

Higher A-Homotopy Vanishing for Coxeter Chamber Graphs

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Abstract

We prove that the standard Cayley graph of every finite-rank Coxeter system has trivial A-homotopy groups in all degrees at least two. The argument replaces the Coxeter group by the right-angled Coxeter group that retains exactly its commuting relations. All short closed walks lift to the resulting cover, while every finite image in its median Cayley graph admits a based contraction through graph maps. A direct lattice-lifting proof checks the exterior base-point condition and explains why degree one is different. For a finite real reflection group, the theorem establishes the all-degree group comparison for the real 3-parabolic arrangement by combining higher-A vanishing with the previously known fundamental-group comparison and classical asphericity of the complement. The right-angled construction and degree-one kernel are credited to earlier work. The result concerns $k = 3$ and does not settle the general k -parabolic comparison for $k > 3$. This is an unrefereed manuscript; no absolute priority claim is made.

1 Statement and context

For a pointed graph (G, v) , the discrete homotopy groups $A_j(G, v)$ are defined using graph maps from integer lattices that are constant outside finite boxes. Throughout this paper, adjacent vertices may have equal images, and homotopies use the Cartesian product of graphs. These are the A-homotopy conventions of [8, Section 2.1]; they are not the conventions of the distinct $\times$ -homotopy theory.

Let (W, S) be a Coxeter system with S finite. Write $G_W = \text{Cay}(W, S)$ for the simple undirected graph with vertices W and edges w – ws , where $s \in S$. The group W itself need not be finite.

Theorem 1.1. *For every finite-rank Coxeter system (W, S) ,*

$$A_j(G_W, e) = 0 \quad (j \geq 2).$$

When W is finite of rank $n \geq 2$, let $\mathcal{C}(W)$ be its Coxeter complex, and let

$$\mathcal{W}_{n,3} = \{\text{Fix}(P) : P \leq W \text{ is an irreducible rank-two parabolic subgroup}\}.$$

Write $\mathcal{M}(\mathcal{W}_{n,3})$ for the complement of the union of these subspaces in the real reflection representation $\mathbb{R}^n$. The chamber corresponding to e is denoted by σ_e .

Corollary 1.2. *For every finite real reflection group of rank $n \geq 2$,*

$$A_j^{n-2}(\mathcal{C}(W), \sigma_e) = 0 \quad (j \geq 2), \tag{1}$$

$$\pi_j(\mathcal{M}(\mathcal{W}_{n,3})) \cong A_j^{n-2}(\mathcal{C}(W), \sigma_e) \quad (j \geq 1). \tag{2}$$

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The degree-one case of (2) is the theorem of Barcelo, Severs and White [4, Theorem 3.4]. The classical higher groups vanish by Davis, Januszkiewicz and Scott [5, Theorem 0.1.9], as applied in [4, Section 6]. The issue addressed here is the discrete higher vanishing in (1). Lutz’s no-triangles-or-squares theorem [8, Theorem 1.1] already implies it when no two distinct simple generators commute, including the irreducible finite dihedral family. Theorem 1.1 also covers the commuting-square cases.

The AIM list asks about extending arrangement comparisons beyond A_1 through higher- A vanishing [1]. The precise interpretation treated here is $k = 3$. Conjecture 6.2 of [4] also concerns $k > 3$, where a different, denser chamber-adjacency graph is involved; that broader comparison is not proved here. Nor do we construct a canonical comparison map between the arrangement complement and a graph realization. In the higher-degree part of the corollary, the displayed group isomorphisms are between zero groups.

2 A right-angled cover and its squares

Retain the generators and commuting relations of W , but discard all longer braid relations. Thus let

$$R = \langle S \mid s^2 = 1 \ (s \in S), \quad st = ts \ (s \neq t, \ m_W(s, t) = 2) \rangle. \quad (3)$$

There is a surjection $\phi : R \rightarrow W$ taking each generator to itself and a graph map

$$p : G_R = \text{Cay}(R, S) \longrightarrow G_W.$$

This is exactly the group construction denoted by W' in [4, Section 4]. In the finite case those authors identify $\ker \phi$ with the fundamental group of the 3-parabolic complement. Neither the construction nor that degree-one identification is new here.

Lemma 2.1. *Every closed walk of at most four steps in G_W , with stationary steps allowed, has a label word equal to 1 in R . Consequently every such walk lifts closed under p once its initial lift is chosen.*

Proof. The sign character sending every standard generator to -1 is well defined, so G_W is bipartite. In particular it has no triangles. Every simple square comes from two distinct commuting generators. Indeed, after translating its initial vertex to e , the two paths to the opposite vertex give distinct reduced words of length two for the same element. The Coxeter word property connects reduced words by braid moves [6]. Only a length-two braid move can fit in such a word; it swaps commuting letters. Thus the square has vertices

$$w, \quad ws, \quad wst = wts, \quad wt, \quad w, \quad s \neq t, \quad m_W(s, t) = 2.$$

For finite W , this square description is also explicit in [4, Section 3].

Delete the stationary steps of a short closed walk. A nonstationary walk of odd length cannot close. A length-two walk is a backtrack. A length-four walk either is a simple square or splits at a repeated vertex into backtracks. Backtracks contribute $ss = 1$ and simple squares contribute $stst = 1$ with $st = ts$. These relations all hold in R .

At each vertex p bijects the closed star onto the closed star downstairs, since distinct standard generators are distinct nonidentity involutions in both groups. Paths, including stationary steps, therefore lift uniquely from a chosen starting vertex. The label calculation proves the assertion. $\square$

Remark 2.2. Lemma 2.1 verifies more than an ordinary graph covering: a lift of three sides of a Cartesian square extends across the fourth side. Together with unique edge lifting this is the covering criterion of Kapulkin and Mavinkurve [7, Lemma 4.8, Definition 4.9 and Remark 4.10]. Their Proposition 4.14 would give $A_j(G_R, e) \cong A_j(G_W, e)$ for $j \geq 2$. For clarity, Section 4 gives the needed lattice lifting directly, including the basepoint condition at infinity.

3 Finite-image contractions in median graphs

A connected graph is *median* if every three vertices have a unique vertex lying on shortest paths between each pair. We use the classical hypercube and retraction characterizations of median graphs; see [3, Theorem 2.1] and [2].

Lemma 3.1. *The graph G_R of (3) is median.*

Proof. The usual coset cube complex has a cube with vertex set rR_T for each set T of pairwise commuting distinct generators, where $R_T = \langle T \rangle \cong (\mathbb{Z}/2)^{|T|}$. Common coset faces are identified. Standard parabolic intersections give cube-face gluings, and the one-skeleton is G_R .

The complex is connected and simply connected. To see the latter, each loop word follows from the presentation (3): involution relations cancel edge backtracks, while commuting relations are filled by the attached squares. At each vertex the link is the clique complex of the commuting graph on S , hence is flag. Since S is finite, the cube complex is finite-dimensional and locally finite. The CAT(0) cube criterion and the median characterization of CAT(0) cube one-skeleta now apply; see [3, Theorems 2.6 and 2.7]. $\square$

Lemma 3.2. *Let M be a median graph and $F \subset V(M)$ a finite set containing b . There is a finite convex median subgraph H containing F and a graph map*

$$h : H \square I_d \longrightarrow H$$

for some finite d , such that $h(-, 0) = \text{id}_H$, $h(-, d) = b$ and $h(b, t) = b$ for every t .

Proof. Embed M isometrically in a weak hypercube. Let J be the set of coordinates in which a point of F differs from b . Each point differs in finitely many coordinates, their number being its graph distance from b , so J is finite. Intersect M with the coordinate face fixing all coordinates outside J to those of b . The resulting subgraph H contains F and has at most $2^{|J|}$ vertices. It is convex: a shortest path in an isometric hypercube subgraph cannot change a coordinate on which its endpoints agree. Thus H is a finite connected median graph.

The finite median retraction theorem supplies an isometric embedding $i : H \hookrightarrow Q_d$ and a nonexpansive retraction $r : Q_d \rightarrow H$ with $ri = \text{id}_H$. Nonexpansive maps are graph maps under the adjacency-or-equality convention. Write $i(b) = (b_1, \dots, b_d)$ and set

$$\begin{aligned} C_t(x_1, \dots, x_d) &= (b_1, \dots, b_t, x_{t+1}, \dots, x_d), & 0 \leq t \leq d, \\ h(x, t) &= rC_t i(x). \end{aligned}$$

Every C_t is a graph map. The values $C_t(x)$ and $C_{t+1}(x)$ differ in at most one coordinate, so they are equal or adjacent. All C_t fix $i(b)$. The displayed formula therefore defines the required Cartesian-product graph map, with the stated endpoints and fixed basepoint. The case $d = 0$ is the one-vertex constant homotopy. $\square$

Corollary 3.3. *Every median graph M satisfies $A_j(M, b) = 0$ for $j \geq 1$.*

Proof. A based lattice map has finite image. Apply Lemma 3.2 to that image together with b and compose with h . The homotopy stays equal to b outside the same finite box. $\square$

Remark 3.4. The lemma asserts a finite-time contraction on a finite subgraph, not on an arbitrary unbounded infinite median graph. Also, changing only one coordinate at each time is important: resetting several coordinates at once need not give adjacent values and would not prove the Cartesian-product homotopy condition.

4 Based lattice lifting

We give a direct proof of Theorem 1.1. This also makes explicit the difference between degree one and higher degrees.

Proof of Theorem 1.1. Fix $j \geq 2$ and a graph map $f : \mathbb{Z}^j \rightarrow G_W$ representing an element of $A_j(G_W, e)$. Choose $N \geq 1$ such that $f(x) = e$ on

$$E_N = \{x \in \mathbb{Z}^j : \|x\|_\infty \geq N\}.$$

Choose $x_0 \in E_N$. Label an oriented lattice edge $x \rightarrow y$ by the unique generator taking $f(x)$ to $f(y)$, or by 1 if these values coincide. Reversing an edge inverts its label; the nonidentity labels are involutions. Lemma 2.1 says that the label product around every elementary square is 1 in R .

The square two-skeleton of the standard cubical lattice is simply connected. It is the two-skeleton of the cubical cell structure on the contractible space $\mathbb{R}^j$, and attaching cells of dimension at least three does not change the fundamental group. Equivalently, lattice paths with fixed endpoints differ by backtrack removals and elementary-square moves. The formula

$$\tilde{f}(x) = \text{the product in } R \text{ of labels along a path from } x_0 \text{ to } x$$

is therefore path independent. It defines a graph map $\tilde{f} : \mathbb{Z}^j \rightarrow G_R$ satisfying $p\tilde{f} = f$ and $\tilde{f}(x_0) = e$.

For $j \geq 2$, the exterior E_N is connected. One can move along an exterior face while holding one coordinate at a value of absolute value at least N ; after moving another coordinate to a boundary value, the first coordinate can be changed as well. These moves connect all exterior faces. Every label on E_N is 1, so $\tilde{f} = e$ throughout E_N . In particular the lift has finite image and satisfies the based exterior condition.

By Lemmas 3.1 and 3.2, this image lies in a finite median subgraph admitting a contraction h that fixes e . Then

$$(x, t) \mapsto p(h(\tilde{f}(x), t))$$

is a graph map $\mathbb{Z}^j \square I_d \rightarrow G_W$ from f to the constant map e . It equals e on E_N at every time. Thus every class in $A_j(G_W, e)$ is zero. $\square$

Remark 4.1. When $j = 1$, the exterior of a finite interval has two components. The lift of a based loop can be constant at different elements of $\ker \phi$ on these two ends. This is the precise obstruction to using the preceding argument to force $A_1(G_W)$ to vanish. For example, if $W = I_2(m)$ with $m \geq 3$, the graph is C_{2m} , its A_1 is $\mathbb{Z}$, and its higher A-groups vanish.

5 The arrangement consequence and verification

Proof of Corollary 1.2. Two maximal chambers of $\mathcal{C}(W)$ share an $(n - 2)$ -face exactly when their labels differ by a standard generator. Thus their adjacency graph is G_W . The graph formulation of A-theory [8] gives

$$A_j^{n-2}(\mathcal{C}(W), \sigma_e) \cong A_j(G_W, e).$$

Theorem 1.1 proves (1).

The degree-one isomorphism (2) is [4, Theorem 3.4]. For the higher classical groups, the 3-parabolic arrangement is a W -stable collection of codimension-two flats. If it is nonempty, its union contains the origin. Its real complement retracts radially onto the sphere complement, a two-sheeted cover of the projective complement in [5, Theorem 0.1.9]. That projective complement is aspherical, and coverings preserve higher homotopy groups. This proves the needed real asphericity, also recorded explicitly in [4, Section 6].

If the arrangement is empty, every irreducible factor of W has rank one. Then $W \cong (\mathbb{Z}/2)^n$, the real complement is $\mathbb{R}^n$, and G_W is a cube. Both have trivial higher groups. This case is treated directly: removing the origin and using radial projection would not be justified for an empty arrangement. Combining classical and discrete higher vanishing completes the proof. $\square$

The supplementary standard-library Python checker gives exact finite regression tests. It enumerates 43 Coxeter models, including A_1 – A_5 , B_2 – B_5 , D_3 – D_5 , $I_2(m)$ for $2 \leq m \leq 24$, seven cubes, and H_3 in exact $\mathbb{Z}[\varphi]$ matrices. It tests 21,739 short words including stationary steps, of which 2,253 are closed, and constructs retractions for 237 median subgraphs of small hypercubes. The resulting homotopies pass 62,438 horizontal and vertical edge checks. Normal and optimized Python runs produce identical output. Negative controls distinguish ordinary graph covers from square-lifting covers, single-coordinate folds from diagonal jumps, and one-dimensional exteriors from higher ones. These finite tests do not replace the proofs above.

The contribution of this note is the higher-A vanishing argument and its full $k = 3$ consequence, assembled from the credited Coxeter, median and covering tools. A bounded literature search found no direct statement of this full vanishing result; it is not a certificate of priority. The manuscript was prepared with AI assistance in proof development, source comparison, exposition and computational checks. It is unrefereed, and neither independent peer review nor proof-assistant certification is claimed.

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