A Four-Point Counterexample to Excision for Directed Cubical Homology of Closure Spaces

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Abstract

Bubenik and Milićević defined cubical singular homology theories of Čech closure spaces from an interval and either the product or inductive product. Excision is known for the product theories and was left open for the three inductive theories. We give a four-point counterexample for the directed interval J_+. For $X = J_+ \boxdot J_+$, an interior cover $\{U, V\}$ has

$$H_1^{(J_+,\boxdot)}(V, U \cap V; \mathbb{Z}) \cong \mathbb{Z},$$

generated by the difference of the two coordinate edges. Under inclusion into (X, U) this class is the boundary of the identity square. Hence the excision map is not injective. The calculation is integral, exact, and accompanied by two independently implemented finite-chain checks. This is an unrefereed note and makes no absolute priority claim.

1 Introduction

An AIM problem attributed to Nikola Milićević asks whether cubical homology of Čech closure spaces satisfies the excision axiom for interior covers [AIM]. The relevant cubical theories were developed by Bubenik and Milićević: an interval $J \in \{J_+, J_1, I\}$ and a product operation determine a normalized cubical chain complex [BM24]. Their Eilenberg–Steenrod treatment proves excision for the three product theories. For the inductive product $\boxdot$, the corresponding small-chain equivalence, and therefore excision by that route, was left open [BM21, Remark 5.8].

The same paper gives a four-point example for which the cover-small cubical chain complex is strictly smaller than the full complex [BM21, Example 5.9]. Strict inequality of chain complexes does not itself show that their homologies differ or that an excision map fails. Here the identity square from that four-point geometry is used differently: it fills a class that remains nonzero on one member of a two-set interior cover. This exhibits a kernel in the actual relative-homology map.

The counterexample concerns $H_*^{(J_+,\boxdot)}$ with integer coefficients. It answers the unrestricted yes/no question negatively, but does not decide the theories associated to $(J_1, \boxdot)$ or $(I, \boxdot)$.

2 The directed inductive cube

A closure space is a set Y equipped with a Čech closure operator c. Its interior operator is

$$i(S) = Y \setminus c(Y \setminus S).$$

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A family $\mathcal{U}$ is an *interior cover* when $Y = \bigcup_{S\in\mathcal{U}} i(S)$ [BM21, Definition 2.12]. A map $f : (Y, c) \to (Y', c')$ is continuous when $f(c(S)) \subseteq c'(f(S))$ for every $S \subseteq Y$.

The directed interval $J_+ = \{0, 1\}$ has

$$c(\{0\}) = \{0, 1\}, \qquad c(\{1\}) = \{1\}.$$

For two closure spaces, the canonical neighborhoods of the inductive product are cross-shaped; see [BM21, Definition 2.26]. Thus, in the inductive square

$$X = J_+ \boxdot J_+ = \{o, p, q, r\}, \qquad o = 00,\ p = 10,\ q = 01,\ r = 11,$$

the singleton closures are

$$\begin{aligned} &c(o) = \{o, p, q\}, \quad c(p) = \{p, r\}, \\ &c(q) = \{q, r\}, \qquad c(r) = \{r\}. \end{aligned} \tag{1}$$

We use the normalized cubical chain complex $C_*^{(J_+,\boxdot)}(Y)$ of [BM21, Definition 3.41]. Its n-cubes are continuous maps $J_+^{\boxdot n} \to Y$, modulo cubes whose two opposite faces agree in some coordinate. With A_i and B_i denoting the faces obtained by setting coordinate i to 0 and 1, respectively, the boundary convention is

$$\partial\sigma = \sum_{i=1}^{n} (-1)^i (A_i\sigma - B_i\sigma). \tag{2}$$

For a subspace $W \subseteq Y$, the relative complex is the quotient by $C_*^{(J_+,\boxdot)}(W)$.

Lemma 2.1 (Finite continuity test)**.** *For a map out of $J_+^{\boxdot n}$, continuity is equivalent to the following local condition: whenever $v_i = 0$, the value at the vertex obtained from v by changing coordinate i to* 1 *lies in the closure of the value at v.*

Proof. On a finite Čech closure space, closure is the union of the closures of the points of a set. In the inductive n-cube, the closure of a vertex v consists of v and the vertices obtained by changing one zero coordinate of v to one. The pointwise condition is therefore equivalent to $f(c(S)) \subseteq c(f(S))$ for every subset S. $\square$

3 Failure of excision

Set

$$U = \{p, q, r\}, \qquad V = \{o, p, q\}, \qquad W = U \cap V = \{p, q\}. \tag{3}$$

By (1),

$$i(U) = X \setminus c(\{o\}) = \{r\}, \qquad i(V) = X \setminus c(\{r\}) = V.$$

Consequently $X = i(U) \cup i(V)$, so $\{U, V\}$ is an interior cover.

Theorem 3.1. *For the interior cover* (3)*, the inclusion of pairs*

$$(V, W) \longrightarrow (X, U)$$

does not induce an injection on $H_1^{(J_+,\boxdot)}(-; \mathbb{Z})$. In particular, $H_*^{(J_+,\boxdot)}$ does not satisfy excision.

Proof. In $C_*(V, W)$, degree zero is freely generated by $[o]$. By lemma 2.1 and the induced closure on V, the relative nondegenerate one-cubes are exactly

$$e_p = (o \longrightarrow p), \qquad e_q = (o \longrightarrow q).$$

Equation (2) gives $\partial e_p = \partial e_q = -[o]$. Hence $\ker \partial_1$ is generated primitively by

$$z = e_p - e_q. \tag{4}$$

It remains essential to compute the image of ∂_2. The four nondegenerate relative two-cubes are listed below in vertex order $(00, 10, 01, 11)$. A zero in the face columns means either a degenerate edge or an edge contained in W.

σ	$A_1\sigma$	$B_1\sigma$	$A_2\sigma$	$B_2\sigma$	$\partial\sigma$
(o, o, o, p)	0	e_p	0	e_p	0
(o, o, o, q)	0	e_q	0	e_q	0
(o, p, p, p)	e_p	0	e_p	0	0
(o, q, q, q)	e_q	0	e_q	0	0

The list is exhaustive by the local condition in lemma 2.1: the only nonidentity directed edges in V are $o \to p$ and $o \to q$. Nondegeneracy in both coordinates leaves exactly the four displayed labelings. The signs in (2) show that every displayed relative boundary is zero. Therefore

$$H_1^{(J_+,\boxdot)}(V, W; \mathbb{Z}) = \mathbb{Z} \cdot [z]. \tag{5}$$

Now consider the identity square

$$\tau : J_+ \boxdot J_+ \longrightarrow X.$$

It is continuous and nondegenerate. Its B_1 and B_2 faces lie in U, whereas $A_1\tau = e_q$ and $A_2\tau = e_p$. Thus in the relative complex $C_*(X, U)$,

$$\partial\tau = -e_q + e_p = z.$$

The inclusion sends the nonzero infinite-order class $[z]$ from (5) to zero. It is therefore not injective and cannot be the excision isomorphism. $\square$

Corollary 3.2. *The same four-point cover gives failure after extending coefficients to any nonzero commutative ring.*

Proof. The source calculation has matrices $\partial_1 = (-1 \;\; -1)$ and $\partial_2 = 0$, while the identity-square column is $(1, -1)^{\mathsf{T}}$. These identities persist over every coefficient ring, and $(1, -1)$ is nonzero when $1 \neq 0$. $\square$

4 Verification, literature boundary, and scope

The relative chain calculation is small enough to check by hand, but it was also subjected to two exact exhaustive implementations. The first builds inductive-cube singleton closures and generic face operators, enumerating all continuous cubes through dimension two. The second shares no project code: it tests $f(c(S)) \subseteq c(f(S))$ for every subset S of the domain, constructs the relative boundary matrices directly, and performs exact rank reduction. Both recover

$$\operatorname{rank} C_*(V, W) = (1, 2, 4), \quad \partial_1 = (-1 \;\; -1), \quad \partial_2 = 0,$$

and the identity-square boundary $(1, -1)$.

The subsequent simplicial singular-homology excision theorem for closure spaces [Mil25] does not concern the inductive cubical complex used here. A targeted search through 29 August 2026 located no source stating the four-point kernel in theorem 3.1. This negative search is not a proof of priority, and the note deliberately makes no "first" claim.

The result has a narrow scope. It proves that the directed inductive theory $H_*^{(J_+,\boxdot)}$ fails one Eilenberg–Steenrod axiom. It neither classifies all interior covers nor settles excision for the other two inductive interval choices.

AI-assisted tools were used to enumerate the frozen problem corpus, search for references, test the finite chain complexes, and help prepare this manuscript. The submitting author must inspect the proof and sources, assume responsibility for every claim, and comply with the applicable submission disclosure policy. The note is not peer reviewed.

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