component of $[\tilde{X}_C, \tilde{Y}_C]$ relative to $\tilde{Z}$ is $\langle C \rangle - \langle Cz \rangle$.*

Proof. The determinant and commutator identities follow by multiplication, using $z = 1 + t^2$. For $k = Cz/t$, the scalar entries in (4) are

$$-(X_C)_{21}(X_C Y_C)_{21}(Y_C)_{21} = Ck^2, \quad -(Y_C)_{21}(Y_C X_C)_{21}(X_C)_{21} = Czk^2.$$

Thus $w(X_C, Y_C) = \langle C \rangle$ and $w(Y_C, X_C) = \langle Cz \rangle$. The equality $X_C Y_C = Z Y_C X_C$, together with $w(Z, Y_C X_C) = 0$, compares the two lifted products. Multiplying on the right by $(\tilde{Y}_C \tilde{X}_C)^{-1}$ gives the claimed component. $\square$

Take $D = -s/z$. The four matrices above are (X_1, Y_1, Y_D, X_D) . Their commutators are Z, Z^{-1} , respectively, so they satisfy the defining relation over R . Swapping the two elements of a commutator takes its inverse; moreover $\widetilde{Z^{-1}} = (\tilde{Z})^{-1}$ and $w(Z, Z^{-1}) = 0$. Lemma 2.1 therefore gives

$$\begin{aligned} w(\rho_K) &= \langle 1 \rangle - \langle z \rangle + \langle Dz \rangle - \langle D \rangle \\ &= \langle 1, -z, -s, s/z \rangle = \langle 1, -z, -s, sz \rangle. \end{aligned}$$

In the last equality, the fourth coordinate is multiplied by z .

For an alternative direct check, successively apply the cocycle to the eight-letter relator and its prefixes. Its four nonzero scalar coefficients, in their order of occurrence, are

$$\frac{z^2}{t^2}, \quad -\frac{z}{t^2}, \quad -\frac{s^3}{t^2 z^2}, \quad \frac{s^3}{t^2 z}. \quad (6)$$

These are $1, -z, -s, s/z$ multiplied by the respective squares of $z/t, 1/t, s/(tz), s/t$. These factors, and every denominator used in the calculation, are units of R . Consequently the calculation specializes at every real point under consideration; an unspecified generic open set is not being substituted for all such points.

3 Nonvanishing and real specializations

The polynomial $\pi = t^2 + 1$ is irreducible in $\mathbb{R}(s)[t]$. The residue field of the discrete valuation ring $\mathcal{O}$ is $\mathbb{C}(s)$, in which s is not a square: its valuation at $s = 0$ is odd.

Lemma 3.1. *The form q is anisotropic over K and its Witt class has order two. Its second Witt residue at π is $\langle -1, s \rangle \neq 0$ in $W(\mathbb{C}(s))$.*

Proof. Write the value of q on a vector as

$$(x_0^2 - sx_2^2) + \pi(-x_1^2 + sx_3^2).$$

For any nonzero pair (u, v) , the π -valuation of $u^2 - sv^2$ is $2 \min\{v_\pi(u), v_\pi(v)\}$. Indeed, after division by that even power, cancellation in the residue field would imply that s is a square. Thus the two nonzero summands in the displayed expression have opposite valuation parity and cannot cancel. If one coordinate pair is zero, the other summand is nonzero. This proves anisotropy, and in particular $q \neq 0$ in $W(K)$.

The second residue sends a unit coefficient to zero and a coefficient πu with u a unit to $\langle \bar{u} \rangle$. Applied to $\langle 1, -\pi, -s, s\pi \rangle$ it gives $\langle -1, s \rangle$. This binary form is anisotropic over $\mathbb{C}(s)$ by the same nonsquare test. The preceding argument proves nonvanishing directly, without needing a residue exact sequence.

Finally, the invertible matrix $N = \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}$ satisfies $N^\top N = zI_2$. Hence $\langle 1, 1 \rangle \simeq \langle z, z \rangle$, giving $2\langle 1, -z \rangle = 0$ and $2q = 0$. Since $q \neq 0$, its order is exactly two. $\square$

At a real point, $z = 1 + t^2 > 0$, and $\langle 1, -z \rangle$ is hyperbolic over $\mathbb{R}$. Thus the specialized class is zero in $W(\mathbb{R})$, not just a form with a numerically small signature. The ordinary real Toledo invariant, which is obtained from the real Witt class by signature and normalization, is therefore zero. This conclusion is independent of the usual sign and scale conventions.

To prove the last assertion of Theorem 1.1, observe that

$$\mathrm{tr}(B_1) = 1 + z^{-1} = \frac{\pi + 1}{\pi}, \quad v_\pi(\mathrm{tr}(B_1)) = -1.$$

Trace is invariant under $\mathrm{GL}_2(K)$ -conjugacy, while a matrix over $\mathcal{O}$ has integral trace. The pole excludes every claimed integral conjugate. The divisor has no real points, and π has already been inverted in R , so this is consistent with regularity of the family on $\mathrm{Spec} R$.

Corollary 3.2. *The real Toledo function does not determine the generic Witt class of an algebraic family of genus-two surface-group representations in SL_2 .*

Proof. Compare ρ with the constant trivial family on the same base. Their real Toledo functions are both identically zero, whereas their generic Witt classes are $q \neq 0$ and zero. $\square$

4 Relation to prior work and verification

The realization theorem of Dymara–Januszkiewicz [2, Theorem 11.6(b)] already realizes every element of $I^2(K)$ of anisotropic dimension at most four on a genus-two surface. Here $I(K)$ is the fundamental ideal of the Witt ring. Our form q lies in $I^2(K)$, so that theorem, followed by spreading to an affine open, already implies an existential version of this example. Their Lemma 11.2 and Markov-surface construction motivated the pair (5). The point of this note is the short rational presentation, the calculation on the entire stated real parameter domain, and the visible nonreal obstruction. We make no claim to a new range theorem or first-existence result.

The accompanying source package contains a SymPy script that checks the determinant, commutator, full-relator, square-factor, norm-matrix and trace identities exactly. Twelve symbolic checks pass in Python 3.12 and 3.13 with SymPy 1.14.0. These checks supplement the proof above; they are neither a proof-assistant formalization nor a test of novelty. This is an unrefereed note. AI assistance was used in literature retrieval, algebraic exploration, symbolic checking and preparation of the exposition; the author is responsible for the manuscript and its claims.

References

- [1] American Institute of Mathematics, *Representations of Surface Groups*, problem list, Question 4.5 and Comment 4.6 (Marc Burger), p. 3. <https://aimath.org/WWN/surfacegroups/surfacegroups.pdf>.
- [2] J. Dymara and T. Januszkiewicz, *Tautological characteristic classes II: the Witt class*, *Mathematische Annalen* **390** (2024), 4463–4496. [doi:10.1007/s00208-024-02851-7](https://doi.org/10.1007/s00208-024-02851-7). Preprint: [arXiv:2403.05255v1](https://arxiv.org/abs/2403.05255).