

The Quaternionic Toledo Spectrum for Split F_4 over Products of Surfaces

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Abstract

Let F be the connected linear split real group of type F_4 and let Q_ρ be its canonical oriented three-plane bundle, obtained from a maximal compact reduction of a flat F -bundle. We determine the range of $\langle p_1(Q_\rho), [\Sigma_g \times \Sigma_h] \rangle$ over all representations of a product of two closed oriented surface groups. For $g, h \geq 2$, put $G = g - 1$, $H = h - 1$, and $P(R, S) = \{ab : a, b \in \mathbb{Z}, |a| \leq R, |b| \leq S\}$. The range is $2P(2G, 2H) \cup 2P(3G, H) \cup 2P(G, 3H)$, with sharp absolute bound $8GH$. The number is zero if either genus is at most one. The argument retains nonreductive images, disconnected image closures and central lifting obstructions. Its main steps are a full Levi reduction, control of the ordered real normalizers of commuting semisimple factors, and descent of integral degree-two classes to the original surfaces. Genuine symplectic two-plane representations realize every value. The theorem concerns this target and this source family, not arbitrary four-manifolds or all quaternionic symmetric targets.

1 The invariant and the main theorem

The quaternionic Toledo problem raised in the 2007 AIM workshop on representations of surface groups includes representations of four-manifold groups; products of two surfaces were specifically suggested as test cases [1, Question 4.2 and Comments 4.3–4.4]. Here we treat the split real target of type F_4 . Write

$$F = F_{4(4)}, \quad K = (\mathrm{Sp}(3) \times \mathrm{Sp}(1)) / \langle (-I, -1) \rangle.$$

We use the connected *linear* group. In fact all real points of the split algebraic F_4 are connected: the root datum is simply connected and adjoint, and the real points are generated by their connected real root groups [5, Proposition 2.5]. Algebraic simple connectedness is not topological simple connectedness.

The adjoint action of the last compact factor gives $q : K \rightarrow \mathrm{SO}(3)$. Since F/K is contractible, a flat F -bundle has a compact reduction unique up to homotopy and hence a well-defined associated oriented three-plane bundle Q_ρ . On a closed oriented four-manifold M put

$$\tau(\rho) = \langle p_1(Q_\rho), [M] \rangle \in \mathbb{Z}. \tag{1}$$

This is the integral characteristic normalization of the quaternionic invariant [12, Section II.A]; a metric normalization can differ by a fixed scalar. All products below have the product orientation.

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For nonnegative integers R, S define

$$P(R, S) = \{ab : a, b \in \mathbb{Z}, |a| \leq R, |b| \leq S\}.$$

Here $2P(R, S)$ means multiplication of the set by 2.

Theorem 1.1. *For $g, h \geq 2$, put $G = g - 1$ and $H = h - 1$. The exact range of (1) over representations $\pi_1(\Sigma_g) \times \pi_1(\Sigma_h) \rightarrow \mathbf{F}$ is*

$$S_4(G, H) = 2P(2G, 2H) \cup 2P(3G, H) \cup 2P(G, 3H). \quad (2)$$

Thus the sharp absolute bound is $8GH$. If $\min(g, h) \leq 1$, the range is $\{0\}$. No reductivity, connected-closure or lifting hypothesis is imposed.

An exact range says more than a sharp bound. For $g = h = 2$ the range is

$$\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}.$$

For $g = h = 3$ it contains 32 but not 14, 26, 28 or 30.

The degree-four characteristic relations used here are classical; see Herrera–Nagatomo [11] and Tshishiku [16, Sections 4.11 and 5.11]. Bucher–Gelandner’s multiplicative Euler-number theorem for flat bundles over products of surfaces [4] is a close precedent and supplies an input below. It does not apply directly to Q_ρ : this compact-reduction bundle is not asserted to be flat. The point of the present calculation is the restriction and realization of its values for every original $\mathbf{F}$ -representation. The complex dual-pair classification is imported from Gaetz [8]; all real component and integral-descent arguments needed here are given below. Theorem 1.1 addresses one target and the product-source case, not the whole AIM program.

This manuscript was developed with substantial AI assistance in the arguments, computational checks and drafting. It has not been independently refereed or formally verified. No absolute priority claim is made.

2 Bundle reductions on the original source

Let u, v be the integral orientation classes pulled back from the two surfaces. Thus $u^2 = v^2 = 0$ and $\langle uv, [\Sigma_g \times \Sigma_h] \rangle = 1$. We distinguish transferring a zero or a homogeneous inequality from descending a discrete set of integer values.

Lemma 2.1 (Central lifting covers). *Let $g, h \geq 1$, and let $\tilde{L} \rightarrow L$ be a central covering of Lie groups with finite kernel A of exponent m . Every representation of the product surface group into L lifts after a finite normal product cover. The cover can be chosen so that its pullback kills the entire obstruction in $H^2(\Sigma_g \times \Sigma_h; A)$.*

Proof. Take on each factor the kernel of the homomorphism to $H_1(\Sigma; \mathbb{Z}/m)$. Its degree is $m^{2\text{genus}(\Sigma)}$. Every A -valued degree-one character restricts to zero. Pullback of the degree-two orientation class multiplies by the degree and is zero with coefficients in A . Surface integral homology is free, so the universal coefficient and Kunneth formulas identify degree two with the two pure surface summands and $\text{Hom}(H_1(\Sigma_g; \mathbb{Z}) \otimes H_1(\Sigma_h; \mathbb{Z}), A)$. The covering maps send each H_1 into mH_1 , killing the mixed summand as well. The pulled-back central extension therefore splits. This lifts the whole product representation, not just its factors with possibly nontrivial cross commutators. Further normal product covers can kill any finite component images while retaining the lift. $\square$

Lemma 2.2 (Full Levi reduction). *Let L be the real Zariski closure of a representation into the real identity component of a linear algebraic group. It may be disconnected. Projection along its unipotent radical onto a real Levi complement preserves the underlying ambient principal bundle and all integral characteristic classes on the original source. If the source is a product, the two projected algebraic image closures are commuting normal reductive subgroups.*

Proof. In characteristic zero the full group has a real splitting $L = U \rtimes L_0$, including its component group [6, Proposition 5.4.1]. The real unipotent group $U(\mathbb{R})$ is connected and contractible by exponential and logarithm. Intersecting with the ambient identity component gives $L' = U(\mathbb{R}) \rtimes L'_0$. The inclusion $BL'_0 \rightarrow BL'$ is a homotopy equivalence because its homogeneous fiber is $U(\mathbb{R})$. Its homotopy inverse is induced by the projection. Composing these classifying maps with the ambient inclusion proves the bundle assertion. No contraction of U is asserted to be a path of homomorphisms.

The two original image closures commute because commutation is a polynomial identity; they are normal and their product is the full closure. Their projections have these properties and are closed algebraic subgroups. Their unipotent radicals would be connected normal unipotent subgroups of L_0 , so are trivial. At Lie-algebra level each simple derived ideal belongs to exactly one labelled source: its two commuting ideal projections sum to that simple ideal, and precisely one can be nonzero. The common central torus is retained. $\square$

Lemma 2.3 (Source support). *If the noncompact simple ideals of the reductive image closure are all supplied by one surface, every rational degree-four ambient characteristic class vanishes on the product. In particular, for F a nonzero number needs noncompact simple contributions from both surfaces, and a sphere or torus factor gives zero.*

Proof. A sphere factor has trivial fundamental group and gives immediate factorization through the other surface. Otherwise first kill the finite component images. The real connected reductive group is finitely centrally covered by the product of its connected central torus and its connected linear simple factors. Lemma 2.1 lifts on a normal product cover; further finite product covers kill the finite image of the opposite source in each simple factor. Each such factor now comes from its labelled surface. In degree four the rational cohomology of the classifying space of this connected reductive product consists of degree-four summands and products of degree-two summands. This follows from compact reduction and the even-degree rational cohomology of compact connected groups. Positive-degree rational classes from compact flat factors and from the central torus vanish by their flat compact connections. All remaining classes come from one surface and their degree-four products vanish. Rational transfer descends zero. An abelian torus source cannot supply a noncompact simple ideal, proving its assertion. $\square$

Lemma 2.4 (Circle purity and component reversal). *Let $q : K \rightarrow \mathrm{SO}(3)$ be a homomorphism for a maximal compact subgroup of a connected real reductive linear group. On a product of surfaces, a nonzero p_1 number of its associated three-plane bundle forces an oriented plane reduction*

$$Q = \mathbb{R} \oplus E, \quad e(E) = au + bv \quad (a, b \in \mathbb{Z}), \quad \langle p_1(Q), [M] \rangle = 2ab.$$

More generally, if a compact image normalizes a circle and its Euler class on a finite normal product cover is pure (a linear combination of the two surface classes), any nontrivial plane-orientation character forces the downstairs Pontryagin number to be zero.

Proof. After Lemma 2.2, choose a maximal compact subgroup C of the full reductive image; it meets every component and is a homotopy equivalence into that image. Compact containment

conjugates it into K [14, Propositions 1.2.3 and 1.2.7]. Lift the connected central product as in Lemma 2.3. The two labelled connected compact images in $\mathrm{SO}(3)$ commute. A connected subgroup of $\mathrm{SO}(3)$ is trivial, a circle, or all of $\mathrm{SO}(3)$. An image equal to $\mathrm{SO}(3)$ kills all the other images; its characteristic class comes from one surface. If at most one label contributes a circle, its Euler square is also zero; central flat torus contributions are zero. Nonzero evaluation therefore forces the connected compact image to be one circle.

The full image lies in its normalizer, which is $O(2)$ embedded as $A \mapsto \mathrm{diag}(\det A, A)$. On a normal product cover killing components and lifting the central product, the Euler class is pure: classifying spaces of connected groups have $H^1 = 0$, the simple factor classes are supported on their assigned surfaces, and central flat classes vanish. Every product deck transformation preserves the two orientation classes. If some holonomy reverses the plane, naturality of the Euler class with its orientation local system simultaneously sends that class to its negative. It is thus zero rationally, and so is its square downstairs by transfer.

Otherwise the plane is oriented already on M and its axis is trivial. Pullback is injective on rational H^2 and on each Kunneth summand, including $H^1 \otimes H^1$. Purity therefore descends. Since $H^2(M; \mathbb{Z})$ is free, $e(E) = au + bv$ with integer coefficients. The identity $p_1(Q) = e(E)^2$ gives the formula. $\square$

3 Characteristic classes of the standard module

Let W be the faithful real 26-dimensional representation of F . It has invariant quadratic form of signature $(14, 12)$ and compact summands U, V of those dimensions. On the compact double cover, with standard complex modules E_6, A_2 , the classical branching [11, Section 2] is

$$U_{\mathbb{C}} = \Lambda_0^2 E_6, \quad V_{\mathbb{C}} = E_6 \otimes A_2, \quad \mathfrak{p}_{\mathbb{C}} = \Lambda_0^3 E_6 \otimes A_2. \quad (3)$$

Diagonal signs act trivially on these modules and on Q ; the last module fixes which compact factor is quaternionic scalar. One can construct the split real representation from the positive compact module by replacing V by iV : the compact involution acts as $+1$ on U , -1 on V , and as the Cartan involution on the Lie algebra. This gives signature $(14, 12)$ rather than the other noncompact real form of F_4 .

Let $\pm x_i$ ($1 \leq i \leq 3$) and $\pm y$ be compact weights, and put $S = \sum x_i^2$. Pairing opposite complex weights gives

$$p_1(U) = 4S, \quad p_1(V) = 2S + 6y^2, \quad p_1(Q) = 4y^2, \quad p_1(W) = 6S + 6y^2.$$

Restriction to a maximal torus is rationally injective, so universally

$$p_1(Q) = \frac{2}{3}p_1(W) - p_1(U), \quad 3(p_1(V) - p_1(Q)) = p_1(W). \quad (4)$$

For a flat F -bundle $p_1(W) = 0$ rationally, whence $p_1(Q) = -p_1(U) = p_1(V)$ rationally. Neither compact summand is assumed flat. The same flat relation is visible in the degree-four kernel computed in [16, Section 5.11].

4 Small-index product representations

We use the following classical surface inputs with their indicated normalizations. A flat oriented real two-plane over Σ_g has Euler number in $[-G, G]$, and every integer in that interval occurs [13]; the statement is also recorded in [4, p. 3]. For a flat complex Hermitian bundle of signature (p, q)

the degree of its negative bundle has absolute value at most $\min(p, q)G$ [3, Corollary 3.27]. For a real symplectic bundle of rank $2r$, the degree of its compatible complex bundle is bounded by rG [10, Theorem 2.5 and Proposition 3.34]. The latter also follows by complexifying to a Hermitian bundle of signature (r, r) . Witt degeneration, described next, extends the reductive formulations to arbitrary representations without changing these degrees.

4.1 Tensor factors and Witt degeneration

An irreducible finite-dimensional complex representation of a product is $A \otimes B$ for genuine representations of its factors [4, Lemma 6.2]. Indeed, the translates under the second group of a minimal first-group invariant subspace span a direct sum of copies of that subspace; Schur's lemma identifies the commuting action with an action on its multiplicity space. If this tensor has an invariant nonsingular Hermitian or complex bilinear form, the form factors. Matrix coefficients on the second factor are intertwiners from the first irreducible to its conjugate dual or dual; a nonzero coefficient is nonsingular and the intertwiner space is one-dimensional. Normalizing its scalar makes each factor form Hermitian, or respectively symmetric or alternating.

An invariant isotropic subspace I gives a Witt filtration $I \subset I^\perp \subset V$. In a compatible splitting $I \oplus V_0 \oplus I^*$ (conjugate dual in the Hermitian case), conjugation by $\text{diag}(tI, I, t^{-1}I)$ removes the upper off-diagonal blocks as $t \rightarrow 0$. Every relation and commutator persists, including at the endpoint. The family of associated bundles over $M \times [0, 1]$ has constant characteristic classes. A hyperbolic Hermitian pair contributes a negative complex bundle smoothly isomorphic to the flat I , so its rational Chern classes vanish. A hyperbolic real orthogonal pair contributes the flat real I to the negative bundle: its Pontryagin classes vanish rationally, but its Euler class need not vanish. A symplectic hyperbolic pair has compatible bundle $I_{\mathbb{C}}$ and hence torsion c_1 , which is zero on this M . Induction leaves nondegenerate irreducibles. These assertions use smooth graph identifications, not flatness of arbitrary negative subbundles.

Lemma 4.1 (Unitary support). *For a product representation in $U(k, 1)$, the negative line has $c_1 = au, bv$ or 0 , where $|a| \leq G$, $|b| \leq H$. For a representation in $U(k, 2)$ with negative bundle B , the pair of degree-four numbers*

$$\left(\langle c_1(B)^2, [M] \rangle, \langle c_1(B)^2 - 4c_2(B), [M] \rangle \right)$$

is $0, (2ab, -2ab)$ or $(0, -8ab)$, with the same bounds on a, b . Both entries vanish if $k = 1$. For $Sp(k, 1)$ the canonical adjoint quaternionic-line Pontryagin number is zero on a product of surfaces.

Proof. For Hermitian tensor signatures (p, q) and (r, s) , the negative index is $ps + qr$. At index one, a nondefinite irreducible is a one-surface module tensored with a definite line from the other surface. The latter is a flat unitary character. The isotropic case has a flat negative line. This proves purity and the stated surface degree bounds. An index-two nondegenerate module either splits into two index-one blocks, is a one-surface module with a definite line twist, has a definite two-dimensional factor tensored with an index-one factor, or is a tensor of two $(1, 1)$ factors. These exhaust $ps + qr \leq 2$. In the two-block case the nonzero opposite-source roots are au, bv , giving $(2ab, -2ab)$. Definite factors have flat metric connections and leave curvature on one surface. In the two-indefinite-factor case the negative roots are $au - bv$ and $-au + bv$, giving $(0, -8ab)$. Isotropic pairs leave at most an index-one remainder, whose Euler square is zero. With only one positive direction none of the two-source nonzero cases fits.

For $Sp(k, 1) \subset SU(2k, 2)$ the negative complex two-plane is a quaternionic line, so $c_1 = 0$. This excludes the nonzero two-block case. A remaining nonzero tensor $A \otimes B$ has two indefinite

complex two-dimensional factors. It is not unitarizable and cannot pair invariantly with the positive definite complement. The invariant complex alternating form therefore restricts nonsingularly to it. It factors into one symmetric and one alternating form. The symmetric two-dimensional factor is in $O(2, \mathbb{C})$ and becomes reducible on a subgroup of index at most two, since $SO(2, \mathbb{C})$ is a torus. A reducible $U(1, 1)$ surface representation has degree zero: its invariant line is either nondegenerate, giving flat line summands, or isotropic, giving the degeneration above. Transfer forces the original factor degree to be zero, a contradiction. $\square$

Lemma 4.2 (Symplectic rank budget). *For $1 \leq n \leq 3$, a representation of the product surface group into the linear group $\mathrm{Sp}(2n, \mathbb{R})$ has compatible first Chern class $Au + Bv$, with integers A, B satisfying, for some integer $0 \leq k \leq n$,*

$$|A| \leq kG, \quad |B| \leq (n - k)H. \quad (5)$$

Every such class occurs. The same statement for $n = 1$ includes the flat oriented real-plane case after determinant normalization.

Proof. Symplectic Witt degeneration gives a completely reducible real product representation without changing its integral c_1 . Let A_1, A_2 be the real algebras spanned by the two commuting images, and $A = A_1 A_2$. The algebra A is semisimple. If J is the radical of A_1 , then JA_2 is a nilpotent ideal of A , so $J = 0$; the same holds for A_2 . Thus restriction to either original surface is completely reducible, without a finite cover. Decompose into first-source real isotypic components. The second source preserves each. Non-self-dual components pair with their duals, have an invariant Lagrangian, and contribute zero c_1 .

For a self-dual real irreducible U , write $D = \mathrm{End}_{A_1}(U) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. If $D = \mathbb{R}$, the actual block is $U \otimes_{\mathbb{R}} T$. Its form is a tensor of an alternating form and a symmetric form. If the alternating factor is U and the symmetric factor has signature (p, q) , a compatible bundle is $(U_J \otimes (T_+)_{\mathbb{C}}) \oplus (\overline{U_J} \otimes (T_-)_{\mathbb{C}})$. The complexifications of real bundles have torsion first Chern class; consequently this block has $c_1 = (p - q)c_1(U_J)$, supported on the first surface. Interchanging the form types gives support on the second surface.

If $D = \mathbb{C}$, its central complex structure J satisfies $J^* = J$ or $J^* = -J$ for the symplectic adjoint. In the first case $\omega(x, y) - i\omega(Jx, y)$ is a nonsingular complex alternating form. A complex symplectic group retracts to a compact semisimple group, whose determinant character is trivial; the block contributes zero. In the second case $\omega(x, Jy) - i\omega(x, y)$ is an invariant Hermitian form and the actual complex tensor block is $A_0 \otimes_{\mathbb{C}} T_0$ with Hermitian tensor form. If their dimensions are a, b , then $ab \leq n \leq 3$, so at least one is a definite one-dimensional factor. Its flat unitary character has zero Chern class, and the block's c_1 is supported on the other surface.

If $D = \mathbb{H}$, the bound on real dimension forces U to have real dimension four and multiplicity one. Identify it with $\mathbb{H}$. The two actions lie in the left and right quaternion algebras. Multiplication by a quaternion a has real determinant $|a|^4$; preservation of the alternating form forces norm one in each factor. Both images are compact, so $c_1 = 0$.

Each nondegenerate block of symplectic rank r_i is thus supported on one surface or has zero class. Restrict to that surface and apply the surface bound $r_i G$ or $r_i H$. Allocate zero blocks arbitrarily. The integer sum of ranks allocated to the first surface is k ; this proves (5), including absence of a mixed Kunneth term on the original source. Conversely take k genuine $\mathrm{SL}(2, \mathbb{R})$ surface planes from the first factor and $n - k$ from the second. Milnor's consecutive Euler intervals sum to all integers in (5). Their symplectic direct sum gives the claim. $\square$

5 The split symplectic subgroup

Proposition 5.1. *The actual connected subgroup*

$$H_s = (\mathrm{Sp}(6, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})) / \langle (-I, -I) \rangle \subset \mathbf{F}$$

has exact product-surface spectrum $S_4(G, H)$, including representations which do not lift to the displayed product.

Proof. Use the F_4 roots $\pm e_i \pm e_j$, $\pm e_i$, and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. For $\alpha = e_1 + e_2$, the perpendicular roots form C_3 with coordinates $(e_1 - e_2)/2, (e_3 + e_4)/2, (e_3 - e_4)/2$. Together with $\pm\alpha$ this is a closed $C_3 + A_1$ subsystem. Real Chevalley generators give an algebraic homomorphism $\mathrm{Sp}(6, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathbf{F}$ (also [7, Table 32]). Its faithful branching is $W = \Lambda_0^2 E \oplus (E \otimes L)$. Each individual central minus sign is nontrivial on the tensor summand; their product is trivial throughout. The kernel is exactly the diagonal signs.

On $\mathrm{U}(3) \times \mathrm{SO}(2)$ let x_1, x_2, x_3, y be the standard complex and oriented plane weights, $C = \sum x_i$, and $T = \sum_{i < j} x_i x_j$. The invariant form on $\Lambda^2 E$ is

$$(u \wedge v, w \wedge z) = \omega(u, w)\omega(v, z) - \omega(u, z)\omega(v, w).$$

Its real $(1, 1)$ part is positive of dimension nine, its real $(2, 0) + (0, 2)$ part is negative of dimension six, and the removed symplectic line is positive. The primitive signature is $(8, 6)$. The tensor summand has signature $(6, 6)$. For a fixed $\varepsilon = \pm 1$, the positive weights of W are $x_i - x_j$ ($i < j$), two real zeros, and $x_i - \varepsilon y$; the negative weights are $x_i + x_j$ and $x_i + \varepsilon y$. Consequently

$$p_1(U) = 3 \sum x_i^2 - 2T - 2\varepsilon C y + 3y^2, \quad p_1(W) = 6 \sum x_i^2 + 6y^2.$$

Equation (4) gives

$$p_1(Q) = (C + \varepsilon y)^2. \tag{6}$$

This is integral on $B\mathrm{U}(3) \times B\mathrm{U}(1)$, whose fourth cohomology is torsion-free. The sign will not affect the symmetric value sets.

On the quotient compact group $K_s = (\mathrm{U}(3) \times \mathrm{U}(1)) / \langle (-I, -1) \rangle$ the character $\chi(A, z) = \det(A) z^\varepsilon$ is well-defined since $3 + \varepsilon$ is even. Put $\zeta = c_1(\chi)$. The map $[(A, z)] \mapsto (zA, z^2)$ identifies K_s with $\mathrm{U}(3) \times \mathrm{U}(1)$. Thus $H^4(BK_s; \mathbb{Z})$ is torsion-free. Rational injectivity under the finite compact covering and (6) give the universal integral identity $p_1(Q)|_{BK_s} = \zeta^2$.

For a lifted pair, Lemma 4.2 supplies the rank-three and rank-one degree sets. Their Minkowski sum is exactly the rank-four rectangle union

$$R_4 = \bigcup_{k=0}^4 \{(A, B) \in \mathbb{Z}^2 : |A| \leq kG, |B| \leq (4 - k)H\}.$$

Indeed budgets add; conversely split any integer in a consecutive interval of length kG into one bounded by G and one bounded by $(k - 1)G$, assigning the rank-one block to that source (or to the second if $k = 0$). This also proves realizability of each pair.

For an arbitrary H_s representation, Lemma 2.1 gives a product lifting cover of degrees d, e . Upstairs $\zeta = A'u' + B'v'$ and, for one integer k , $|A'| \leq kdG$, $|B'| \leq (4 - k)eH$. Downstairs ζ is an actual integral class. Injectivity on the mixed Kunneth summand proves it has the form $Au + Bv$, with $A' = dA$, $B' = eB$ and $A, B \in \mathbb{Z}$. Thus the *same integer* budget k descends. Its square evaluates to $2AB$, giving precisely (2). Conversely four independent genuine $\mathrm{SL}(2, \mathbb{R})$ surface planes, three in the symplectic factor and one in the last factor, realize all the rectangles before quotienting. The $2 + 2$ allocation realizes both $8GH$ and $-8GH$. $\square$

6 Orthogonal upper sets and the split G_2 bound

We need an upper set for a genuine linear orthogonal representation, not for an unspecified projective quotient. For a negative oriented four-plane bundle V , put

$$t_{\pm} = \langle p_1(V) \pm 2e(V), [M] \rangle.$$

Its adjoint three-plane bundles $\Lambda_{\pm}^2 V$ have these classes: with oriented two-plane roots x, y , their rotation weights are $x + y, x - y$.

Lemma 6.1 (Orthogonal negative planes). *For a genuine orthogonal product representation with negative index two, its negative plane, if orientable, has one of the Euler classes*

$$S_g : au \ (|a| \leq 2G), \quad S_h : bv \ (|b| \leq 2H), \quad B : au + bv \ (|a| \leq G, \ |b| \leq H). \quad (7)$$

If the plane is nonorientable, its Euler class on an orienting product cover is zero rationally. A B class with $ab \neq 0$ requires at least two positive directions. A flat real plane has only a one-surface class with the smaller bound G or H .

Proof. First a surface representation in $\mathrm{SO}_0(p, 2)$ has negative-plane Euler degree at most $2G$. After a double surface cover lift to $\mathrm{Spin}_0(p, 2)$. In a full complex Clifford module, take Euclidean Hermitian generators γ_i , replace the two negative generators by $c_a = i\gamma_a, c_b = i\gamma_b$, and put $B = ic_a c_b$. This is a Hermitian involution with two eigenspaces of equal dimension r . The spin generators preserve its form of signature (r, r) . The negative-plane rotation acts on one eigenspace with half its angle. Every positive bivector has trace zero on that eigenspace, as conjugation by a positive Clifford generator changes its sign. Thus the negative spinor degree is $(r/2)$ times the plane degree. The $U(r, r)$ surface inequality gives $2G$, and the bound descends.

For a flat real two-plane, apply Lemma 4.2 for $n = 1$ after positive determinant normalization and, if necessary, an orienting product cover. For a general negative two-plane, Witt degeneration gives a flat plane or reduces to a nondegenerate irreducible of negative index two; two negative lines have zero Euler class.

Here is the real-form exhaustion used both here and below. For a real orthogonal irreducible its commuting division algebra is $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with the adjoint involution of the symmetric form. For $D = \mathbb{C}$, a skew-adjoint complex structure gives a pseudo-unitary realification; a self-adjoint one gives a complex orthogonal structure. In the latter case compact reduction to $O(d, \mathbb{R})$ makes the negative d -plane the real bundle whose complexification is the flat complex orthogonal bundle. Its Pontryagin classes vanish rationally. Its even-rank Euler class also vanishes, since the complex Pfaffian of the flat orthogonal connection restricts to the real Euler polynomial. Orienting covers handle disconnected compact groups.

For $D = \mathbb{H}$, the adjoint involution is quaternionic conjugation or has a self-adjoint imaginary unit. Its composition with quaternionic conjugation is inner; the involution condition makes the conjugating quaternion real or purely imaginary. The second case is complex orthogonal as above; the first is quaternionic Hermitian and has negative index divisible by four.

For $D = \mathbb{R}$ the complexification is irreducible and factors as $A \otimes B$. Both its bilinear form and its antilinear real structure factor by the one-dimensional intertwiner spaces. Normalize the two antilinear squares to both $+I$ or both $-I$, and normalize the factor forms compatibly. The possibilities are:

Factor type	Negative index	Possibilities at index at most four
Real symmetric, signatures $(p, q), (r, s)$	$ps + qr$	Both indefinite, neither $(1, 1)$: only $(2, 1), (2, 1)$, up to simultaneous sign reversal.
Real alternating, dimensions $2r, 2s$	$2rs$	$(r, s) = (1, 1), (2, 1), (1, 2)$.
Quaternionic alternating, signatures $(p, q), (r, s)$	$4(ps + qr)$	One factor is a compact quaternionic line when the index is nonzero.
Quaternionic symmetric, complex dimensions $2r, 2s$	$2rs$	One factor is $O^*(2)$ and cannot be complex irreducible.

For the last row, $ib(x, jy)$ has signature (r, r) since j reverses its sign. An $O^*(2)$ matrix has positive real complex determinant by quaternionic linearity and determinant ± 1 by orthogonality, hence determinant one. It lies in the abelian $SO(2, \mathbb{C})$ and is reducible. In the first row a $(1, 1)$ factor preserves a null line after a subgroup of index at most four and is treated by Witt degeneration. A compact real symmetric factor has a flat metric connection, leaving all curvature on the other surface and giving zero degree-four classes. The same argument applies to the compact quaternionic factor in the third row.

At index two the list leaves a one-surface orthogonal plane with bound $2G$ or $2H$, a unitary negative line with the smaller bounds from Lemma 4.1, or a real alternating two-plane tensor two-plane. In the last case compact rotations on the negative plane add their angles, giving $au + bv$ with Milnor bounds and positive index two. These statements first hold on normal orienting product covers. Purity and the reversal argument of Lemma 2.4 prove the nonorientable assertion. Otherwise the actual integral Euler class downstairs is pure, and its coefficients pull back multiplied by the separate cover degrees. Dividing their bounds gives (7), not a divided discrete spectrum. $\square$

Define the following sets with all parameters integral:

$$\begin{aligned}
\mathcal{A} &= \{(2ef, -2ef) : |e| \leq 2G, |f| \leq 2H\}, \\
\mathcal{M}_g &= \{(2(a+c)b, 2(a-c)b) : |a| \leq G, |c| \leq 2G, |b| \leq H\}, \\
\mathcal{M}_h &= \{(2a(b+d), 2a(b-d)) : |a| \leq G, |b| \leq H, |d| \leq 2H\}, \\
\mathcal{T} &= \{(2(a+c)(b+d), 2(a-c)(b-d)) : |a|, |c| \leq G, |b|, |d| \leq H\}.
\end{aligned}$$

Each set is invariant under interchanging its two output coordinates.

Proposition 6.2. *For every representation into $SO_0(n, 4)$ with $n \geq 3$,*

$$(t_+, t_-) \in \mathcal{A} \cup \mathcal{M}_g \cup \mathcal{M}_h \cup \mathcal{T}.$$

For $n = 3$ the set $\mathcal{T}$ is unnecessary. Each individual coordinate belongs to $S_4(G, H)$.

Proof. We give the remaining negative-index-four cases of the preceding exhaustion. In the complex orthogonal case both classes vanish. In the quaternionic Hermitian case the negative bundle is a quaternionic line: one adjoint class is zero and the other has zero number by Lemma 4.1. Compact-factor cases vanish by their one-surface curvature.

For a pseudo-unitary realification with negative complex rank two B , the two classes are $c_1(B)^2$ and $c_1(B)^2 - 4c_2(B)$, up to interchange. Lemma 4.1 puts the numbers in $\mathcal{A} \cup \mathcal{T}$: the axial pair $(0, -8ab)$ is obtained by equal or opposite parameters in $\mathcal{T}$. A nonzero case needs at least four positive real directions.

For real symmetric $(2, 1)$ tensors, the negative four-plane is the sum of the positive plane of each factor tensored with the negative line of the other. If these planes are oriented, their roots

are eu, fv with $|e| \leq 2G$, $|f| \leq 2H$; twisting by a flat real line changes no rational Euler class. The pair is $(2ef, -2ef)$. If a plane is not oriented, its Euler class on the orienting surface cover is zero by the same deck argument, and the pair is zero. These tensors have positive index five. A symmetric $(1, 1)$ factor instead gives, on a normal product cover, a maximal null bundle $L \otimes B$ with L an oriented real line from one surface and B a four-bundle from the other. Since L is smoothly trivial, the negative four-plane after degeneration is one-surface and has zero numbers.

For a real alternating $\mathrm{Sp}(4, \mathbb{R})$ tensor $\mathrm{SL}(2, \mathbb{R})$, compatible bundles of ranks two and one have degrees t, b with $|t| \leq 2G$, $|b| \leq H$. The negative four-plane is the real bundle underlying their complex tensor, up to orientation. Its classes are $p_1 = 2tbuv$, $e = \pm tbuv$, giving $(4tb, 0)$ up to interchange. Split $t = a + c$ with $|a|, |c| \leq G$ and set $d = b$ to obtain this pair in $\mathcal{T}$. Reverse tensor order is identical. These tensors have positive index four.

An irreducible negative-index-three summand has zero p_1 : the unitary and alternating cases have even index, quaternionic Hermitian cases have index divisible by four, and the other cases are one-surface or complex orthogonal, or reduce through a $(1, 1)$ null line to a flat negative three-bundle. Now decompose a representation with no isotropic invariant space orthogonally. Partitions $3 + 1$ and $1 + 1 + 1 + 1$ give zero. A partition $2 + 1 + 1$ gives $(2ab, 2ab)$, $|a| \leq G$, $|b| \leq H$, by Lemma 6.1, and belongs to $\mathcal{M}_g$. For $2 + 2$, the two plane orientation characters agree because the whole negative bundle is oriented. If nontrivial both cover Euler classes vanish. Otherwise their actual roots are $x = au + bv$, $y = cu + dv$, giving

$$(t_+, t_-) = (2(a + c)(b + d), 2(a - c)(b - d)). \quad (8)$$

Opposite one-surface roots give $\mathcal{A}$; equal supports give zero; one B root and one one-surface root give $\mathcal{M}_g$ or $\mathcal{M}_h$; two B roots give $\mathcal{T}$.

Finally let a maximal invariant null bundle have rank $s > 0$. Witt degeneration gives $V = I \oplus V_0$ smoothly, with I flat and V_0 having no isotropic subrepresentation. For $s = 4$ the exact flat Euler inclusion of [4, Corollary 1.9] is $\langle e(I), [M] \rangle = ab$, $|a| \leq G$, $|b| \leq H$; the pair is $(2ab, -2ab) \in \mathcal{A}$. For $s = 3$ it is zero. For $s = 2$ the plane orientation characters again agree. If nontrivial, both cover Euler classes vanish. Otherwise put $x = e(I)$, $y = e(V_0)$. The pair is $(\langle y^2 + 2xy, [M] \rangle, \langle y^2 - 2xy, [M] \rangle)$. The flat plane root is au , $|a| \leq G$, or its second-source counterpart. Inserting each of (7) for y gives $\mathcal{A}$, $\mathcal{M}_g$ or $\mathcal{M}_h$. For $s = 1$ the Euler product is zero and only a negative-plane p_1 in the index-three quotient can contribute, giving $(2ab, 2ab)$.

When $n = 3$, a nonzero irreducible index-four block needs at least four positive directions, and $s = 4$ is impossible. Two B roots with both coefficients nonzero would each need two positive directions. If one coefficient is zero, reclassify that root as S_g or S_h . Thus $\mathcal{T}$ adds no cases at $n = 3$. Projections of $\mathcal{A}, \mathcal{M}_g, \mathcal{M}_h$ are the three rectangles in (2); the projection of $\mathcal{T}$ lies in the first rectangle. This proves the assertion. $\square$

Corollary 6.3. *For any product representation into center-free linear split G_2 , the two adjoint three-plane bundles of its maximal compact $SO(4)$ have numbers in $6P(G, H)$ and $2P(G, H)$, in a fixed order.*

Proof. Use split octonions $\mathbb{H} \oplus \mathbb{H}$ with norm $|a|^2 - |b|^2$. The compact action is $(a, b) \mapsto (qaq^{-1}, pbq^{-1})$ for unit quaternions p, q , with diagonal sign kernel. This is maximal compact $SO(4)$: the algebra involution $(a, b) \mapsto (a, -b)$ induces the ambient Cartan involution, and its fixed automorphisms are exactly these quaternion actions. The imaginary seven-space has positive three-space $U = \mathrm{Im} \mathbb{H}$ and negative four-space $V = \mathbb{H}$, giving a genuine representation in $SO_0(3, 4)$ [2, Section 6]. For compact angles θ, ϕ , the plane weights of V are $\theta - \phi, \theta + \phi$ and that of U is 2ϕ . The two adjoint

classes are $P_+ = 4\theta^2 = p_1(V) + 2e(V)$ and $P_- = 4\phi^2 = p_1(V) - 2e(V)$, with

$$P_+ + 3P_- = 2p_1(U \oplus V).$$

Flatness makes their numbers satisfy $t_+ + 3t_- = 0$. Proposition 6.2 for $n = 3$ applies. In $\mathcal{A}$ this forces zero; in $\mathcal{M}_g$ it forces $4(2a - c)b = 0$, and in $\mathcal{M}_h$ it forces $4a(2b - d) = 0$. Every nonzero pair is consequently $(6ab, -2ab)$ with $|a| \leq G$, $|b| \leq H$. The sign-symmetric sets give the claim. $\square$

7 The real involution centralizers

Lemma 7.1 (Peirce restriction). *The connected stabilizer of a standard primitive idempotent in the split Albert algebra is the genuine $\text{Spin}_0(5, 4)$. If V_4 is the negative vector four-plane, with a fixed orientation, then*

$$p_1(Q)|_{B \text{Spin}_0(5,4)} = p_1(V_4) + 2e(V_4) \quad \text{in rational cohomology.}$$

Every product-surface representation in this subgroup consequently has its F_4 number in $S_4(G, H)$.

Proof. Write the split octonions as $\mathcal{O} = \mathbb{H} \oplus \mathbb{H}$, with

$$(x, y)(z, w) = (xz + \bar{w}y, wx + y\bar{z}), \quad \overline{(x, y)} = (\bar{x}, -y), \quad N(x, y) = |x|^2 - |y|^2.$$

In $J = H_3(\mathcal{O})$ fix $e = \text{diag}(0, 0, 1)$. Its trace-zero module splits into three summands. The first is the positive line spanned by $\text{diag}(1, 1, -2)$. The second is the nine-space

$$X(r, c) = \begin{pmatrix} r & c & 0 \\ \bar{c} & -r & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with form $r^2 + N(c)$ of signature $(5, 4)$, and the sixteen-space $Y(a, b)$ with entries $Y_{13} = a$, $Y_{23} = b$ and their conjugate transposes, of signature $(8, 8)$. The primitive-idempotent stabilizer is the spin group of this Peirce zero nine-space [9, Proposition 7]. Its real identity component is $\text{Spin}_0(5, 4)$; the spin representation, together with the vector representation, makes the action faithful.

To determine the orientation and class, its Clifford action on the half-Peirce space is

$$\Gamma_{(r,c)}(a, b) = (ra + cb, \bar{c}a - rb), \quad \Gamma_x \Gamma_y + \Gamma_y \Gamma_x = 2B(x, y)I.$$

The identity follows by alternativity, since $c, \bar{c}$ generate an associative subalgebra. The corresponding Jordan derivations $[L_x, L_y]$ act on the vector space by $z \mapsto B(y, z)x - B(x, z)y$ and on the half space by $[\Gamma_x, \Gamma_y]/4$. Orient the negative vector directions by $(0; (0, 1)), (0; (0, i)), (0; (0, j)), (0; (0, k))$. Their Clifford volume element Ω acts on each octonion coordinate by $(x, y) \mapsto (x, -y)$, directly from the multiplication formula. Hence it is $+1$ on the positive spin eight-space and -1 on the negative one.

Let α_1, α_2 be positive vector rotation weights and β_1, β_2 the negative ones in that orientation. A spin rotation generator is $\Gamma_x \Gamma_y / 2$ with square $-I/4$. Nonempty even Clifford monomials have zero trace: conjugation by a contained generator negates them. On the whole spin sixteen-space the diagonal Pontryagin coefficients are therefore two and the cross coefficients zero. Projecting to its positive half with $(I + \Omega)/2$ halves the diagonal coefficients; the only nonzero cross trace is for the two negative rotations, whose product is $\Omega/4$. Adding the vector and scalar pieces gives

$$\begin{aligned} p_1(W) &= 3(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2), \\ p_1(U) &= 2\alpha_1^2 + 2\alpha_2^2 + \beta_1^2 + \beta_2^2 - 2\beta_1\beta_2. \end{aligned}$$

Equation (4) now yields $(\beta_1 + \beta_2)^2$, which is $p_1(V_4) + 2e(V_4)$. These rotations span a maximal compact torus, so rational injectivity proves the universal identity on the actual spin group, without a bundle-lifting assumption. Projection to genuine $SO_0(5, 4)$ and Proposition 6.2 give the asserted upper set. $\square$

Proposition 7.2. *Every representation into the full real centralizer $C_F(t)$ of a nonidentity involution has number in $S_4(G, H)$. There are four real conjugacy classes of such involutions, with the following full centralizers. Here Δ_2 denotes diagonal signs.*

Representative in K	Identity component of $C_F(t)$	π_0
$[(\text{diag}(-1, 1, 1), 1)]$	$(\text{Sp}(2, 1) \times \text{Sp}(1))/\Delta_2$	1
$[(\text{diag}(-1, -1, 1), 1)]$	$\text{Spin}_0(5, 4)$	1
$[(-I, 1)]$	K	1
$[(iI, i)]$	H_s	C_2

In the last row a representation meeting the other component has number zero. The identity component already realizes all of S_4 .

Proof. Every finite subgroup is conjugate into K , and the global Cartan map $K \times \mathfrak{p} \rightarrow F$ is a diffeomorphism [14]. Uniqueness of this decomposition shows that two elements of K conjugate in F are conjugate in K , and that

$$C_F(t) = C_K(t) \exp(\mathfrak{p}^t)$$

is a product diffeomorphism. Thus real component groups can be computed in K , without inferring them from complex connectedness.

Lift t to $(x, y) \in \text{Sp}(3) \times \text{Sp}(1)$. Its square is $(I, 1)$ or $(-I, -1)$. In the first case choose $y = 1$ by the diagonal kernel and diagonalize x to $\text{diag}(-I_r, I_{3-r})$. The nontrivial possibilities are $r = 1, 2, 3$. Their compact centralizers are the connected groups $(\text{Sp}(r) \times \text{Sp}(3-r) \times \text{Sp}(1))/\Delta_2$. In the second case independent symplectic sign changes give $x = iI$, $y = i$. The compact centralizer has identity component $(\text{U}(3) \times \text{U}(1))/\Delta_2$ and other coset represented by $w = [(jI, j)]$, with $w^2 = 1$. It conjugates both complex structures to their negatives and reverses the canonical $SO(3)$ circle.

From (3) and the compact adjoint modules $\text{Sym}^2 E_6 \oplus \text{Sym}^2 A_2$ one obtains

Class	$\text{tr}(t W)$	$\dim \mathfrak{k}^t$	$\dim \mathfrak{p}^t$
$r = 1$	2	16	8
$r = 2$	-6	16	20
$r = 3$	2	24	0
Complex structures	2	10	14

For example $\text{tr}(x|E_6) = 6 - 4r$ and $\text{tr}(x|\Lambda_0^2 E_6) = ((6 - 4r)^2 - 6)/2 - 1$; the tensor summand adds $2(6 - 4r)$. In the complex-structure case E_6 has eigenvalues $i, -i$, three each, and A_2 has one of each. The same exterior-power characters give the two fixed dimensions.

The compact row $r = 3$ has no fixed tangent space, so its full centralizer is K . The central spin minus sign of the actual Peirce subgroup acts as $+1$ on its scalar plus vector ten-space and -1 on its spin sixteen-space. Its trace identifies $r = 2$, and its dimension 36 equals the centralizer dimension. Compact connectedness proves that this is the whole real centralizer. Similarly the genuine H_s has a central involution of trace 2, dimension 24 and maximal compact dimension 10, identifying the last row and its identity component.

For $r = 1$, the regular real embedding $\mathfrak{sp}(2, 1) + \mathfrak{sp}(1) \subset \mathfrak{f}_{4(4)}$ exists by [7, Table 32]. Both real factors are topologically simply connected, so the Lie homomorphism integrates to their genuine

product. Its complex regular branching is (3); faithfulness of W makes its kernel exactly Δ_2 . Its central involution has trace 2 and its compact dimension is 16, identifying $r = 1$. This imports existence of an embedding, not a classification of all real conjugacy classes.

The compact row has zero number by flatness. For the first row, on its compact cover $\mathrm{Sp}(2) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$, the $\mathfrak{sp}(2)$ map is zero and at most one $\mathfrak{sp}(1)$ can map nontrivially, necessarily by its adjoint action. One selection is flat compact; the other is the quaternionic line of $\mathrm{Sp}(2, 1)$ and vanishes by Lemma 4.1. Lemma 2.1 and transfer handle its central quotient. The Peirce row is Lemma 7.1. Proposition 5.1 treats H_s .

For the extra component of the last row, Q has a canonical $O(2)$ plane. On H_s its Euler class is $\pm\zeta$: its character and ζ have equal squares by (6), hence are equal up to sign after rational torus restriction, and then integrally in degree two. On a normal product cover killing the component character this class is pure by Proposition 5.1. Some deck transformation reverses its orientation, so Lemma 2.4 forces zero. Finally H_s realizes S_4 , proving the assertion about the union. $\square$

8 From arbitrary images to seven ordered pairs

Proposition 8.1. *Let $\mathfrak{a}$ be a nonzero semisimple algebraic subalgebra of $\mathfrak{f}_4(\mathbb{C})$ whose centralizer $\mathfrak{b}$ has nonzero derived algebra. Set $\mathfrak{a}' = \mathfrak{z}(\mathfrak{b})$. Then $(\mathfrak{a}', \mathfrak{b})$ is a reductive dual pair, with common center of dimension zero or one. The full normalizer of $\mathfrak{a}$ preserves its two members individually. If the common center is one-dimensional, this normalizer centralizes a canonical involution. If $\mathfrak{a}$ is real, so is that involution.*

Proof. Conjugate a compact form of the connected semisimple group of $\mathfrak{a}$ into a compact form of $F_4(\mathbb{C})$. The complex centralizer is the complexification of its compact centralizer, so it is reductive in the ambient algebra and has toral center. Repeating the argument makes $\mathfrak{a}'$ reductive. Double centralizing gives $\mathfrak{a} \subset \mathfrak{a}'$ and $\mathfrak{z}_{\mathfrak{f}_4(\mathbb{C})}(\mathfrak{a}') = \mathfrak{b}$; the common center is $\mathfrak{a}' \cap \mathfrak{b}$. These centralizers are algebraic and are preserved by the full normalizer.

The commuting Cartans of $\mathfrak{a}, \mathfrak{b}$ have total rank at most four. If the center of $\mathfrak{b}$ had dimension at least two, its nonzero derived algebra would give rank at least three. Equality of the rank bound would force $\mathfrak{a} = A_1$ and their combined Cartan to be an ambient Cartan. The two nonzero weight spaces of this A_1 would be ambient root spaces. But the centralizer of a long-root A_1 in F_4 is C_3 , and that of a short-root A_1 is A_3 ; both have zero center. In the explicit root model, the roots centralizing $e_1 - e_2$ are the eighteen perpendicular roots, forming C_3 . For e_1 , the six perpendicular short roots $\pm e_2, \pm e_3, \pm e_4$ do not centralize its root vectors; removing them leaves the rank-three $D_3 = A_3$ system. Its coroots span the whole centralizing Cartan. This proves the bound.

In the one-dimensional case the connected center of the centralizer group is an algebraic torus with a unique nonidentity two-torsion point t . Every normalizer element fixes t . Complex conjugation also fixes it when $\mathfrak{a}$ is real, even for an anisotropic real torus. The real involution then lies in F , whose real points are connected, and its full real centralizer is covered by Proposition 7.2. $\square$

When the common center is zero, Gaetz's complex classification [8, Section 9.2.2, Tables 8 and 10] gives exactly

$$\begin{aligned} (A_1^8, G_2^1), \quad (A_1^6, A_1^6), \quad (A_2, \tilde{A}_2), \quad (C_3, A_1), \\ (A_1^3, A_1^8 + A_1), \quad (A_3, \tilde{A}_1), \quad (B_2, 2A_1). \end{aligned} \tag{9}$$

Superscripts are embedding indices; tildes denote short-root types. This is a complex classification input, not a real normalizer theorem. Our pair is ordered, so its two members cannot be interchanged.

Lemma 8.2 (Five involution overgroups). *The full ordered real normalizer of each pair in (9) other than $G_2 + A_1$ and $A_2 + \tilde{A}_2$ centralizes a nonidentity real involution.*

Proof. For (C_3, A_1) the connected complex subgroup is $(\mathrm{Sp}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))/\Delta_2$, with center of order two. Its nontrivial central element is fixed by the full normalizer and by the real structure. For $(A_1^3, A_1^8 + A_1)$ the index-one ideal in the second factor is distinguished from the index-eight ideal by its embedding index. It is real and normalizer-invariant. In the tabulated embedding it is the long-root A_1 with centralizer C_3 , reducing to the preceding case.

The remaining three pairs lie in the standard complex Peirce B_4 stabilizer, where $W = \mathbb{C} \oplus V_9 \oplus S_{16}$. Their vector restrictions are respectively $6 + 3$, $5 + 4$, and $3 \otimes 3$ for the pairs $(A_3, \tilde{A}_1)$, $(B_2, 2A_1)$ and (A_1^6, A_1^6) . For the first two regular rank-four embeddings, V_9 has no invariant and all spin weights $(\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2$ are nonzero on their Cartan. For the tensor pair, four positive vector weights are $(2, 2), (2, 0), (2, -2), (0, 2)$. Each coordinate of a spin weight is a sum of three signs and is nonzero; the spin restriction is $(4, 2) + (2, 4)$. Its two A_1 indices are six, since their vector Cartan-square trace is 24, versus 4 for a long-root A_1 . Thus these are the indexed pairs in (9), and in each case the invariant subspace in W is one-dimensional.

For the standard idempotent that line is spanned by $v = 3e - I = \mathrm{diag}(-1, -1, 2)$, with $\mathrm{tr}(v^2) = \mathrm{tr}(v^3) = 6$. On any conjugate invariant line this uniquely determines v : scaling by c preserves both values only if $c^2 = c^3 = 1$. The full normalizer and real structure preserve both Jordan invariants and hence fix v , not merely its line. Therefore $e = (v + I)/3$ is a real primitive idempotent fixed by the normalizer. Its Peirce involution is $+1$ on the zero and one eigenspaces and -1 on the half eigenspace. Peirce multiplication rules make it a nonidentity real Albert automorphism. Every automorphism fixing e preserves those spaces and centralizes the involution. This proves the full-group containment. $\square$

9 The full normalizer of the $G_2 + A_1$ pair

Proposition 9.1. *For every actual real ordered pair of complex type (G_2^1, A_1^8) in F , every product-surface representation into its full normalizer has number in $6P(G, H) \cup 2P(G, H) \subset S_4(G, H)$.*

Proof. We first determine the full complex group, including its components. In the complex Albert algebra, entrywise octonion automorphisms commute with conjugation by scalar orthogonal three-by-three matrices. They give an actual embedding

$$H_{\mathbb{C}} = G_2(\mathbb{C}) \times \mathrm{SO}(3, \mathbb{C}) \longrightarrow F_4(\mathbb{C}).$$

Indeed, on scalar symmetric matrices an element of the kernel makes the orthogonal matrix scalar; determinant one excludes $-I$ in dimension three. The remaining imaginary octonion entries force the G_2 element to be the identity. On the trace-zero Albert module,

$$W = S \oplus T, \quad S = \mathrm{Sym}_0^2(\mathbb{C}^3), \quad T = \mathrm{Im}(\mathbb{O}_{\mathbb{C}}) \otimes \Lambda^2(\mathbb{C}^3).$$

These are distinct irreducible modules of dimensions 5 and 21. For an A_1 coroot their squared-weight traces are 40 and 56; the long-root F_4 reference trace is 12. For a long G_2 coroot the trace on its seven-space is 4, occurring three times. The embedding indices are therefore 8 and 1, identifying the specified pair in (9).

The two invariant polynomials $f_j(X) = \mathrm{tr}(X^j)$, $j = 2, 3$, are nonzero on each summand. On S , take $\mathrm{diag}(2, -1, -1)$, with $f_2 = f_3 = 6$. On T , choose quaternion units i, j, k in the complex

octonions and take

$$X = \begin{pmatrix} 0 & k & -j \\ -k & 0 & i \\ j & -i & 0 \end{pmatrix}; \quad f_2(X) = 6, \quad f_3(X) = -6.$$

This computation lies in an associative quaternion subalgebra. Schur's lemma makes any centralizer element scalar, say a, b , on S, T . Preservation of both invariants gives $a^2 = a^3 = b^2 = b^3 = 1$, hence $a = b = 1$. An Albert automorphism fixes the unit, so faithfulness on W proves that the full centralizer is trivial. The two distinct simple factors have no diagram automorphisms; all their automorphisms are inner and implemented by $H_{\mathbb{C}}$. Consequently their full ordered normalizer is exactly $H_{\mathbb{C}}$.

For any real conjugate of the pair, its two distinct complex factors are separately stable under the real structure. The complex product isomorphism descends as an isomorphism of real algebraic groups, not merely as an isogeny. Thus the full real normalizer is

$$N = H_G(\mathbb{R}) \times H_A(\mathbb{R}).$$

There is no central lifting obstruction. The real G_2 forms are the compact and split octonion automorphism groups [2, Section 6]. Their linear real groups are connected, in the split case by real root-group generation [5, Proposition 2.5]. The adjoint A_1 factor is $\mathrm{SO}(3)$ or the full two-component $\mathrm{SO}(2, 1)$, as follows from its adjoint action on the real Killing form. Its maximal compact is respectively $\mathrm{SO}(3)$ or $O(2)$. We do not require that every abstract combination of these forms occurs in $\mathbf{F}$.

Choose a maximal compact product $K_G \times K_A$ in N and conjugate it into K . The canonical bundle is computed by $\psi : K_G \times K_A \rightarrow \mathrm{SO}(3)$. Compact G_2 maps trivially: its simple Lie algebra has dimension 14. For split G_2 , $K_G = \mathrm{SO}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\Delta_2$, and ψ is trivial or one of its two adjoint projections. Two nonzero simple images cannot commute in $\mathfrak{so}(3)$. A nonzero projection is onto $\mathrm{SO}(3)$, whose centralizer in itself is trivial, so it kills the entire K_A , including its components. Its characteristic number belongs to $6P(G, H)$ or $2P(G, H)$ by Corollary 6.3, on the original source.

If ψ kills K_G , compact H_A gives a flat bundle and zero rational p_1 . For full $\mathrm{SO}(2, 1)$, pass to its component-killing finite cover. The flat Lorentz three-bundle reduces to an oriented plane E and an oriented line, so $e(E)^2 = p_1(E) = 0$ rationally. An $\mathrm{SO}(2)$ homomorphism into $\mathrm{SO}(3)$ has integer rotation weight m , making the selected class $m^2 e(E)^2 = 0$. Transfer descends zero; no discrete spectrum is divided by a cover degree. This proves all cases. Finally $6ab = 2(3a)b$ belongs to the middle rectangle of (2), and $2ab$ to the first. $\square$

10 The full normalizer of the two A_2 factors

Proposition 10.1. *For every actual real ordered pair of complex type $A_2 + \tilde{A}_2$ in $\mathbf{F}$, every representation into its full normalizer satisfies*

$$\tau \in 2P(2G, H) \cup 2P(G, 2H) \subset S_4(G, H).$$

A representation meeting a component outside the connected pair subgroup has zero number.

10.1 The complex group and its components

Use the first Tits model $J(M_3(\mathbb{C}), 1)$, with elements (x, y, z) , unit $e = (I, 0, 0)$, and structure data

$$\begin{aligned} N(x, y, z) &= \det x + \det y + \det z - \operatorname{tr}(xyz), & T(x, y, z) &= \operatorname{tr} x, \\ T((x, y, z), (x', y', z')) &= \operatorname{tr}(xx') + \operatorname{tr}(yy') + \operatorname{tr}(zz'), \\ (x, y, z)^\# &= (x^\# - yz, z^\# - xy, y^\# - zx). \end{aligned}$$

Matrix $\#$ is the adjugate. These classical formulas apply to matrix algebras without a division hypothesis [15, pp. 7–8, 15]. For $p, q \in \operatorname{SL}(3, \mathbb{C})$ put

$$\Phi(p, q)(x, y, z) = (pxp^{-1}, pyq^{-1}, qzp^{-1}). \quad (10)$$

It preserves all the displayed structure data. Cyclicity handles the mixed trace, and $(AB)^\# = B^\#A^\#$ handles the adjoint. These identities imply preservation of the Jordan product through

$$2v \circ w = (v + w)^\# - v^\# - w^\# + T(v)w + T(w)v - (T(v)T(w) - T(v, w))e.$$

The kernel consists exactly of (zI, zI) , $z^3 = 1$: triviality on x forces p scalar, and triviality on y then forces $q = p$. The actual connected subgroup and its traceless module are

$$H_{\mathbb{C}} = (\operatorname{SL}(3, \mathbb{C}) \times \operatorname{SL}(3, \mathbb{C})) / \Delta\mu_3, \quad W = (8, 1) \oplus (3, 3^*) \oplus (3^*, 3).$$

If e_i, f_i are the standard weights, with each triple summing to zero, half the squared-weight sum is

$$p_1(W) = 6 \sum e_i^2 + 3 \sum f_i^2. \quad (11)$$

Long-coroot squared traces are 24 and 12 respectively, so the first factor has index two and the second index one. This fixes the short/long ordering of the pair.

We need the whole centralizer. On the three distinct irreducible summands of W its elements are scalars a, b, c . The first summand contains $\operatorname{diag}(2, -1, -1)$, with squared trace 6 and determinant 2, forcing $a^2 = a^3 = 1$. The vectors $(0, I, 0)$ and $(0, 0, I)$ have norm one and pairing three. Thus $b^3 = c^3 = bc = 1$. Conversely each transformation $(x, y, z) \mapsto (x, by, b^{-1}z)$, $b^3 = 1$, is $\Phi(bI, I)$. It follows that

$$C_{F_4(\mathbb{C})}(H_{\mathbb{C}}) = Z(H_{\mathbb{C}}) = \mu_3.$$

Modulo inner automorphisms, each factor has only the contragredient involution. Applying it to one factor alone produces irreducible tensor summands absent from W and so cannot preserve the ambient module. Simultaneous duality is realized by the actual Albert involution

$$t(x, y, z) = (x^\top, z^\top, y^\top).$$

It preserves the unit, trace and pairing; the mixed cubic becomes $\operatorname{tr}(yzx) = \operatorname{tr}(xyz)$. Transposition of the adjoint formula, including the reversed mixed products, shows $(tv)^\# = t(v^\#)$. Polarization proves it is an algebra automorphism. Directly, $t\Phi(p, q)t^{-1} = \Phi(p^{-\top}, q^{-\top})$. Different embedding indices exclude a factor interchange. After removing inner automorphisms and this simultaneous duality, a normalizer element is in the centralizer already contained in $H_{\mathbb{C}}$. Therefore the full ordered normalizer is

$$N_{\mathbb{C}} = H_{\mathbb{C}} \rtimes \langle t \rangle, \quad t^2 = 1. \quad (12)$$

10.2 Actual real groups

The first Tits model over $\mathbb{R}$ is split. Transporting a real pair to this standard complex model transports conjugation to $\sigma = \text{Int}(\eta)\sigma_0$, with $\eta \in N_{\mathbb{C}}$. Its lift to the simply connected cover of $H_{\mathbb{C}}$ is therefore inner semilinear on both factors or outer semilinear on both, never one of each.

For an inner semilinear involution $g \mapsto P\bar{g}P^{-1}$, one has $P\bar{P} = \lambda I$. Conjugation and the determinant show $\lambda \in \mathbb{R}$ and $\lambda^3 = |\det P|^2 > 0$. After scaling, $v \mapsto P\bar{v}$ is an antilinear involution. A basis of its real fixed space identifies the group with $\text{SL}(3, \mathbb{R})$. For an outer semilinear involution $g \mapsto P\bar{g}^{-\top}P^{-1}$, squaring gives $P = \lambda\bar{P}^{\top}$, $|\lambda| = 1$. A scalar multiple of P is Hermitian. Hermitian congruence gives $\text{SU}(3)$ or $\text{SU}(2, 1)$; reversing the entire form does not change the group.

The finite central isogeny is surjective on real points. Its obstruction lies in $H^1(C_2, \mu_3)$, with either trivial or inversion action. For trivial action a cocycle has order dividing both two and three; for inversion every cocycle is a coboundary, since $z \mapsto z^{-2}$ permutes μ_3 . Thus the real connected pair group H is actually $\text{SL}(3, \mathbb{R}) \times \text{SL}(3, \mathbb{R})$, or a product of two unitary real forms modulo diagonal μ_3 . These Lie groups are connected, as are their maximal compacts $\text{SO}(3)$, $\text{SU}(3)$ and $\text{U}(2)$ before the quotient. The full real normalizer has $|N/H| \leq 2$. We neither assume that (12) splits over $\mathbb{R}$ nor assert that every abstract pair of signatures occurs in $\mathbb{F}$.

10.3 Compact selection and the integral character

Choose a maximal compact K_N with $K_H = K_N \cap H$, and conjugate it into K . Restrict q to $\psi : K_N \rightarrow \text{SO}(3)$. For the split pair, ψ on $\text{SO}(3) \times \text{SO}(3)$ is zero or one adjoint projection. The associated class is the p_1 of a standard flat $\text{SL}(3, \mathbb{R})$ three-plane and is rationally zero. For a unitary pair, pull back to compact factors $\text{SU}(3)$ or $\text{U}(2)$. The $\mathfrak{su}(3)$ ideals map to zero; at most one $\mathfrak{su}(2)$ ideal maps onto $\mathfrak{so}(3)$. If it does, every other Lie summand maps to zero. The resulting bundle is $\mathfrak{su}(E_+)$ for the relevant positive two-plane. Negating its Hermitian form views it as the negative two-plane of $\text{SU}(1, 2)$, whose number is zero by the $k = 1$ case of Lemma 4.1. A normal product cover as in Lemma 2.1 kills the entire μ_3 obstruction; transfer handles a nonliftable representation and subsequently any outer components in these zero cases.

It remains that ψ kills all semisimple compact ideals. Its connected image is trivial or a circle. In the latter case over BH there is an integral Euler class ζ with $p_1(Q) = \zeta^2$. If both factors are $\text{SU}(2, 1)$, let A, B be the positive two-plane first Chern classes on the compact $\text{U}(2) \times \text{U}(2)$ cover. Its characters are integral determinant powers, so

$$\zeta \mapsto mA + nB, \quad m, n \in \mathbb{Z}, \quad m + n \equiv 0 \pmod{3}. \quad (13)$$

The congruence follows because $zI_3 \in \mu_3$ acts on each positive determinant by z^2 .

The ambient identity (4) gives a useful uniform bound. On a compact circle $p_1(U)$ has nonnegative coefficient, the sum of squared real rotation weights. Set $(e_1, e_2, e_3) = (u, u, -2u)$ and $f_i = 0$ in (11). Its coefficient is 36, while $\zeta = 2mu$. Thus $4m^2 \leq 24$ and $|m| \leq 2$. On the other central circle the coefficient is 18, yielding $4n^2 \leq 12$ and $|n| \leq 1$. Consequently

$$(m, n) \in \{(0, 0), \pm(1, -1), \pm(2, 1)\}. \quad (14)$$

This is coefficient positivity on BS^1 , not positivity of an arbitrary four-manifold evaluation of $p_1(U)$. It applies to every actual real embedding. A compact $\text{SU}(3)$ factor has no character; if only one noncompact unitary factor remains, its degree-two class has one-source support and hence square zero.

10.4 Descent and outer components

Suppose first that ρ takes values in H . On a normal product cover killing the central obstruction, each genuine $\mathrm{SU}(2, 1)$ factor has positive determinant class equal to the negative of its negative-line class. Lemma 4.1 makes each zero or one-source, with degree at most the corresponding genus minus one. Let the cover degrees be d_g, d_h . When the two supports differ, in one ordering,

$$\zeta' = ma u' + nb v', \quad |a| \leq d_g G, \quad |b| \leq d_h H.$$

If the supports agree the square is zero. In all cases the mixed $H^1 \otimes H^1$ component vanishes. Injectivity of rational pullback on each Kunneth summand, and integrality and torsion-freeness on the original product, give

$$\zeta = cu + dv, \quad c, d \in \mathbb{Z}, \quad d_g c = ma, \quad d_h d = nb$$

in the different-support case. Hence $|c| \leq 2G$, $|d| \leq H$, or the exchanged bounds. The number $2cd$ belongs to the asserted set. We have descended the integral coefficients before the number, not divided an upstairs discrete spectrum.

Finally consider a component outside H . In the unitary case its simultaneous outer action negates A and B . To see this for every real representative, diagonalize the Hermitian form to a real J . On $\mathrm{SU}(J)$, $g^{-\top} = J\bar{g}J^{-1}$, so an arbitrary outer automorphism has form $g \mapsto D\bar{g}D^{-1}$. Reality implies $D^*JD = \lambda J$ with $\lambda \in \mathbb{R}$. The unequal positive and negative indices in dimension three exclude $\lambda < 0$ by inertia. After scaling, D is an isometry, and the induced positive two-plane map is conjugate-linear. It negates its first Chern class. The finite compact central cover is rationally injective, so the outer component sends ζ to $-\zeta$.

Take the normal product cover killing the two component characters. Its Euler class ζ_0 is pure by the connected case. Every product deck transformation fixes u', v' , while one has outer holonomy and negates ζ_0 . Thus $\zeta_0 = 0$ rationally. Transfer gives zero downstairs. Negation by itself would not annihilate ζ_0^2 ; purity and the deck action are both needed. This completes the proof of Proposition 10.1.

11 Completion of the spectrum calculation

Proof of Theorem 1.1. Apply the full real Levi reduction of Lemma 2.2 to the Zariski closure of ρ . It preserves the original principal F-bundle. The two source closures are commuting normal reductive groups. If all noncompact simple factors have one source label, Lemma 2.3 gives zero. Otherwise let $\mathfrak{a}$ be the nonzero semisimple derived algebra of the first connected source closure. Its centralizer contains the nonzero semisimple algebra from the second source. The full reductive image normalizes $\mathfrak{a}$, hence normalizes the ordered dual pair of Proposition 8.1.

If the common center is one-dimensional, the entire image centralizes the canonical real involution. Its number is in S_4 by Proposition 7.2. In the zero-center case the pair is one of (9). Five types reduce to the same conclusion by Lemma 8.2. The remaining two are covered by Propositions 9.1 and 10.1. These arguments retain the full real normalizers, their components and their actual central quotients; thus they exhaust arbitrary original images. This proves the upper inclusion.

Proposition 5.1 realizes every element of S_4 using four genuine symplectic two-planes and the actual split subgroup of F . The first rectangle attains both signs of $8GH$, and the other two have absolute bound $6GH$. If a source is a sphere the representation factors through the other surface. For a torus source its connected image closure is abelian and supplies no noncompact simple factor; Lemma 2.3 gives zero. This proves the low-genus statement and the theorem. $\square$

Remark 11.1. The three rectangles are an exact arithmetic range, not an interval of all even numbers below the bound. No classification of connected components of the representation variety is asserted. The argument uses classical structure and surface inequalities, together with the real-normalizer and integral character calculations above. Finite weight and matrix checks can test these calculations but do not replace their general proofs.

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