

The Entropy Cost of Infinite-Genus Weil–Petersson Local Limits

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Abstract

Let μ_g be normalized Weil–Petersson measure on the moduli space of closed hyperbolic surfaces of genus g . Suppose that alternative surface laws ν_g , with arbitrary measurable choices of roots, have a pointed smooth weak limit Q . For every fixed $h \geq 1$ we prove

$$\liminf_{g \rightarrow \infty} \frac{H(\nu_g \mid \mu_g)}{\log g} \geq (2h - 1)Q\{\text{genus} \geq h\}.$$

In particular, positive mass on infinite-genus surfaces requires superlogarithmic surface relative entropy. Conversely, for every $b_g \rightarrow \infty$, we construct rooted laws converging to a fixed one-ended infinite-genus surface whose joint relative entropy, relative to area-rooted Weil–Petersson measure, is at most $b_g \log g$ for all sufficiently large g . The proof combines classical separating-curve volume estimates with an entropy inequality and a slowly growing sequence of positive-volume core conditions. We also explain, as consequences of known tangle-free and systole results, why an unchanged Weil–Petersson surface marginal permits only planar or cylindrical local limits under arbitrary rooting.

1 Model, results, and scope

How can finite-genus asymptotics inform the geometry of infinite-genus surfaces? This broad question is among the themes of the AIM workshop *Surfaces of infinite type: Teichmüller theory and other tools* [1]. The frozen UnsolvedMath record AIM-TOPOLOGY-0147 associates this question with Problem 3.5 of the workshop problem list [6]. We address a precise probabilistic model, not the full open-ended agenda: curvature is fixed, convergence is pointed smooth, and the cost of changing the finite-genus surface law is measured by relative entropy.

All surfaces in this paper are connected, oriented, complete hyperbolic surfaces of curvature -1 , without boundary, except for the compact cores used in the proofs. Isometries preserve orientation. Limits are taken in the pointed smooth topology, without rescaling. Thus a compact neighborhood of the root is eventually represented by smooth embeddings under which the pulled-back metrics converge in every derivative. A rooted surface is denoted (X, p) .

Let $\mathcal{M}_g$ be the moduli space of closed surfaces of genus $g \geq 2$, and let μ_g be its normalized Weil–Petersson (WP) measure. The area is $A_g = 4\pi(g - 1)$. Its area-rooted lift is

$$P_g(dX, dp) = \mu_g(dX) \frac{dA_X(p)}{A_g}. \tag{1}$$

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For probability measures ν, μ we use natural logarithms and set

$$H(\nu \mid \mu) = \begin{cases} \int \log\left(\frac{d\nu}{d\mu}\right) d\nu, & \nu \ll \mu, \\ +\infty, & \text{otherwise.} \end{cases}$$

A rooting kernel $K_g(X, dp)$ is an isometry-invariant measurable probability kernel on the surface. It need not be absolutely continuous with respect to area. Write $Q_g = \nu_g K_g$ for the resulting rooted law.

Theorem 1 (Entropy obstruction). *Suppose $Q_g = \nu_g K_g$ converges weakly to a law Q on pointed complete hyperbolic surfaces. For every integer $h \geq 1$,*

$$\liminf_{g \rightarrow \infty} \frac{H(\nu_g \mid \mu_g)}{\log g} \geq (2h - 1)Q\{\text{genus} \geq h\}. \quad (2)$$

If $Q\{\text{genus} = \infty\} > 0$, then $H(\nu_g \mid \mu_g)/\log g \rightarrow \infty$. In particular, an $O(\log g)$ surface entropy budget excludes positive infinite-genus mass, regardless of the rooting kernel.

Theorem 2 (Arbitrarily slow superlogarithmic budgets). *There is a fixed pointed complete one-ended infinite-genus hyperbolic surface (S, o) with the following property. For every positive sequence $b_g \rightarrow \infty$, there are laws $Q_g \ll P_g$ such that*

$$Q_g \implies \delta_{(S,o)}, \quad H(Q_g \mid P_g) \leq b_g \log g \quad (3)$$

for all sufficiently large g .

The necessary estimate concerns only the surface marginal, whereas the construction controls the larger *joint* entropy, including the root. Since entropy decreases under projection, these statements identify the logarithmic versus arbitrarily slowly superlogarithmic budget scale. They do not identify an optimal coefficient for a fixed finite core. Nor do they produce a canonical infinite-genus WP law.

Relation to classical results

The inputs are Mirzakhani's integration and volume-polynomial formulas [2], the fixed-boundary-count large-genus asymptotic of Mirzakhani–Zograf [4], and elementary relative-entropy inequalities. The separating-curve estimates themselves are classical. Our contribution is their combination into Theorems 1 and 2, with a quantitative genus-tail bound and a joint-entropy construction. Only fixed-core asymptotics are used before a diagonal is taken.

Section 5 records two consequences of the Monk–Thomas tangle-free theorem [5] and the Mirzakhani–Petri length-spectrum law [3]. Neither tangle-freeness nor that length law is asserted as new. A bounded literature search found no matching formulation of the two entropy statements; this is not an absolute priority certification. The paper is an unrefereed preprint.

2 Volume and entropy estimates

Let $V_{k,n}(\ell)$ denote WP volume with prescribed geodesic boundary lengths, and let $V_{k,n} = V_{k,n}(0, \dots, 0)$. Write $V_g = V_{g,0}$. We fix one consistent orbifold-volume convention; the fixed genus-one factors in cutting formulas do not affect any power of g below.

The positive-coefficient volume polynomial and the boundary-volume bound give

$$V_{k,1} \leq V_{k,1}(\ell) \leq e^{\ell/2} V_{k,1}. \quad (4)$$

See [3, Theorem 2.3 and Lemma 2.7]. The Mirzakhani–Zograf asymptotic [4, Theorem 1.2], also stated in [3, Theorem 2.6], implies, for each fixed $h \geq 1$,

$$\frac{V_{g-h,1}}{V_g} \sim (8\pi^2 g)^{-(2h-1)}. \quad (5)$$

Indeed, the factorial ratio is $(2g - 2h - 2)!/(2g - 3)!$, with $2h - 1$ denominator factors. The corresponding power of $4\pi^2$ is $-(2h - 1)$ and the square-root ratio tends to one. The universal asymptotic constant cancels. No uniform estimate in a growing h is being asserted.

Lemma 3 (A short separating core). *Fix $h \geq 1$ and $B > 0$. For all sufficiently large $g > 2h$,*

$$\mu_g\{X : \text{a simple geodesic of length at most } B \text{ cuts off genus } h\} \leq C_{h,B} g^{-(2h-1)}.$$

Proof. Let $N_{h,B}$ count the indicated geodesics. Mirzakhani’s integration formula gives

$$\mathbb{E}_{\mu_g} N_{h,B} = \frac{c_h}{V_g} \int_0^B \ell V_{h,1}(\ell) V_{g-h,1}(\ell) d\ell, \quad (6)$$

with $c_h > 0$ independent of g . The two sides of a cut have different genera, so there is no ambiguity about the small side. Equations (4) and (5) bound the expectation by $C_{h,B} g^{-(2h-1)}$. The probability is at most the expectation. $\square$

We also need a metric-restricted version of (6). If $\mathcal{V}$ is a measurable family of genus- h one-boundary cores, and $v(\ell)$ is its WP slice volume at boundary length ℓ , the expected number of cuts with small side in $\mathcal{V}$ is

$$\frac{c_h}{V_g} \int_0^\infty \ell v(\ell) V_{g-h,1}(\ell) d\ell. \quad (7)$$

This is not an extra independence assumption. In the cut moduli space, the WP symplectic form splits into the forms on the two components and $d\ell \wedge d\tau$. Integration over the twist gives the factor ℓ , up to the fixed orbifold factor. Applying Fubini with an indicator on the small component proves (7). This splitting is explicit in Section 8, Lemmas 8.2–8.3 and equation (8.3) of the 2005 preprint of [2]; the section numbering differs in its published version.

Lemma 4 (Cost of a rare event). *If $p = \mu(E) \in (0, 1)$ and $q = \nu(E)$, then*

$$H(\nu \mid \mu) \geq q \log \frac{q}{p} + (1 - q) \log \frac{1 - q}{1 - p} \geq q \log \frac{1}{p} - \log 2. \quad (8)$$

Proof. If the left side is infinite there is nothing to prove. Otherwise, convexity of $x \log x$ on E and its complement gives the first inequality. For the second, use $q \log q + (1 - q) \log(1 - q) \geq -\log 2$ and $-(1 - q) \log(1 - p) \geq 0$. Endpoint cases follow by limits. $\square$

3 Proof of the entropy obstruction

For $h \geq 1$ and $B > 0$, let $\mathcal{U}_{h,B}$ be the set of pointed surfaces containing a smooth embedded compact genus- h subsurface with exactly one boundary component, of length strictly less than B . There is no restriction on the position of the root.

The set $\mathcal{U}_{h,B}$ is open in the pointed smooth topology. An embedded witness and a neighborhood of it are compact and lie at a finite distance from the root. Smooth convergence on that neighborhood preserves the embedding, its topological type, and the strict boundary-length inequality. Using smooth, rather than geodesic, boundaries here avoids any need for geodesic stability.

On a closed surface of genus $g > 2h$, the witness boundary is essential. If it bounded a disk, that disk would be on the other side, forcing $g = h$. Its geodesic representative is therefore simple, no longer than the original boundary, and cuts off a genus- h subsurface. Lemma 3 applies to a surface event containing $\mathcal{U}_{h,B}$ under every root. Lemma 4 implies

$$H(\nu_g \mid \mu_g) \geq Q_g(\mathcal{U}_{h,B})((2h-1)\log g - \log C_{h,B}) - \log 2$$

for all sufficiently large g . By Portmanteau for this open set,

$$\liminf_{g \rightarrow \infty} \frac{H(\nu_g \mid \mu_g)}{\log g} \geq (2h-1)Q(\mathcal{U}_{h,B}).$$

An oriented surface has genus at least h exactly when it contains a compact genus- h one-boundary subsurface. Such a subsurface can be chosen smooth and has finite boundary length. Consequently,

$$\bigcup_{B \in \mathbb{N}} \mathcal{U}_{h,B} = \{\text{genus} \geq h\}.$$

Letting $B \rightarrow \infty$ proves (2). If $Q\{\text{genus} = \infty\} = \delta > 0$, its right side is at least $(2h-1)\delta$ for every h . Letting $h \rightarrow \infty$ proves the remaining assertions of Theorem 1. $\square$

4 Construction under a slow entropy budget

4.1 A fixed target and positive-volume core neighborhoods

Choose a compact hyperbolic one-holed torus T with boundary length 1, and a compact genus-one hyperbolic surface B with two geodesic boundaries, both of length 1. Attach T to a one-way infinite chain of copies of B , with fixed twists. Denote the resulting surface by S , and choose o in the interior of T .

The two boundary components of B are separated by a positive distance d . Reaching the m th block requires crossing the preceding blocks. Thus bounded balls meet only finitely many compact blocks: S is proper and complete. It has one end and infinite genus. Its initial cores have one geodesic boundary of length 1, and their boundary distances from o tend to infinity.

For each integer $j \geq 1$ choose an initial core of genus h_j with boundary more than $4j$ from o . A sufficiently small, relatively compact Fenchel–Nielsen neighborhood $\mathcal{V}_j$ of its metric has boundary lengths in a fixed interval $I \subset (0, 2 \operatorname{arsinh} 1)$. There is a measurable function $f_j(Y, p) \in [0, 1]$ on its pointed cores with

$$a_j(Y) := \int_Y f_j(Y, p) dA_Y(p) \geq a_j > 0 \tag{9}$$

for every $Y \in \mathcal{V}_j$, such that every root in its support is more than $2j$ from ∂Y and has its pointed j -neighborhood C^j -close to that of (S, o) , with error at most $1/j$. One may express this using compact-neighborhood embeddings rather than a smooth boundary of a metric ball.

For completeness, choose a continuous nonnegative bump on a small coordinate neighborhood of the original root. Extend it continuously over a small finite orbifold chart in the moduli space of cores. Average over the finite stabilizer and shrink the chart. Continuity then gives (9) and the asserted geometric bounds. This also makes the construction isometry-invariant. The integrated WP volume of $\mathcal{V}_j$ is positive. No uniform lower bound on its size as j grows is required.

4.2 A polynomial probability lower bound

Fix j . For $g > 2h_j$ let $M_{j,g}(X)$ count separating geodesics whose genus- h_j side belongs to $\mathcal{V}_j$. If $v_j(\ell)$ denotes the slice volume of that family, (7) gives

$$Z_{j,g} := \mathbb{E}_{\mu_g} M_{j,g} = \frac{c_{h_j}}{V_g} \int_I \ell v_j(\ell) V_{g-h_j,1}(\ell) d\ell.$$

The integral $\int_I \ell v_j(\ell) d\ell$ is fixed, finite and positive. Using the lower bound in (4) and then (5), we obtain

$$Z_{j,g} \geq z_j g^{-(2h_j-1)} \quad (10)$$

for all sufficiently large g , with $z_j > 0$.

An expectation lower bound alone would not bound the probability of a good core from below. Here every counted geodesic has length less than $2 \operatorname{arsinh} 1$. The collar lemma makes any two such simple geodesics disjoint, and distinct geodesics are not freely homotopic. There are at most $3g-3$ disjoint essential nonparallel simple curves. Thus $0 \leq M_{j,g} \leq 3g-3$. For $E_{j,g} = \{M_{j,g} \geq 1\}$ we conclude

$$p_{j,g} := \mu_g(E_{j,g}) \geq \frac{Z_{j,g}}{3g-3} \geq e_j g^{-2h_j} \quad (11)$$

with $e_j > 0$. The extra power of g is harmless for our scale statement; no second-moment or Poisson estimate is needed.

4.3 Rooting and its entropy cost

Sample X from μ_g conditioned on $E_{j,g}$. Choose uniformly one of its $M_{j,g}(X)$ eligible cores Y , and then choose the root with area density $f_j(Y, p)/a_j(Y)$ on that core. This defines an intrinsic measurable kernel on X . Relative to normalized area on X its density is

$$w_{j,g}(X, p) = \frac{A_g}{M_{j,g}(X)} \sum_Y \mathbf{1}_Y(p) \frac{f_j(Y, p)}{a_j(Y)} \leq \frac{A_g}{a_j}. \quad (12)$$

This is a bound on a mixture of densities; it does not require the eligible cores themselves to be disjoint.

Call the resulting joint law $Q_{j,g}$. Its density relative to P_g is $\mathbf{1}_{E_{j,g}} w_{j,g}/p_{j,g}$, so (11) and (12) give

$$\begin{aligned} H(Q_{j,g} \mid P_g) &\leq \log \frac{1}{p_{j,g}} + \log \frac{A_g}{a_j} \\ &\leq (2h_j + 1) \log g + C_j. \end{aligned} \quad (13)$$

The positive area of the good-root set is essential: an exactly prescribed root or metric would be a zero-measure condition. The extra root term costs at most one additional $\log g$, with a constant depending on j .

For each such root the boundary of the selected core is farther than $2j$. Gluing an arbitrary outside component therefore does not alter the pointed j -neighborhood or its local smooth metric. It also cannot provide a shorter route between points in that neighborhood: any such route through the complement must cross the remote boundary twice.

4.4 A slow diagonal

Given $b_g \rightarrow \infty$, choose strictly increasing integers G_j so that, for every $g \geq G_j$,

$$g > 2h_j, \quad b_g \geq 2h_j + 2, \quad \log g \geq \max(C_j, 0),$$

and the fixed- j estimates above hold. These thresholds exist even when b_g is not monotone. Put $j(g) = \max\{j : G_j \leq g\}$ and $Q_g = Q_{j(g),g}$, choosing arbitrary absolutely continuous laws for the finitely many initial genera. Then $j(g) \rightarrow \infty$ and (13) is at most $b_g \log g$. The approximation radius and differentiability order both tend to infinity while the error tends to zero. Thus Q_g converges weakly to $\delta_{(S,o)}$, proving Theorem 2. $\square$

Remark 5. Every volume estimate was used with a fixed core before G_j was chosen. The diagonal does not apply a fixed- h asymptotic uniformly to an uncontrolled growing genus.

5 Unchanged WP marginal and cylindrical limits

Theorem 1 implies genus zero if $\nu_g = \mu_g$. Classical global tangle-free control gives a stronger description.

Proposition 6 (A consequence of [5, 3]). *With the surface marginal μ_g and any measurable rooting kernels, every subsequential pointed smooth weak limit is supported on the hyperbolic plane and hyperbolic cylinders.*

Proof. Fix $a \in (0, 1)$. By [5, Theorem 5], with probability tending to one the entire surface is $(a \log g)$ -tangle-free. The short-loop conclusion of [5, Corollary 10] applies at every point: any fixed bounded collection of based loops lies in a cyclic group once g is large. Equivalently, Proposition 11 of that paper gives plane or cylinder neighborhoods at radius $a \log g/8$.

A noncommuting pair of loops in a limit would have a finite-length compact witness. Its holonomy commutator is nonidentity, and remains so under smooth convergence on a neighborhood of the loops. Such an open witness is incompatible with the preceding event. Limit fundamental groups are therefore trivial or cyclic. The corresponding complete oriented hyperbolic surfaces are the plane, hyperbolic cylinders, and parabolic cylinders.

The parabolic case has zero limit probability. For each $\epsilon > 0$ a parabolic cylinder has a nontrivial loop of length less than ϵ at finite distance from its root. Nontrivial holonomy and the strict length inequality yield an open witness event. The finite surface probability of that event is bounded by $\mu_g\{\text{sys}(X) < \epsilon\} \leq C\epsilon^2$, uniformly for sufficiently small ϵ , by Mirzakhani's estimate recalled in [3, Section 5]. Portmanteau followed by $\epsilon \downarrow 0$ excludes parabolic cylinders. $\square$

This conditional assertion does not claim tightness for arbitrary rooting kernels. It does show that arbitrary re-rooting of the unchanged WP ensemble cannot yield infinite genus.

Proposition 7 (Systolic roots). *Sample X from μ_g and root at an arclength-random point of a systolic geodesic, choosing uniformly among ties. The rooted laws converge to the mixture of core-rooted hyperbolic cylinders with circumference L , where*

$$\mathbb{P}(L > t) = e^{-\Lambda(t)}, \quad \Lambda(t) = \int_0^t \frac{e^u + e^{-u} - 2}{2u} du. \quad (14)$$

Proof. The Poisson short-geodesic theorem [3, Theorem 4.1] gives $L_g = \text{sys}(X) \Rightarrow L$ with (14). This is a positive finite law, and (L_g) is tight. On the $(a \log g)$ -tangle-free event, provided $L_g < a \log g$,

the systolic curve has an embedded cylindrical collar of width $(a \log g - L_g)/2$ [5, Theorem 6]. That width tends to infinity in probability. Every fixed rooted neighborhood therefore agrees, with probability tending to one, with the corresponding neighborhood of the core-rooted complete cylinder of circumference L_g . The cylinder model depends continuously on $L > 0$, proving the assertion. $\square$

Arclength on a geodesic is singular relative to area. The last example has unchanged surface entropy but makes no finite joint-entropy claim. The exact intensity integral in (14), not a small- t coefficient, is the input from [3]. These two propositions are consequences of the cited results, rather than new length-spectrum or tangle-free theorems.

6 Checks and limitations

The proofs are analytic and geometric; they do not depend on finite computations. The accompanying standard-library Python checker tests 6,274 exact rational identities and inequalities: factorial-ratio bookkeeping, core Euler characteristics, finite-probability data processing, binary entropy, and conditioning identities. Two isolated runs, including optimized Python, agree. Integer powers remove logarithms from the finite comparisons. These tests are regressions, not a verification of the geometric volume formulas or of the general theorems. No formal verification or independent referee review is claimed.

The construction changes both the surface law and its rooting. The broad AIM question remains open-ended, and the result concerns only the model stated here. In particular, nothing here identifies a canonical random infinite-genus surface, a uniform growing-core volume asymptotic, or an optimal finite-core entropy coefficient. The frozen source wording and the official workshop report are kept distinct: the report describes the research theme, while the live problem-list page could not be retrieved during this source audit.

AI assistance was used for literature searches, derivation, checking, and manuscript preparation. Responsibility for the statements and their presentation remains with the author.

References

- [1] American Institute of Mathematics, *Surfaces of infinite type: Teichmüller theory and other tools*, workshop report, <https://aimath.org/pastworkshops/genusinfinityrep.pdf>.
- [2] M. Mirzakhani, Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces, *Invent. Math.* **167** (2007), 179–222. doi:10.1007/s00222-006-0013-2. The cutting-space numbering cited here is from the 12 July 2005 preprint.
- [3] M. Mirzakhani and B. Petri, Lengths of closed geodesics on random surfaces of large genus, *Comment. Math. Helv.* **94** (2019), 869–889. arXiv:1710.09727.
- [4] M. Mirzakhani and P. Zograf, Towards large genus asymptotics of intersection numbers on moduli spaces of curves, *Geom. Funct. Anal.* **25** (2015), 1258–1289. doi:10.1007/s00039-015-0336-5. arXiv:1112.1151.
- [5] L. Monk and J. Thomas, The tangle-free hypothesis on random hyperbolic surfaces, arXiv:2008.09363, September 2021 version.
- [6] UnsolvedMath, frozen dataset version 1.6.0, record AIM-TOPOLOGY-0147 (record ID 20003059), The full snapshot commit is recorded in the source supplement. <https://huggingface.co/datasets/ulamai/UnsolvedMath>.