Variable Critical Exponents on a Fixed Free-Deck Regular Cover

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Abstract

An AIM problem asks for an infinitely generated Fuchsian group of the first kind whose critical exponent is nonconstant on its quasiconformal Teichmüller space. We give an explicit affirmative construction. For a closed surface S_g, let K be the kernel of the epimorphism $\pi_1(S_g) \to F_g$ that kills a cut system and maps its dual curves to a free basis. For every marked hyperbolic metric m, the group $\Gamma_m = \rho_m(K)$ is infinitely generated and of the first kind. Its free deck group is nonamenable, so Brooks's theorem gives $\delta(\Gamma_m) < 1$. When all curves of the killed cut system are pinched to length ℓ, a compactly supported cutoff in one lifted cell gives

$$\lambda_0(\mathbb{H}^2/\Gamma_m) \leq \frac{g \sinh(1)}{\pi(g-1)}\ell, \qquad 1 - \delta(\Gamma_m) \leq \frac{2g \sinh(1)}{\pi(g-1)}\ell.$$

Thus the exponents tend to one while remaining strictly below one at every finite metric. All structures lie in the same quasiconformal Teichmüller space. This note is unrefereed and makes no absolute priority claim.

1 Introduction

For a Fuchsian group Γ, its critical exponent is

$$\delta(\Gamma) = \inf\left\{ s > 0 : \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty \right\}.$$

Problem 4.3 in the 2014 AIM list *Problems on Thurston Metric* asks for an infinitely generated Fuchsian group Γ_0 of the first kind for which this exponent takes distinct values at points of $\mathcal{T}_{\mathrm{qc}}(\Gamma_0)$ [Su14]. The expanded 2016 list repeats the question and discusses Bishop's nearby work on quasi-Fuchsian δ-stability [Su16, Bis03].

We construct a regular cover with deck group F_g and vary only the marked hyperbolic metric on its compact base. Nonamenability keeps the exponent strictly below one at each finite metric. Pinching curves whose lifts bound a compact fundamental cell makes the spectral bottom tend to zero. The Elstrodt–Patterson–Sullivan formula then forces the exponent to tend to one. The pinching estimate is linear in the pinched length and comes from a fixed-width collar cutoff.

Theorem 1.1. *For every integer $g \geq 2$, there is an infinitely generated Fuchsian group Γ_0 of the first kind for which the critical-exponent function is nonconstant on $\mathcal{T}_{\mathrm{qc}}(\Gamma_0)$.*

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More precisely, there is a fixed normal subgroup

$$K \triangleleft \pi_1(S_g), \qquad \pi_1(S_g)/K \cong F_g,$$

such that for every marked closed hyperbolic metric m on S_g, the group $\Gamma_m = \rho_m(K)$ is infinitely generated and of the first kind, and $\delta(\Gamma_m) < 1$. There is a pinching family m_ℓ in the pullback copy of $\mathcal{T}(S_g)$ inside $\mathcal{T}_{\mathrm{qc}}(\Gamma_0)$ for which

$$\delta(\Gamma_{m_\ell}) \longrightarrow 1.$$

For all sufficiently small ℓ,

$$1 - \delta(\Gamma_{m_\ell}) \le \frac{2g\sinh(1)}{\pi(g-1)}\ell. \tag{1.1}$$

Regular covers with exponents below one and the range of exponents obtained by varying the normal subgroup over a fixed metric are studied in [BTMT12]. Quantitative results for families of normal subgroups appear in [DS19], and conformal measures on regular covers in [Shw21]. We did not locate in these works the fixed-kernel, varying-metric construction used here.

2 The fixed cover

Let S_g be a closed oriented surface of genus $g \ge 2$. Choose a geometric symplectic system

$$\pi_1(S_g) = \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \prod_{i=1}^{g}[a_i, b_i] = 1 \right\rangle \tag{2.1}$$

such that $b_1, \dots, b_g$ are disjoint and jointly nonseparating. Cutting along them gives a sphere P with $2g$ boundary components. Define

$$q : \pi_1(S_g) \twoheadrightarrow F_g = \langle x_1, \dots, x_g \rangle, \qquad q(a_i) = x_i, \qquad q(b_i) = 1, \tag{2.2}$$

and put $K = \ker q$. The surface relation in (2.1) maps to the identity, and the images of the a_i generate F_g, so this is an epimorphism.

For a marked closed hyperbolic metric m, let

$$\rho_m : \pi_1(S_g) \longrightarrow \mathrm{PSL}_2(\mathbb{R})$$

be its holonomy and set

$$\Gamma_m = \rho_m(K), \qquad X_m = \mathbb{H}^2/\Gamma_m. \tag{2.3}$$

Then $X_m \to (S_g, m)$ is a regular cover with deck group F_g.

Lemma 2.1. *For every m, the group Γ_m is infinitely generated and of the first kind. Moreover, the covers X_m obtained from all marked closed metrics lie in one quasiconformal Teichmüller space.*

Proof. The kernel K is a nontrivial non-elementary normal subgroup of the surface group. A non-elementary normal subgroup has the same limit set as its ambient Fuchsian group. Since the latter is cocompact, $\Lambda(\Gamma_m) = S^1$, and Γ_m is of the first kind.

The subgroup has infinite index because its quotient is the infinite group F_g. If it were finitely generated, it would be a geometrically finite Fuchsian group without parabolics and would have compact convex core. First kind makes its convex hull all of $\mathbb{H}^2$, so X_m itself would be compact. Its finite area would then force finite index in the compact base, a contradiction.

For two marked metrics m, m', a marked quasiconformal map $(S_g, m) \to (S_g, m')$ preserves K under the marking and therefore lifts to a quasiconformal map $X_m \to X_{m'}$. Hence all of the covers represent points of $\mathcal{T}_{\mathrm{qc}}(X_{m_0})$ for any chosen base metric m_0. □

3 The strict upper bound at finite metrics

We use two classical spectral facts. Brooks's regular-cover theorem says that for a compact base, the spectral bottom of the cover is zero exactly when its deck group is amenable [Bro85]. Since F_g is nonamenable,

$$\lambda_0(X_m) > 0 \tag{3.1}$$

for every fixed metric m.

For a complete hyperbolic surface, the Elstrodt–Patterson–Sullivan formula is [Sul87, BTMT12]

$$\lambda_0(X_m) = \begin{cases} 1/4, & \delta(\Gamma_m) \le 1/2, \\ \delta(\Gamma_m)(1-\delta(\Gamma_m)), & \delta(\Gamma_m) \ge 1/2. \end{cases} \tag{3.2}$$

In either branch, (3.1) rules out $\delta(\Gamma_m) = 1$. Thus

$$\delta(\Gamma_m) < 1 \tag{3.3}$$

at every finite metric.

4 Pinching and the Rayleigh quotient

Choose marked metrics m_ℓ for which all b_i are geodesics of common length $\ell \to 0$. The collar lemma supplies coordinates

$$(-w(\ell), w(\ell)) \times \mathbb{R}/\mathbb{Z}, \qquad ds^2 = dr^2 + \ell^2 \cosh^2(r)\, d\theta^2, \tag{4.1}$$

where $w(\ell) \to \infty$.

The fundamental group of the cut-open surface P is generated by its boundary classes. All these classes are conjugates of $b_i^{\pm 1}$ and are killed by q. It follows that a copy $\widetilde{P}_\ell$ lifts homeomorphically to X_{m_ℓ}.

For sufficiently small ℓ, every boundary half-collar in this copy has width greater than one. With $r = 0$ at a gluing boundary, define

$$u_\ell(r,\theta) = \min\{r, 1\} \tag{4.2}$$

on each of the $2g$ boundary half-collars, let $u_\ell = 1$ on the rest of $\widetilde{P}_\ell$, and extend it by zero outside that cell. Its trace at every gluing boundary is zero, so the extension is Lipschitz, compactly supported, and belongs to $H_0^1(X_{m_\ell})$.

On a transition strip $0 \le r \le 1$, the area form from (4.1) is $\ell \cosh(r)\, dr\, d\theta$, and $|\nabla u_\ell| = 1$. Hence

$$\int_{X_{m_\ell}} |\nabla u_\ell|^2 = 2g\ell \sinh(1). \tag{4.3}$$

The area of P is $4\pi(g-1)$. Discarding the transition strips gives

$$\int_{X_{m_\ell}} u_\ell^2 \ge 4\pi(g-1) - 2g\ell\sinh(1). \tag{4.4}$$

When

$$\ell \le \frac{\pi(g-1)}{g\sinh(1)}, \tag{4.5}$$

the right side of (4.4) is at least $2\pi(g-1)$. Smooth compactly supported approximations to u_ℓ are admissible in the variational definition of λ_0. Equations (4.3)–(4.4) give

$$\lambda_0(X_{m_\ell}) \leq \frac{g\sinh(1)}{\pi(g-1)}\ell. \tag{4.6}$$

In particular $\lambda_0(X_{m_\ell}) \to 0$.

For small ℓ, the right side of (4.6) is below 1/4. The first branch of (3.2) is then impossible, and $\delta(\Gamma_{m_\ell}) > 1/2$. Therefore

$$1 - \delta(\Gamma_{m_\ell}) = \frac{\lambda_0(X_{m_\ell})}{\delta(\Gamma_{m_\ell})} \leq 2\lambda_0(X_{m_\ell}), \tag{4.7}$$

which proves (1.1) and $\delta(\Gamma_{m_\ell}) \to 1$.

Fix any initial metric m_0. Its exponent is strictly below one by (3.3); a sufficiently late point in the pinching family has a different exponent. By lemma 2.1, the two groups represent points of the same quasiconformal Teichmüller space. This completes the proof of theorem 1.1.

5 Scope and prior-work calibration

The proof gives a one-sided rate, not monotonicity or an exact formula for the exponent along the pinching path. It uses a nonamenable free deck group; amenable regular covers have exponent one and cannot yield this mechanism. No classification of infinite-type surfaces with variable exponent is claimed.

Bishop's deformation theorem concerns the equality between the critical exponent of a quasi-Fuchsian deformation and the Hausdorff dimension of its limit set [Bis03]. Bonfert-Taylor–Matsuzaki–Taylor vary the normal subgroup over a fixed compact hyperbolic metric and prove results near the lower endpoint 1/2 [BTMT12]. The construction here instead fixes the subgroup and varies the Fuchsian base metric. A targeted search did not locate this exact argument or (1.1). Since it is an elementary synthesis of classical ingredients, no absolute priority claim is made.

AI-assisted tools were used for source discovery, algebraic exploration, adversarial proof review, and preparation of this manuscript. The argument is written out explicitly and uses only the cited classical theorems. The named author is responsible for the final manuscript.

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