

Induced-Cover Excision Characterizes Dismantlable Hom Tests

Alper Ferudun*

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Abstract

Fix a nonempty finite graph T , with loops permitted, and a nonzero commutative coefficient ring R . We classify when ordinary relative homology of $\text{Hom}(T, -)$ satisfies excision for every induced graph cover. This occurs exactly when T dismantles to a single looped vertex. In fact, exactness of the degree-zero Mayer–Vietoris sequence for all such covers already characterizes these tests. If T is not dismantlable, we construct an obstruction on at most $|V(T)| + 1$ target vertices. Its defect contains a split copy of R , so changing nonzero coefficients does not remove it. The proof uses the established fold theorem and the isolated identity of a stiff graph, followed by an unlooped-twin or a nonedge-deletion construction. For the proposed edge test K_2 , the two-edge cover of a three-vertex loopless path is a counterexample. This classifies a fixed-test approach to the AIM graph-homology question; it does not rule out other homology theories or restricted gluing axioms.

1 The question and the classification

Problem 5.2 in the AIM list on discrete and combinatorial homotopy theory asks for graph homology, its relation to homotopy groups, and an appropriate variant of the Eilenberg–Steenrod axioms. The list suggests taking homology of $\text{Hom}(K_2, X)$ [1]. We examine the excision axiom for the whole family of fixed finite tests T , under a precise ordinary-cover convention.

All graphs in this note are finite sets equipped with a symmetric adjacency relation. Loops are optional and multiple edges are absent. Graph maps preserve adjacency, including loops. Write $N_G(v) = \{w : w \sim_G v\}$; this contains v only if v is looped. The graph P is one looped vertex. A *fold* deletes a vertex v when $N_G(v) \subseteq N_G(w)$ for some $w \neq v$, sending v to w . Here *dismantlable* means reducible by folds to P . A graph without an available fold is *stiff*.

The polyhedral complex $\text{Hom}(T, X)$ has cells indexed by maps

$$\eta : V(T) \longrightarrow \mathcal{P}(V(X)) \setminus \{\emptyset\}, \quad \eta(t) \times \eta(t') \subseteq E(X) \quad (t \sim_T t').$$

The cell is $\prod_{t \in V(T)} \Delta^{\eta(t)}$ and its faces are coordinatewise subsets. Put $\mathcal{H}_T(X) = |\text{Hom}(T, X)|$. Subgraphs give subcomplexes. For a graph pair (X, Y) define

$$h_n^T(X, Y; R) = H_n(\mathcal{H}_T(X), \mathcal{H}_T(Y); R), \quad n \geq 0, \tag{1}$$

using ordinary unreduced homology, with $H_n(\emptyset; R) = 0$. We write $h_n^T(X; R) = h_n^T(X, \emptyset; R)$. Throughout, R is a commutative ring with identity $1_R \neq 0$.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

An *induced cover* $X = A \cup B$ consists of induced subgraphs whose vertices and edges cover those of X . Equivalently, the vertex sets cover $V(X)$ and no edge joins $A \setminus B$ to $B \setminus A$. The pieces and their intersection may be empty.

Theorem 1. *Let T be a nonempty finite graph and let R be as above. The following are equivalent.*

1. *For every finite induced cover $X = A \cup B$, the inclusion $(B, A \cap B) \rightarrow (X, A)$ induces an isomorphism in (1) for every $n \geq 0$.*
2. *For every such cover, the sequence induced by inclusions*

$$h_0^T(A \cap B; R) \xrightarrow{(i_*, -j_*)} h_0^T(A; R) \oplus h_0^T(B; R) \xrightarrow{k_* + l_*} h_0^T(X; R) \longrightarrow 0 \quad (2)$$

is exact.

3. *The graph T is dismantlable to P .*

If these conditions fail, a counterexample to (2) can be constructed with $|V(X)| \leq |V(T)| + 1$. Its middle defect or final cokernel contains a direct summand isomorphic to R .

The fold equivalence used below is classical [2]; the isolated-identity lemma is due to Dochtermann [3, Lemma 6.5]. The point of the argument is the induced-cover classification and its uniform small witness, not a new Hom construction or fold theorem.

2 Natural reduction to a stiff core

Lemma 2. *If $S = T - v$ is a fold of T , then $\mathcal{H}_T(X)$ and $\mathcal{H}_S(X)$ are naturally homotopy equivalent, with homotopies preserving the subcomplexes for every subgraph $Y \subseteq X$. Consequently properties (1) and (2) of Theorem 1 are invariant under folds of the test graph.*

Proof. Let $r : T \rightarrow S$ be the fold and $i : S \rightarrow T$ the inclusion. Precomposition defines maps on Hom face posets, hence on their realizations. We have $ri = \text{id}_S$. For v folded onto w , the coordinatewise union of η and $\eta \circ ir$ is again a multihomomorphism: only the v coordinate is enlarged, by $\eta(w)$, and neighborhood containment verifies all adjacency conditions, including loops. This order-preserving common enlargement gives a poset homotopy between $(ir)^*$ and the identity, as in the standard fold argument [2, Theorem 3.3]. No new target values are introduced, so the homotopy preserves $\text{Hom}(T, Y)$ and is natural in the target. Thus the induced pair homology maps and all the inclusion diagrams are naturally isomorphic. $\square$

Lemma 3 (Dochtermann). *For a finite stiff graph S , the identity is an isolated component of $\mathcal{H}_S(S)$.*

Proof. Suppose a cell contains the identity and an additional value $w \neq v$ in coordinate v . Every neighbor of v must then be a neighbor of w . This contradicts stiffness. The identity belongs to no positive-dimensional cell, and the complex is finite. This is [3, Lemma 6.5]. $\square$

Repeated folding terminates at a nonempty stiff graph S . Lemma 2 transfers both positive properties and negative inclusion diagrams between S and T . We shall construct witnesses using S , keeping the target size at most $|V(S)| + 1$.

3 The positive direction and the degree-zero test

For $T = P$, the complex $\text{Hom}(P, X)$ is the clique complex on the looped vertices of X . If $X = A \cup B$ is an induced cover, a clique cannot meet both exclusive parts: such vertices would be adjacent. Hence

$$\begin{aligned}\mathcal{H}_P(X) &= \mathcal{H}_P(A) \cup \mathcal{H}_P(B), \\ \mathcal{H}_P(A) \cap \mathcal{H}_P(B) &= \mathcal{H}_P(A \cap B).\end{aligned}$$

The cells outside $\mathcal{H}_P(A)$ are exactly those of $\mathcal{H}_P(B)$ outside $\mathcal{H}_P(A \cap B)$. Their relative cellular boundary maps agree. This proves excision for P , including empty complexes. Lemma 2 proves (3) $\Rightarrow$ (1).

For clarity, (1) $\Rightarrow$ (2) does not presume that an arbitrary Hom functor preserves the graph cover. Set

$$Z = \mathcal{H}_T(X), \quad U = \mathcal{H}_T(A), \quad V = \mathcal{H}_T(B), \quad W = U \cup V.$$

We always have $U \cap V = \mathcal{H}_T(A \cap B)$. Ordinary cellular excision identifies $H_*(V, U \cap V; R)$ with $H_*(W, U; R)$. Its composite with the inclusion into $H_*(Z, U; R)$ is the map assumed to be an isomorphism in (1). Thus $H_*(W, U; R) \rightarrow H_*(Z, U; R)$ is an isomorphism. The two pair long exact sequences, with the identity on $H_*(U; R)$, imply that $H_*(W; R) \rightarrow H_*(Z; R)$ is an isomorphism. Apply the ordinary Mayer–Vietoris sequence to $W = U \cup V$ and use this identification to obtain (2).

4 Two obstructions from a stiff core

Assume that a stiff fold core S is not P . There are two exhaustive cases.

4.1 An unlooped vertex

Choose an unlooped vertex $v \in V(S)$. Adjoin an unlooped twin v' with precisely $N_S(v)$ as its neighbors, producing a graph X . In particular, there is no edge vv' . Take

$$A = S, \quad B = X - v, \quad I = A \cap B = S - v.$$

Both pieces are induced and cover X . Let $a : S \rightarrow A$ be the identity copy and $b : S \rightarrow B$ the isomorphism replacing v by v' . Inside $\text{Hom}(S, X)$ they are endpoints of the same one-dimensional cell: assign $\{v, v'\}$ to v and singleton values elsewhere. Therefore

$$z = ([a], -[b]) \in h_0^S(A; R) \oplus h_0^S(B; R)$$

lies in the kernel of the second arrow of (2).

By Lemma 3, projection onto the identity component of $\mathcal{H}_S(A)$ defines an R -linear map

$$\lambda : h_0^S(A; R) \oplus h_0^S(B; R) \longrightarrow R.$$

Every map through I misses v and hence misses that isolated component. Thus λ annihilates the image of the first arrow, whereas $\lambda(z) = 1_R$. The middle term of (2) is not exact. Moreover λ descends to its middle defect, and the map $r \mapsto r[z]$ splits this projection onto R . This construction uses $|V(S)| + 1$ vertices. It also treats the single unlooped vertex, with I empty.

4.2 A reflexive nonpoint core

Suppose every vertex of S is looped. Since $S \neq P$ is stiff, some distinct vertices u, v are nonadjacent. Otherwise every neighborhood would be $V(S)$, admitting a fold. Set

$$X = S, \quad A = S - u, \quad B = S - v.$$

The only possible edge missed by the two induced subgraphs would be uv , which does not exist. Thus they form an induced cover. Neither image in $\mathcal{H}_S(S)$ meets the isolated identity component, because maps through either piece miss a vertex. That component gives a free R -summand in the cokernel of the second arrow in (2), so the arrow is not surjective. Here the target has $|V(S)|$ vertices.

Lemma 2 transfers these obstructions, including the maps and split coefficient summands, to the original test T . They prove (2) $\Rightarrow$ (3), the witness bound and the final assertion of Theorem 1.

5 Consequences and distinctions

Corollary 4. *No fixed nonempty finite test T gives a theory (1) that is both excisive for all induced covers and has nonzero absolute homology on any loopless graph.*

Proof. A dismantlable graph contains a looped vertex, since vertex deletion cannot create loops. A graph map from such a test to a loopless target does not exist, so its Hom complex is empty. Apply Theorem 1. $\square$

For $T = K_2$, the unlooped edge, the twin witness is the loopless path with vertices 0, 1, 2 and edges 01, 12, covered by its two edges. Each edge Hom complex consists of two points; the Hom complex of the intersection $\{1\}$ is empty. The whole path gives two intervals, with maximal cells $(\{1\}, \{0, 2\})$ and $(\{0, 2\}, \{1\})$. Thus (2) becomes

$$0 \longrightarrow R^4 \longrightarrow R^2 \longrightarrow 0,$$

whose middle kernel is R^2 . This is an induced-cover obstruction, unlike the matching cover of a reflexive four-cycle considered in the frozen research aid accompanying [1].

The cover hypothesis matters. Binary interior covers of closure spaces impose a different condition; our earlier note [8] uses them to compare two homology constructions. The present positive clique case overlaps that familiar construction, but the quantification over all fixed Hom tests is different. Nor do the obstructions rule out long-cylinder gluings as in Matsushita’s proof [4, Theorem 5.1] or the graph double mapping cylinders of Goyal–Santhanam [5, Proposition 3.3]. Minichiello’s homotopy-pushout statement assumes a cross-NDR inclusion of simplicial complexes [6, Proposition 3.7]. These are restricted or derived gluing settings, not the universal ordinary induced covers tested here. Existing graph homology theories with other excision conventions also remain outside the no-go statement; see Talbi–Benayat [7].

6 Exact checks and scope of the result

The accompanying implementation enumerates all 1,098 labeled optional-loop graphs on one through four vertices. It checks 2,374 fold steps as graph maps and elementary cross-homotopies. There are 596 dismantlable cases and 502 explicit negative witnesses: 293 twin constructions and 209 reflexive nonedge constructions. Degree-zero maps are computed from actual graph

homomorphisms and one-coordinate Hom edges, not inferred from the theorem. Separately, 46,758 induced covers of targets on at most four vertices satisfy the looped-clique cover identity.

A second implementation enumerates product-simplex cells and constructs their signed cellular boundaries for all 31 negative sources on at most three vertices. Exact rational ranks reproduce the first computation, and every tested boundary square vanishes. This replay shares the witness data but not the fold or component code. Normal and optimized Python runs in isolated directories give identical results. The general coefficient statement follows from the split-summand proof, not from the rational rank tests.

The theorem classifies the specified fixed-test family; it is not a complete resolution of the broader AIM program. The original Hom construction, fold invariance, stiff-identity lemma and the restricted K2 pullback approach are credited to prior work. A bounded literature comparison found no identical classification, but does not establish absolute novelty or priority. AI assistance was used in developing, checking and presenting the arguments and computational material. This preprint is self-audited and unrefereed; no independent review or proof-assistant certification is claimed.

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