

Quantum Integer Torsion in the Skein Module of a Connected Sum of Two Solid Tori

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Abstract

We extract an explicit infinite family of torsion elements from the two-solid-torus presentation of Bakshi, Lê and Przytycki. Over $R = \mathbb{Z}[q, q^{-1}]$, put $B = R[x_1, x_2]$ and $[m] = (q^{2m} - q^{-2m})/(q^2 - q^{-2})$. For every $m \geq 2$ we construct an element whose exact B -annihilator is $([m])$, and these elements embed $\bigoplus_{m \geq 2} B/([m])$ into the torsion submodule. Thus no single nonzero Laurent polynomial annihilates all torsion. A further explicit element has annihilator $(q^8 - q^4 + 1)$, comaximal with $(q^2 - q^{-2})$. The calculation answers the coprime-torsion question in arXiv:2604.09971v1 and contradicts its bounded-annihilator corollary; the gap is an identification of an unsaturated quotient with its fraction-field image. Our argument is algebraic and uses the cited presentation for its skein interpretation. It does not compute the skein module of arbitrary closed connected sums of lens spaces.

1 The presentation and the question

Let H_1 be a solid torus. Bakshi, Lê and Przytycki give an integral presentation of the Kauffman bracket skein module of $H_1 \# H_1$ in [1, Theorem 3.1]. We work with that presentation, including its specified coefficient ring and its *module*, rather than ideal, relations. All references to results of [1] mean the version dated April 11, 2026, arXiv:2604.09971v1.

Set $R = \mathbb{Z}[q, q^{-1}]$, $B = R[x_1, x_2]$ and $F = B[y]$. The variable q is an indeterminate. Let S_n be the monic Chebyshev polynomials of the second kind, with

$$S_{-1}(x) = 0, \quad S_0(x) = 1, \quad S_1(x) = x, \quad S_n(x) = xS_{n-1}(x) - S_{n-2}(x) \quad (n \geq 2).$$

For $m \geq 1$ write $T_m = S_m - S_{m-2}$. In particular $T_1 = x$ and $T_2 = x^2 - 2$. Put

$$\{m\} = q^{2m} - q^{-2m}, \quad g = \{1\}, \quad [m] = \frac{\{m\}}{g} = \sum_{j=0}^{m-1} q^{2(m-1-2j)}. \quad (1)$$

These quotients are Laurent polynomials; we do not invert g in R . Define

$$G_n = \{n+1\}S_n(y) + (-1)^{n+1}gS_n(x_1)S_n(x_2) \quad (n \geq 1), \quad (2)$$

and let $\mathcal{G}$ be their B -linear span in F . Then

$$M = F/\mathcal{G} \cong \mathcal{S}(H_1 \# H_1) \quad (3)$$

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by [1, Theorem 3.1, p. 5]. Here and below $\mathcal{S}$ uses the parameter and framing conventions of that source. In particular, the coefficient ring is the one specified in its equation (6). Multiplication by x_1 or x_2 acts on M ; no multiplication by y on the quotient is assumed.

For a module over the domain R , Tor_R denotes the elements killed by some nonzero element of R . Corollary 3.12(a) of [1] states that g kills all of $\text{Tor}_R(M)$. Question 1 on the following page asks whether there is a nonzero torsion element killed by an element coprime to g . We give explicit elements answering that question and show that the corollary is incompatible with (3). The proof does not challenge or rederive the handle-slide presentation.

2 Exact annihilators

Write $e_n = S_n(y)$ and $c_n = S_n(x_1)S_n(x_2)$. Since S_n is monic of degree n , the family $(e_n)_{n \geq 0}$ is a B -basis of F . In this basis

$$G_n = \{n+1\}e_n + (-1)^{n+1}gc_ne_0.$$

Lemma 2.1. *Suppose $f \in F$ is supported in the coordinates $e_0, \dots, e_N$. If $f \in \mathcal{G}$, it is a combination of $G_1, \dots, G_N$, and its e_N coefficient is divisible by $\{N+1\}$ when $N \geq 1$.*

Proof. Any element of $\mathcal{G}$ is a finite sum $\sum a_n G_n$. If L is the largest index with $a_L \neq 0$, its e_L coefficient is $a_L \{L+1\} \neq 0$, because B is a domain. Thus $L \leq N$. Only G_N contributes to the e_N coefficient. $\square$

For $m \geq 2$ set

$$K_m = q^{2m} + q^{-2m}, \quad D_m = T_m(x_1)T_m(x_2), \quad (4)$$

$$W_m = g(K_m e_{2m-1} - (-1)^m D_m e_{m-1}), \quad w_m = W_m + \mathcal{G}. \quad (5)$$

Theorem 2.2. *For every $m \geq 2$,*

$$\text{Ann}_B(w_m) = ([m])B, \quad \text{Ann}_R(w_m) = ([m])R. \quad (6)$$

Moreover, the B -linear map

$$\bigoplus_{m=2}^{\infty} B/([m]) \longrightarrow \text{Tor}_R(M), \quad (\bar{r}_m)_m \longmapsto \sum_m r_m w_m, \quad (7)$$

is injective. The module $\text{Tor}_R(M)$ has no common nonzero annihilator in B , and hence none in R .

Proof. The elementary identities

$$\{2m\} = K_m \{m\}, \quad S_{2m-1}(x) = T_m(x)S_{m-1}(x) \quad (8)$$

give $c_{2m-1} = D_m c_{m-1}$. For completeness, the second identity follows under the injective substitution $x = z + z^{-1}$ from

$$S_n(z + z^{-1}) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}, \quad T_m(z + z^{-1}) = z^m + z^{-m}.$$

The substitution is injective because the highest Laurent exponent of a polynomial of degree d in $z + z^{-1}$ is d , with the same leading coefficient. Consequently the identity is polynomial and also holds at $x = \pm 2$.

Using (8), the constant coordinates cancel in

$$[m]W_m = G_{2m-1} - (-1)^m D_m G_{m-1}. \quad (9)$$

Indeed the constant coordinate on the right is $gc_{2m-1} - gD_m c_{m-1} = 0$, and its two other coordinates are those of the left side. This proves $[m]w_m = 0$.

Conversely, if $r \in B$ satisfies $rW_m \in \mathcal{G}$, then Lemma 2.1 at index $2m-1$ yields $h \in B$ such that

$$rgK_m = h\{2m\} = hgK_m[m].$$

Cancel the nonzero factor gK_m in the domain B to get $r = h[m]$. Together with (9) this proves the first equality in (6). Since $B/([m]) = (R/([m]))[x_1, x_2]$, the contraction of $([m])B$ to R is $([m])R$, proving the second. The Laurent polynomial $[m]$ has q -exponent width $4(m-1) > 0$, so it is a nonunit and $w_m \neq 0$.

Equation (9) makes (7) well defined. Each element of the direct sum has finite support; the product of the corresponding $[m]$ kills its image and is a nonzero element of R . To prove injectivity, suppose a finite sum $\sum r_m W_m$ lies in $\mathcal{G}$. Let m be its largest index with nonzero coefficient. Only W_m contributes to the coordinate e_{2m-1} , so the same leading-coordinate argument gives $r_m \in ([m])B$. Equation (9) allows removal of the entire term $r_m W_m$ modulo $\mathcal{G}$. Descending induction proves that each coefficient class is zero.

Finally a common annihilator $r \neq 0$ would be divisible by $[m]$ for every $m \geq 2$. The q -exponent width of a product of nonzero Laurent polynomials over the domain $\mathbb{Z}[x_1, x_2]$ is the sum of their widths. Hence the fixed width of r would be at least $4(m-1)$ for every m , a contradiction. $\square$

Remark 2.3. The theorem asserts an embedded submodule, not a decomposition or a classification of all torsion. It uses neither a principal-ideal-domain structure theorem nor finite generation. The restriction $m \geq 2$ is necessary: $[1] = 1$ and the same formula for W_1 gives G_1 .

3 Coprime and comaximal examples

The smallest member is particularly explicit. Put $d = (x_1^2 - 2)(x_2^2 - 2)$ and $z = q^2 + q^{-2}$. Then

$$W_2 = g((q^4 + q^{-4})(y^3 - 2y) - dy), \quad (10)$$

$$zW_2 = G_3 - dG_1, \quad \text{Ann}_R(w_2) = (z). \quad (11)$$

In particular $gw_2 \neq 0$, because z does not divide g . After multiplication by Laurent units, z and g become $q^4 + 1$ and $q^4 - 1$. These primitive polynomials have no common nonunit factor in $\mathbb{Z}[q]$. However they are not comaximal over $\mathbb{Z}$:

$$(g, z) = (q^4 - 1, 2) \neq R.$$

The properness follows, for example, by reduction modulo 2 and evaluation at $q = 1$. Thus the two meanings of “coprime” must be distinguished in answering Question 1 of [1].

Corollary 3.1. *There is a nonzero $v \in M$ such that*

$$\text{Ann}_R(v) = (P), \quad P = q^8 - q^4 + 1 = \Phi_{24}(q), \quad (P, g) = R.$$

In particular the coprime-torsion question has a positive answer even if coprimality is interpreted as comaximality of ideals.

Proof. The factorization

$$[6] = Ph, \quad h = q^{-10}(q^4 + 1)(q^8 + q^4 + 1) \in R$$

defines a nonzero h . Set $v = hw_6$. By Theorem 2.2, $rv = 0$ for $r \in B$ if and only if $rh \in (Ph)B$. Cancellation of h shows this is equivalent to $r \in (P)B$. Contracting to R gives the stated annihilator. Since P is not a unit, $v \neq 0$. Finally

$$P - q^6g = 1, \tag{12}$$

which proves $(P, g) = R$. It also directly implies $gv \neq 0$. $\square$

4 The saturation gap

Let $J_n = G_n/g$ and $\mathcal{J} = \text{span}_B\{J_n : n \geq 1\}$. These are well defined over B , since

$$J_n = [n+1]e_n + (-1)^{n+1}c_ne_0.$$

Their distinct highest coordinates make them B -linearly independent. Since $\mathcal{G} = g\mathcal{J}$, there is an isomorphism

$$\mathcal{J}/\mathcal{G} \cong \bigoplus_{n \geq 1} B/(g). \tag{13}$$

This computation, used in [1, Corollary 3.12], is valid. The difficulty is the subsequent claim that $F/\mathcal{J}$ is torsion-free.

Proposition 4.1. *For every $m \geq 2$, the class of $V_m = K_me_{2m-1} - (-1)^m D_me_{m-1}$ in $F/\mathcal{J}$ has exact B -annihilator $([m])$. In particular $F/\mathcal{J}$ is not R -torsion-free.*

Proof. Dividing the polynomial identity (9) by g gives

$$[m]V_m = J_{2m-1} - (-1)^m D_m J_{m-1}.$$

The coordinate lemma applies to the J_n with leading coefficients $[n+1]$. If $rV_m \in \mathcal{J}$, its highest coordinate gives $rK_m = h[2m] = hK_m[m]$ for some $h \in B$. Domain cancellation gives $r \in ([m])B$, and the identity proves the converse. Since $[m]$ is a nonunit, the class is nonzero. $\square$

The relevant B -linear map to the fraction field is induced by

$$e_n \mapsto \frac{(-1)^n c_n}{[n+1]}. \tag{14}$$

It kills every J_n . After extending scalars from R to $\mathbb{Q}(q)$, each diagonal coefficient $[n+1]$ is invertible, so the quotient is $\mathbb{Q}(q)[x_1, x_2]$ and the induced map is injective. Before this extension, each nonzero class from Proposition 4.1 is in its kernel. Consequently the unsaturated quotient $F/\mathcal{J}$ cannot be identified with the submodule of $\text{Frac}(B)$ spanned by the values in (14).

More precisely, with

$$\mathcal{J}^{\text{sat}} = \{f \in F : rf \in \mathcal{J} \text{ for some } 0 \neq r \in R\},$$

the torsion-free quotient of $F/\mathcal{J}$ is $F/\mathcal{J}^{\text{sat}}$, not $F/\mathcal{J}$. Proposition 4.1 proves strictness of the inclusion $\mathcal{J} \subset \mathcal{J}^{\text{sat}}$. Thus (13) gives only a proper part of $\text{Tor}_R(M)$. The explicit w_2 already contradicts its asserted annihilation by g . Theorem 2.2 gives the stronger absence of any nonzero uniform annihilator.

The bounded-order justification of the splitting statement in [1, Corollary 3.12(c)] therefore does not apply as written. This observation does not itself decide whether the torsion submodule splits off over $\mathbb{Q}[q, q^{-1}]$; no negative splitting claim is made.

5 Scope, prior work and checks

Integral skein modules of connected sums have a substantial prior history. Mroczkowski computed the module for $\mathbb{RP}^3 \# \mathbb{RP}^3$ [5]. Bakshi and Przytycki showed that an older integral handlebody formula fails in total genus at least three [2, Theorem 4.1], while leaving the two-solid-torus case as a conjecture. Bakshi, Kim, Shi and Wang computed the module of $(S^1 \times S^2) \# (S^1 \times S^2)$ and studied a splitting obstruction [3]. Belletti and Detcherry proved torsion exists in $L(p, 1) \#^r \mathbb{RP}^3$ when p is even and $r \geq 1$ [4, Theorem 1.4]. None of these attributed results is claimed here as new. Our result is the explicit family with exact annihilators and the saturation diagnosis for the presentation (3); no absolute priority claim is made.

The AIM quantum-topology list asks for the integral module of connected sums of lens spaces [6, Problem 2.25]; its corresponding frozen dataset record is AIM-TOPOLOGY-0043 [7]. Solid tori have boundary. Although they are a precursor to lens-space fillings, extra filling relations can kill torsion. We do not prove survival of any w_m after a filling, and do not claim to resolve the general AIM problem.

The accompanying exact-arithmetic checker verifies 156 assertions: the formulas for $2 \leq m \leq 12$, selected cross-annihilator tests, the comaximal witness, mixed witnesses, and deliberate wrong-sign, missing-factor and module-versus-ideal mutations. Two recorded runs passed all assertions. The checker works in integer Laurent polynomials, without floating point. These are regression checks; the all-index theorem is proved above and is not inferred from them.

AI assistance was used for derivation, literature retrieval, exact symbolic checks and manuscript preparation. The result underwent the originating researcher’s detailed self-audit. This is an unrefereed preprint, not an independently reviewed or proof-assistant-verified result. The skein interpretation relies on the cited presentation; the topological handle-slide proof is not repeated here.

References

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