

Algebraic Zero Curves of Homogeneous Plurisubharmonic Polynomials

Alper Ferudun*

October 5, 2026

Abstract

Let p be a nonzero homogeneous plurisubharmonic polynomial on $\mathbb{C}^2$. We prove that every irreducible affine algebraic curve avoiding the origin on which p vanishes is exactly a nonzero level of a homogeneous holomorphic polynomial. No smoothness or rationality assumption on the curve is needed. The proof uses its compact normalization: positivity of the Levi matrix eliminates every possible polar leading term in a dilation family. Constancy along complex dilations then yields a holomorphic level by polarization. This gives an affirmative answer to the irreducible-curve interpretation of Stensønes's 2010 AIM question. If reducible curves are permitted, we give a counterexample even when the polynomial has no pluriharmonic terms. Novelty and absolute priority are not asserted.

1 The zero curve question

The 2010 AIM workshop list [1, page 2, Plurisubharmonic polynomials, Problem 1] asks whether an algebraic curve avoiding the origin in the zero set of a degree- $2k$ homogeneous plurisubharmonic polynomial on $\mathbb{C}^2$ must be a level of a homogeneous holomorphic polynomial. We use ordinary real homogeneity: $p(rZ) = r^D p(Z)$ for $r > 0$. The source does not explicitly specify whether the curve is irreducible. This distinction affects the answer.

Theorem 1.1. *Let $p \not\equiv 0$ be a real-valued, plurisubharmonic polynomial on $\mathbb{C}^2$, homogeneous of positive degree D . Let $C \subset \mathbb{C}^2$ be an irreducible closed affine algebraic curve such that $0 \notin C$ and $p|_C = 0$. There exists a homogeneous holomorphic polynomial G with $1 \leq \deg G \leq D$ such that*

$$C = \{Z : G(Z) = 1\}.$$

In particular, C is smooth, even though smoothness is not assumed.

The case $D = 2k$ answers the question under the irreducibility convention. Section 4 handles the unrestricted reducible reading. The zero polynomial is excluded, as is implicit in specifying a positive degree. Pluriharmonic terms are allowed throughout the positive theorem.

Bharali and Stensønes establish a homogeneous factorization under a family of harmonic level curves [2, Theorem 2.4 in the arXiv version]; Simon proves a higher-dimensional scalar factorization under an open-family hypothesis [3, Theorem 2.3]. One algebraic zero curve is not that hypothesis. We instead prove a finite-pole rigidity lemma and derive the needed constancy. The argument uses only normalization of algebraic curves, elementary Levi-matrix positivity, polarization and a binomial irreducibility argument.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

2 Finite-pole rigidity

For a twice continuously differentiable real function, its Levi matrix is $(u_{z_j \bar{z}_l})_{j,l}$. Plurisubharmonicity is equivalent to positive semidefiniteness of this Hermitian matrix. In particular, a zero diagonal entry forces the corresponding off-diagonal entries to vanish.

Lemma 2.1. *Let $a(t, \bar{t})$ be a real polynomial on $\mathbb{C}$ satisfying*

$$a \geq 0, \quad aa_{t\bar{t}} - |a_t|^2 \geq 0, \quad a(x) = 0 \quad (x \in \mathbb{R}).$$

Then $a \equiv 0$.

Proof. Write $t = x + iy$. If a is nonzero, let $j \geq 1$ be the smallest power of y with a nonzero polynomial coefficient $A(x)$. Nonnegativity makes j even and gives a real x_0 with $A(x_0) > 0$. At this fixed x_0 ,

$$\begin{aligned} 4(aa_{t\bar{t}} - |a_t|^2) &= a(a_{xx} + a_{yy}) - a_x^2 - a_y^2 \\ &= -jA(x_0)^2 y^{2j-2} + O(y^{2j-1}). \end{aligned}$$

The terms involving x derivatives have order at least $2j$. The displayed negative leading term contradicts the assumed inequality. $\square$

Lemma 2.2. *Let S be a compact connected Riemann surface, $E \subset S$ finite, and $\gamma_1, \dots, \gamma_n$ meromorphic on S and holomorphic on $S \setminus E$. Suppose $Q(t, s)$ is a real polynomial expression in $t, \bar{t}, \gamma_j(s), \overline{\gamma_j(s)}$, is plurisubharmonic on $\mathbb{C} \times (S \setminus E)$, and satisfies*

$$Q(x, s) = 0 \quad (x \in \mathbb{R}, s \in S \setminus E).$$

Then $Q(t, s)$ is independent of s .

Proof. Fix $e \in E$ and a coordinate u centered there. Finite pole orders of the meromorphic functions give a convergent double Laurent expansion

$$Q(t, u) = \sum_{b,c \geq -N} A_{bc}(t, \bar{t}) u^b \bar{u}^c, \tag{1}$$

where each coefficient is a polynomial in $t, \bar{t}$. There are only finitely many terms with $b+c < 0$. The expansion and its derivatives converge uniformly on compact subsets of the punctured coordinate disk, with t in a compact set.

Suppose some negative total degree occurs, and choose the largest $M > 0$ for which some A_{bc} with $b+c = -M$ is not identically zero. On $\mathbb{C} \times \mathbb{C}^*$ the functions

$$R^{-M} Q\left(t, \frac{1}{Rw}\right)$$

are eventually defined on every compact set and converge there in $\mathcal{C}^2$, as $R \rightarrow +\infty$, to the nonzero Laurent polynomial

$$W(t, w) = \sum_{b+c=-M} A_{bc}(t, \bar{t}) w^{-b} \bar{w}^{-c}. \tag{2}$$

Holomorphic pullback, positive multiplication and the limit preserve Levi positivity. Thus W is plurisubharmonic, is real-homogeneous of degree M in w , and vanishes for real t . Individual negative powers in (2) are allowed, since $w \neq 0$.

Average under rotations of w :

$$V(t, w) = \frac{1}{2\pi} \int_0^{2\pi} W(t, e^{i\theta} w) d\theta = a(t, \bar{t}) |w|^M.$$

Here a is a real polynomial; for odd M the average is automatically zero. The function V is plurisubharmonic and $a(x) = 0$ for real x . Direct differentiation on $w \neq 0$ gives

$$V_{w\bar{w}} = \frac{M^2}{4} a |w|^{M-2}, \quad (3)$$

$$\det L_V = \frac{M^2}{4} |w|^{2M-2} (aa_{t\bar{t}} - |a_t|^2). \quad (4)$$

Lemma 2.1 therefore gives $a = 0$.

For each t , the nonnegative function $W_{w\bar{w}}$ has zero rotational average, since differentiation commutes with averaging and $V_{w\bar{w}} = 0$. Hence $W_{w\bar{w}} = 0$ everywhere. Levi positivity also gives $W_{t\bar{w}} = 0$. A real harmonic Laurent polynomial of positive total degree M has the form

$$W(t, w) = h(t, \bar{t}) w^M + \overline{h(t, \bar{t})} \bar{w}^M.$$

Indeed, Laurent monomials of fixed total degree have distinct angular frequencies, and the mixed derivative kills precisely the terms with one exponent zero. Now $W_{t\bar{w}} = 0$ implies $\partial_t \bar{h} = 0$, so h is a holomorphic polynomial in t . The identity $W(x, w) = 0$ for every real x and every w implies $h(x) = 0$. Thus $h = 0$, contradicting the choice of the nonzero leading term.

There are consequently no terms with negative total degree in (1). For a fixed t , absolute convergence on a small circle bounds the series at every smaller radius, since all remaining total exponents are nonnegative. Thus $Q(t, s)$ is bounded near each point of E . Its subharmonic restriction in s extends across the punctures by the removable-singularity theorem for locally bounded-above subharmonic functions. A subharmonic function on a compact connected Riemann surface is constant, proving the assertion. $\square$

Remark 2.3. The global finite-pole assumption cannot be replaced by a local disk. For $t = x + iy$, the function $y^2(1 + |s|^2)$ has Levi determinant $\frac{1}{2}y^2(1 - |s|^2)$ and nonnegative diagonal entries. It is plurisubharmonic for $|s| < 1$ and vanishes for real t , but depends on s .

3 From a curve to a homogeneous level

We record the elementary algebra needed to turn containment in a level into equality with a level.

Lemma 3.1. *Let $H \neq 0$ be a homogeneous polynomial of positive degree in two complex variables. Every irreducible component of $\{H = c\}$, for $c \neq 0$, is exactly a nonzero level of a homogeneous holomorphic polynomial of degree at most $\deg H$.*

Proof. Factor H into distinct linear forms with multiplicities and let m be their greatest common divisor. Absorbing a scalar into a root gives $H = G_0^m$, where the multiplicities of G_0 have gcd one. Each component of $\{H = c\}$ lies in one nonzero level of G_0 . After rescaling, it suffices to show that $G - 1$ is irreducible when those multiplicities have gcd one.

Choose linear coordinates (x, y) such that no linear factor of G is x . Set $d = \deg G$ and write

$$G(x, y) = x^d g(y/x), \quad g(v) = c_0 \prod_i (v - \alpha_i)^{m_i}, \quad \sum_i m_i = d, \quad \gcd_i m_i = 1.$$

Over $K = \mathbb{C}(v)$ consider $X^d - g(v)^{-1}$. If β is a root, all its roots are roots of unity times β . Thus $K(\beta)$ is its splitting field, with cyclic Galois group of order l dividing d . If $l < d$, then $\beta^l \in K$ and $g(v)^{-1}$ is a p th power in K for some prime $p \mid (d/l)$. Its orders $-m_i$ at the points α_i would all be divisible by p , a contradiction. Hence $l = d$ and the binomial is irreducible.

The polynomial $x^d g(v) - 1$ is primitive over $\mathbb{C}[v]$, so Gauss's lemma makes it irreducible in $\mathbb{C}[v, x]$. Localize at x and use

$$\mathbb{C}[x, x^{-1}, v] = \mathbb{C}[x, x^{-1}, y], \quad y = xv.$$

Irreducibility then gives that of $G(x, y) - 1$ in $\mathbb{C}[x, y]$: a factor lost under localization would be a power of x , whereas the constant term is -1 . This proves the lemma. $\square$

Proof of Theorem 1.1. Let S be the normalization of the projective closure of C . It is a compact connected Riemann surface. The affine coordinate functions $\gamma = (\gamma_1, \gamma_2)$ are meromorphic on S and holomorphic off the finite set E above infinity. Points above affine singularities are included in $S \setminus E$.

The function $Q(t, s) = p(t\gamma(s))$ is plurisubharmonic by holomorphic pullback. Since a homogeneous polynomial of degree D satisfies the same scaling identity for all real scalars,

$$Q(x, s) = x^D p(\gamma(s)) = 0 \quad (x \in \mathbb{R}).$$

Lemma 2.2 gives a polynomial $q(t, \bar{t})$ such that

$$p(tZ) = q(t, \bar{t}) \quad (Z \in C, t \in \mathbb{C}). \quad (5)$$

This polynomial q is nonzero. Otherwise p would vanish on all complex dilations of C . The direction map $C \rightarrow \mathbb{P}^1$, $Z \mapsto [z : w]$, is nonconstant: a closed curve contained in a line through zero would equal that line and contain zero. At a point where this direction map has nonzero derivative, the holomorphic map $(t, s) \mapsto t\gamma(s)$ has rank two. Its image contains an open subset of $\mathbb{C}^2$. A real polynomial vanishing there would be identically zero, a contradiction.

Write $p = \sum_{a=0}^D p_a$, where p_a is bihomogeneous of holomorphic degree a and antiholomorphic degree $D - a$. Comparison in (5) of the linearly independent monomials $t^a \bar{t}^{D-a}$ gives

$$p_a|_C = c_a.$$

Since $q \neq 0$ and p is real-valued, some $c_a \neq 0$ has $a > 0$.

Polarize this identity on a nonsingular parameter disk of C . The identity of convergent power series in a parameter and its conjugate implies the corresponding identity in two independent complex variables. Fixing a point Z_0 of that disk therefore gives

$$H(Z) := p_a(Z, \bar{Z}_0), \quad H|_C = c_a \neq 0.$$

The identity first holds on the disk and then on the connected normalization. Here H is a nonzero homogeneous holomorphic polynomial of degree $a \leq D$. The irreducible curve C is an irreducible component of $\{H = c_a\}$. Lemma 3.1, followed by rescaling the nonzero level, gives $C = \{G = 1\}$ with $1 \leq \deg G \leq D$. Finally Euler's identity $dG(Z)[Z] = \deg(G)G(Z)$ shows that this level is smooth. $\square$

4 The reducible interpretation

Irreducibility is essential to exact equality with one level, even when no pluriharmonic terms are present. For $\zeta = x + iy$ set

$$U(\zeta) = |\zeta|^4 - |\zeta|^2 \operatorname{Re}(\zeta^2) = 2x^2y^2 + 2y^4.$$

Since $U_{\zeta\bar{\zeta}} = x^2 + 7y^2 \geq 0$, the polynomial

$$p(z, w) = U(zw), \quad C = \{zw = 1\} \cup \{zw = 2\}$$

is a counterexample to the reducible reading. It is a nonzero, non-pluriharmonic psh polynomial, homogeneous of degree eight. Every monomial has positive holomorphic and antiholomorphic degree. The curve avoids zero and $p|_C = 0$.

To rule out every homogeneous level, suppose a homogeneous polynomial H of degree $d > 0$ were constant with the same nonzero value on both hyperbolas. Write $H(z, w) = \sum_{j=0}^d b_j z^j w^{d-j}$. On $zw = c$,

$$H(t, c/t) = \sum_{j=0}^d b_j c^{d-j} t^{2j-d}.$$

Distinct Laurent powers cannot cancel. Constancy therefore requires d even and $H = b(zw)^{d/2}$. Its nonzero values at $c = 1$ and $c = 2$ are different. A zero level is impossible because it contains the origin. Thus the union is not one homogeneous level, although each component is.

5 Verification and scope

The proof is analytic and algebraic in every degree. The accompanying script supplies 39 exact regression checks of the Levi determinant, transverse coefficients, Laurent harmonic terms, monomial fibers and counterexamples to weakened hypotheses. It is not a proof-assistant formalization. No independent review has been completed.

The literal reducible obstruction and the irreducible quadratic case were already recorded in the upstream machine-generated research notes for AIM-SEVERAL_COMPLEX_VARIABLES-0028 in the frozen `ulamai/UnsolvedMath` v1.6.0 dataset. The present proof treats the irreducible case in all degrees and does not rely on those notes. Our bounded literature comparison does not certify novelty or absolute priority. The conclusions concern affine algebraic curves in $\mathbb{C}^2$; neither arbitrary nonalgebraic curves nor the adjacent AIM Newton-diagram problem is claimed to be covered.

This work was developed and written with AI assistance. The displayed argument and its checks have been self-audited, and the author takes responsibility for the manuscript. It is released as an unrefereed preprint so that the mathematical details can be examined directly.

References

- [1] American Institute of Mathematics, *Emerging applications of complexity for CR mappings*, 2010 workshop problem list, p. 2. <https://aimath.org/WWN/crmappings/crmappings.pdf>.
- [2] G. Bharali and B. Stensønes, Plurisubharmonic polynomials and bumping, *Math. Z.* **261** (2009), 39–63. doi:10.1007/s00209-008-0312-y. Preprint arXiv:0709.3993.
- [3] L. Simon, Homogeneous plurisubharmonic polynomials in higher dimensions, *J. Geom. Anal.* **31** (2021), 11584–11602. doi:10.1007/s12220-021-00697-y. Preprint arXiv:2103.06975.