

Binary Interior Excision and Nonuniqueness of Graph Homology

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Abstract

We give an explicit nonuniqueness example for the $(J_1, \times)$ Eilenberg–Steenrod axioms on Čech closure spaces, where J_1 is the indiscrete two-point interval and $\times$ is the categorical product. Apply the known clique and closed neighborhood complex constructions to the mutual singleton graph of a closure space, and take integral relative simplicial homology using intrinsic subspace complexes. For every binary interior cover, both constructions preserve the relevant union and intersection, so excision follows from an actual isomorphism of relative chain complexes. The two theories have coefficient group $\mathbb{Z}$, satisfy the long exact sequence and arbitrary coproduct additivity, and are invariant under the same pair homotopies. Nevertheless, on the undirected four-cycle they give the homology of S^1 and S^2 , respectively. Their restrictions to finite graph pairs also respect homotopies using the ordinary interval. The complexes and general closure-space nonuniqueness are established prior work; our focus is the relative binary-cover argument and its graph consequence. This unrefereed note does not establish uniqueness or nonuniqueness for every strengthened axiom system.

1 The fixed axiom package

The AIM workshop problem attributed to Antonio Rieser asks whether Eilenberg–Steenrod axioms, or a suitable variant, determine homology uniquely for closure spaces or graphs [1, Problem 4.1]. Bubenik and Milićević already state nonuniqueness on the category of closure spaces and ask about subcategories with uniqueness [2, Remark 5.21]. Here we give a concrete graph counterexample for their Definition 5.18 with the indiscrete interval J_1 and categorical product. The crucial excision hypothesis is a *binary interior cover*; neither arbitrary-cover descent nor agreement with ordinary singular homology on topological spaces is part of the package.

The clique and Čech constructions, symmetric singleton reduction, and homotopy-to-contiguity framework occur in [3, Definitions 6.7–6.8 and Theorem 6.17]. For an undirected graph, the second complex below is the established closed neighborhood complex [4, Definition 1.1]. Palacios Vela also studies Čech homology of closure spaces using interior covers [5]. We use intrinsic subspace complexes and quotient chains directly, so no exactness assertion about an inverse limit is needed. We do not claim these complexes, or general closure-space nonuniqueness, as new constructions.

A *Čech closure space* is a set X with a map $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that

$$c(\emptyset) = \emptyset, \quad S \subseteq c(S), \quad c(S \cup T) = c(S) \cup c(T).$$

Idempotence is not required. Put $i(S) = X \setminus c(X \setminus S)$. A map $f : X \rightarrow Y$ is continuous if $f(c_X(S)) \subseteq c_Y(f(S))$ for every $S \subseteq X$. The subspace closure on $A \subseteq X$ is $c_A(S) = A \cap c_X(S)$. An

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elementary homotopy of pairs $(X, A) \rightarrow (Y, B)$ is a continuous $F : X \times J_1 \rightarrow Y$ with $F(A \times J_1) \subseteq B$; homotopy means the equivalence relation generated by these elementary homotopies.

The four axioms considered here are homotopy invariance, excision for $X = i(A) \cup i(B)$, the dimension axiom at a point, and the natural long exact sequence of a pair. We impose, in addition, coefficient group $\mathbb{Z}$ in degree zero and arbitrary coproduct additivity.

2 The two complexes and their relative chains

Define the reflexive symmetric relation

$$xR_X y \iff y \in c_X(\{x\}) \text{ and } x \in c_X(\{y\}).$$

Continuity preserves R , and R_A is its restriction to A . No preservation of infinite unions by c is assumed. Let $K(X)$ be the simplicial complex of finite cliques of R_X . Let $D(X)$ consist of the finite sets σ having a *witness* $w \in X$ such that $wR_X v$ for all $v \in \sigma$. Both complexes contain the empty face, also when $X = \emptyset$. Their vertex set is X .

A continuous vertex map induces simplicial maps on both complexes: it sends a clique to a clique, and a witness w to the witness $f(w)$. Thus both constructions are functorial. The first is the clique or Vietoris–Rips functor; the second is the intrinsic Čech functor after passing to the mutual singleton graph.

For $L = K, D$, define

$$E_n^L(X, A) = H_n(C_*(L(X); \mathbb{Z})/C_*(L(A); \mathbb{Z})), \quad n \geq 0, \quad (1)$$

and let $E_n^L = 0$ for $n < 0$. Here C_* denotes ordinary unreduced oriented simplicial chains. The subspace inclusion gives a simplicial inclusion $L(A) \subseteq L(X)$. Importantly, a witness for $D(A)$ must lie in A . The complex $D(A)$ is generally not the induced subcomplex $D(X)|_A$. For example, for a three-vertex path and its two endpoints A , $D(A)$ is two isolated vertices, whereas $D(X)|_A$ contains their edge.

Lemma 1 (Binary interior-cover identities). *If $i_X(A) \cup i_X(B) = X$, then, for $L = K, D$,*

$$L(X) = L(A) \cup L(B), \quad L(A) \cap L(B) = L(A \cap B). \quad (2)$$

Proof. We first record that

$$w \in i_X(A), \quad wR_X v \implies v \in A. \quad (3)$$

Otherwise $w \in c_X(\{v\}) \subseteq c_X(X \setminus A)$, a contradiction. Also $w \in A$.

If $\sigma \in D(X)$ is nonempty, choose a witness w . The cover places w in $i_X(A)$ or $i_X(B)$. In the first case (3) puts both σ and w in A , so w witnesses $\sigma \in D(A)$. The other case gives $D(B)$. This proves the union identity.

For the intersection identity, suppose nonempty $\sigma \in D(A) \cap D(B)$ has witnesses $a \in A$ and $b \in B$. If either belongs to $A \cap B$, it remains a witness in that subspace. Otherwise $a \in A \setminus B$ and $b \in B \setminus A$. Choose $x \in \sigma$. If $x \in i_X(A)$, then $x \in c_X(\{b\}) \subseteq c_X(X \setminus A)$, a contradiction. If $x \in i_X(B)$, use a instead. The cover exhausts the cases. The reverse inclusion is immediate.

For a nonempty clique, choose any vertex w of it and apply (3); the clique lies in A or B . The intersection identity for cliques follows by restricting the vertex relation. Empty faces cause no exception. $\square$

Corollary 2 (Excision at the chain level). *Under the hypotheses of Lemma 1, inclusion induces an isomorphism of chain complexes*

$$\frac{C_*(L(B))}{C_*(L(A \cap B))} \xrightarrow{\cong} \frac{C_*(L(X))}{C_*(L(A))}.$$

In particular, $(B, A \cap B) \rightarrow (X, A)$ induces isomorphisms in (1) for every n .

Proof. By (2), the simplices in $L(X) \setminus L(A)$ are exactly those in $L(B) \setminus L(A \cap B)$. Use the same orientation for a simplex in either quotient. Inclusion identifies these bases and respects their boundaries. $\square$

3 Homotopy and the remaining axioms

Lemma 3. *Elementary $(J_1, \times)$ homotopies of closure-space pairs induce equal maps on both theories (1).*

Proof. For the categorical closure product,

$$c_{X \times J_1}(\{(x, s)\}) = c_X(\{x\}) \times J_1. \quad (4)$$

Indeed, product neighborhoods contain products of neighborhoods. A point belongs to the closure of a singleton if and only if every one of its neighborhoods contains that singleton, and the only neighborhood of a point of J_1 is all of J_1 .

Let F be an elementary homotopy with endpoint maps f, g . If $xR_X y$, equation (4) shows that (x, s) and (y, t) are mutual singleton neighbors for every $s, t \in J_1$. Their images are therefore mutual neighbors. For a clique σ , this makes $f(\sigma) \cup g(\sigma)$ a clique. For $\sigma \in D(X)$ with witness w , the point $f(w)$ is a common mutual neighbor of $f(\sigma) \cup g(\sigma)$. Thus the induced simplicial maps are contiguous.

If F preserves (A, B) , the argument works on the intrinsic subcomplexes as well: the clique vertices stay in B , and a witness $w \in A$ maps to a witness $f(w) \in B$. To see the relative chain consequence, choose vertex orders, with subspaces inheriting the ambient order, and use oriented simplices. For an increasing generator set

$$P_n[v_0, \dots, v_n] = \sum_{j=0}^n (-1)^j [f(v_0), \dots, f(v_j), g(v_j), \dots, g(v_n)],$$

where a repeated vertex gives zero. Contiguity makes every term admissible. Expansion of the alternating boundary gives $\partial P + P\partial = g_{\#} - f_{\#}$: consecutive mixed terms cancel. Moreover P carries $C_*(L(A))$ into $C_{*+1}(L(B))$. It therefore descends to relative chains and proves the assertion. $\square$

Theorem 4. *The theories E^K, E^D satisfy the four $(J_1, \times)$ Eilenberg–Steenrod axioms of [2, Definition 5.18] on all Čech closure-space pairs. Both have $E_0(*) = \mathbb{Z}$ and preserve arbitrary coproducts of pairs.*

Proof. Corollary 2 gives excision. Lemma 3 and finite zigzags give homotopy invariance. The short exact sequence

$$0 \longrightarrow C_*(L(A)) \longrightarrow C_*(L(X)) \longrightarrow C_*(L(X))/C_*(L(A)) \longrightarrow 0$$

gives the natural long exact sequence of every pair, with its usual connecting homomorphisms. All maps arise from the chain functor. The empty space has zero chains in nonnegative degrees. At

a point each complex is one vertex, so its degree-zero homology is $\mathbb{Z}$ and its positive homology is zero.

A coproduct has no mutual singleton relations between different summands. Every nonempty clique or witnessed simplex lies in one summand. Absolute and relative chain complexes therefore split as direct sums, and direct sums commute with homology. $\square$

4 The four-cycle and the interval restriction

An undirected simple graph is viewed as a closure space via $c(S) = \bigcup_{v \in S} N[v]$, where $N[v]$ is the closed neighborhood. Maps may identify adjacent vertices; subspaces are induced subgraphs. The mutual singleton graph is the original graph with reflexive loops.

Theorem 5. *The restrictions of E^K and E^D to finite undirected graph pairs are not naturally isomorphic. In fact,*

$$E_1^K(C_4) = \mathbb{Z}, \quad E_1^D(C_4) = 0, \quad E_2^K(C_4) = 0, \quad E_2^D(C_4) = \mathbb{Z}.$$

Both degree-zero groups are $\mathbb{Z}$, and all other positive-degree groups vanish.

Proof. Let C_4 have edges 01, 12, 23, 30. Its clique complex has only the four vertices and four edges, and its realization is a circle. Its closed neighborhoods are exactly the four three-element subsets of $\{0, 1, 2, 3\}$. Every proper subset is a face of $D(C_4)$, but the four-element set is not. Consequently $D(C_4) = \partial\Delta^3$, with realization a two-sphere. Their integral homology gives the asserted groups and rules out a natural isomorphism. $\square$

This example is smallest for the absolute homology of this particular pair of theories. Every connected graph on at most three vertices has a dominating vertex: its clique complex is a cone, and its closed neighborhood complex is a full simplex. Additivity treats disconnected graphs. This is not a minimality assertion among all homology theories.

Proposition 6. *On finite undirected graph pairs, both theories are invariant under homotopies using the ordinary topological interval $I = [0, 1]$ and the categorical closure product.*

Proof. Let $F : G \times I \rightarrow H$ be a homotopy between finite graph pairs. The empty source is immediate. For a vertex x and fixed $t \in I$, continuity and the product neighborhood basis give a relative open interval $J_{x,t}$ about t such that

$$F(N_G[x] \times J_{x,t}) \subseteq N_H[F(x, t)].$$

Here $N_G[x]$ and $N_H[F(x, t)]$ are the respective smallest neighborhoods. Intersect over the finitely many x to obtain a neighborhood J_t of t . For $s \in J_t$ and xR_Gy , we have $F(y, s)R_HF(x, t)$. By symmetry all cross-endpoint relations hold. Each time slice is continuous, so F_t, F_s define an elementary J_1 -product homotopy: on the finite cylinder, continuity is equivalent to preservation of the singleton relation. It preserves the pair because F does.

Partition I by finite-zigzag equivalence of its time-slice maps under such elementary pair homotopies. Each class is open by the neighborhoods J_t ; its complement is a union of open classes. Connectedness of I gives a single class. Lemma 3 now applies to the endpoint maps. $\square$

Remark 7. Proposition 6 is restricted to finite graph pairs. On a T_1 topological space the mutual singleton graph is discrete, so both theories have free degree-zero homology on all points. The identity and a constant map of the topological interval induce different maps, although they are ordinarily homotopic. Thus these are not ordinary-interval-invariant theories on the whole category of closure spaces.

The binary-cover condition cannot be dropped. The two edge subsets of a three-vertex path cover its vertices but not by interiors. Their neighborhood complexes miss the three-vertex face of the whole path. Nor does binary excision imply arbitrary finite-cover descent here: the four closed neighborhoods of C_4 form an interior cover, whereas the interior of any proper subset of C_4 has at most one vertex. No two proper subsets form a binary interior cover. These distinctions leave room for stronger axioms in the broader AIM question.

5 Finite checks

The accompanying standard-library Python code constructs clique and witnessed complexes independently using bitsets and set quantifiers. It checks all 4,166 closure relations on at most four points, 139,426 binary interior covers and 278,852 excision identities. On all pairs of three-point closures it tests 45,462 continuous maps and 288,750 elementary cylinders, giving 577,500 contiguity checks. The four-cycle boundary ranks agree over $\mathbb{Q}$ and fields of orders 2, 3, 5, 7. Four negative controls reject incorrect cover or subcomplex substitutions. Normal and optimized Python runs agree byte for byte. These finite checks are regression tests, not proofs of the general statements; in particular the integral sphere and circle identifications establish the absence of torsion.

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